

POSITIVE SYMMETRIC QUOTIENTS AND THEIR SELFADJOINT EXTENSIONS

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ABSTRACT. We define a quotient B/A of bounded operators A and B on a Hilbert space H with a kernel condition $\ker A \subset \ker B$ as the mapping $Au \rightarrow Bu$, $u \in H$. A quotient B/A is said to be positive symmetric if $A^*B = B^*A \geq 0$. In this paper, we give a simple construction of positive selfadjoint extensions of a given positive symmetric quotient B/A .

1. INTRODUCTION

Let A and B be bounded (linear) operators on a Hilbert space H with a kernel condition $\ker A \subset \ker B$. Then a quotient B/A (of bounded operators A and B) is defined by the mapping $Au \rightarrow Bu$, $u \in H$. W. E. Kaufman [7, 8, 9] investigated various properties about quotients. In [7], he showed that “a linear operator T on H is closed if and only if T is represented as a quotient B/A using A and B such that $A^*H + B^*H (= \{A^*x + B^*y : x, y \in H\})$ is closed in H ”, so that every closed operator is included in the class of quotients. Moreover, he proved that if T is a densely defined closed operator, then T is represented as $T = B/(1 - B^*B)^{1/2}$ using a unique pure contraction B , i.e., an operator such that $\|B\| \leq 1$ and $\ker(1 - B^*B) = \{0\}$. Another study of quotients is given by the first-named author [4, 5, 6]. In particular, he showed [4] explicit formulae for constructing quotients which correspond to sums, products, adjoints and closures (if they exist) of given quotients.

Positive selfadjoint extensions of a closed positive symmetric operator have been investigated by J. von Neumann, K. Friedrichs, M. Krein, T. Ando, K. Nishio and so on. In [10], M. Krein showed that the class of all positive selfadjoint extensions of a given densely defined closed positive symmetric operator has two extremal extensions, that is, the smallest and the largest in the ordering of positive selfadjoint operators. He also showed that the smallest extension coincides with the one obtained by J. von Neumann, and that the largest extension coincides with the one obtained by K. Friedrichs. In [1], T. Ando and K. Nishio gave a necessary and sufficient condition for a (not necessarily densely defined) closed positive symmetric operator to admit a positive selfadjoint extension. Recently, Z. Sebestyén and

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J. Stochel [13] gave a necessary and sufficient condition for existence of a positive selfadjoint operator whose restriction to a linear subspace is a given linear operator.

The main purpose of this paper is to show a simple construction of positive selfadjoint extensions of a given positive symmetric quotient B/A , i.e., $A^*B = B^*A \geq 0$. We first give a necessary and sufficient condition (Theorem 3.2) for a given positive symmetric quotient to admit a positive bounded extension. Secondly, using this theorem, we give a necessary and sufficient condition for a given positive symmetric quotient to admit a (not necessarily bounded) positive selfadjoint extension.

2. PRELIMINARIES

In this section, we state some necessary articles in advance. The next lemma is a basic tool for our discussions.

Lemma 2.1 ([2], [3] Douglas’s majorization theorem). *Let L and M be bounded operators on H . Then the following conditions are equivalent:*

- (i) $LH \subset MH$.
- (ii) $LL^* \leq \alpha MM^*$ for some constant $\alpha > 0$.
- (iii) $L = MN$ for some bounded operator N on H .

In (iii), under the condition $\ker N^* \supset \ker M$, the operator N is uniquely determined, and the identity $\|N\|^2 = (\alpha_{M^*,L} :=) \inf\{\alpha > 0 : LL^* \leq \alpha MM^*\}$ holds.

We define the graph $G(S)$ of a linear operator S having domain $\text{dom } S$ by $\{(u, Su) \in H \times H : u \in \text{dom } S\}$. Then, a linear operator \tilde{S} is said to be an extension of S (i.e. $S \subset \tilde{S}$) if the graph inclusion $G(S) \subset G(\tilde{S})$ holds.

Let K be a subspace of H and let $A : K \rightarrow H$ be linear and bounded, that is, $\sup\{\|Au\|/\|u\| : u \in K, u \neq 0\} < \infty$. Then, we define the natural extension \tilde{A} to H of A by

$$\tilde{A}u = \begin{cases} \lim_{n \rightarrow \infty} Au_n & \text{for } u \in \overline{K}, u_n \rightarrow u, \\ 0 & \text{for } u \in K^\perp, \end{cases}$$

where \overline{K} is the closure of K and K^\perp is the orthogonal complement of K .

The identity $L = MN$ means that M is an extension of L/N , i.e., $L/N \subset M$ from the definition of the quotient L/N . In the above lemma, we have $L^* = N^*M^*$, so that N^* is an extension of L^*/M^* ; more precisely, if $\ker N^* \supset \ker M$, then N^* is the natural extension of L^*/M^* .

According to P. A. Fillmore and J. P. Williams [3], the identity $A^*H + B^*H = (A^*A + B^*B)^{1/2}H$ holds for bounded operators A and B on H . Since $B^*H \subset (A^*A + B^*B)^{1/2}H$, there exists a unique bounded operator Y such that $B^* = (A^*A + B^*B)^{1/2}Y$ and $\ker Y^* \supset \ker(A^*A + B^*B)^{1/2}$. Concerning the operator Y , the following fact holds [4].

Lemma 2.2 ([4, Lemma 2.3]). *Let A and B be bounded operators on H . Then*

$$B^{*(-1)}(A^*H) (= \{u \in H : B^*u \in A^*H\}) = (1 - Y^*Y)^{1/2}H,$$

where Y is the operator given as above.

For a quotient B/A , let us define its graph $G(A, B)$ by $G(A, B) = \{(Au, Bu) : u \in H\}$ in the product Hilbert space $H \times H$. Then if AH is dense in H , the adjoint $(B/A)^*$ of (B/A) has the graph $G(A, B)^* = \{(x, y) : B^*x = A^*y\}$. This implies that the domain of the adjoint $(B/A)^*$ is the subspace $B^{*(-1)}(A^*H)$. For convenience

sake, we put $A_* = (1 - Y^*Y)^{1/2}$, where Y is the operator given as above. Then, since $B^*A_*H = B^*B^{*(-1)}(A^*H) \subset A^*H$, it follows from Lemma 2.1 that there exists a unique bounded operator B_\times satisfying the conditions $B^*A_* = A^*B_\times$ and $\ker B_\times \supset \ker A^*$. Thus, we can see that the adjoint $(B/A)^*$ is the mapping from $x = A_*u$ to $y = B_\times u$ for any $u \in H$. That is,

Lemma 2.3 ([4, Theorem 4.1]). *Suppose that AH is dense in H . Then*

$$(B/A)^* = B_\times/A_*.$$

Let B/A and D/C be quotients. If $G(A, B) \subset G(C, D)$, then we say that D/C is an extension of B/A , and write $B/A \subset D/C$. Concerning this relation, we have the following two lemmas.

Lemma 2.4 ([4, Lemma 2.2]). *Let B/A and D/C be quotients. Then $B/A \subset D/C$ if and only if $A = CN$ and $B = DN$ for some bounded operator N .*

Lemma 2.5 ([6, Lemma 2.6]). *Let B/A and D/C be quotients, and suppose that CH is dense in H . Then the following facts hold:*

- (i) $B/A \subset (D/C)^*$ if and only if $D^*A = C^*B$.
- (ii) $B/A \subset (B/A)^*$ if and only if $B^*A = A^*B$.
- (iii) $B/A = (B/A)^*$ if and only if $B^*A = A^*B$, $B^{*(-1)}(A^*H) = AH$.

From the above lemma, it is reasonable to define a (not necessarily densely defined) quotient B/A to be symmetric if $A^*B = B^*A$ and to be positive symmetric if $A^*B = B^*A \geq 0$.

Concerning closability of a quotient we have

Lemma 2.6 ([5, Lemma 2.3]). *Let B/A be a quotient, and let Y and A_* be the operators given as before, i.e., Y is the unique solution of $B^* = (A^*A + B^*B)^{1/2}Y$, $\ker Y^* \supset \ker(A^*A + B^*B)^{1/2}$ and $A_* = (1 - Y^*Y)^{1/2}$, respectively. Then the following conditions are equivalent:*

- (i) B/A is closable, i.e., $Ax_n \rightarrow 0$ and $Bx_n \rightarrow y$ for a sequence $\{x_n\}$ in H imply $y = 0$.
- (ii) $(1 - Y^*Y)^{1/2}H (= A_*H)$ is dense in H .

The product of two quotients is again a quotient, and as a formula we have

Lemma 2.7 ([4, Theorem 3.2]). *Let B/A and D/C be quotients. Then*

$$(B/A)(D/C) = BM/CN,$$

where M and N are bounded operators satisfying the conditions $AM = DN$ and $NH = D^{-1}(AH)$.

3. BOUNDED EXTENSIONS

We define a quotient B/A to be bounded if

$$\sup\{\|Bu\|/\|Au\| : u \in H, Au \neq 0\} < \infty.$$

Then we have

Lemma 3.1. *Let B/A be a quotient. Then the following conditions are equivalent:*

- (i) B/A is bounded.
- (ii) There exists a constant $\alpha > 0$ such that $B^*B \leq \alpha A^*A$.

(iii) B/A admits a bounded extension, that is, $B/A \subset C$ for some bounded operator C .

In (iii), if we add the condition $\ker C \supset \ker A^*$, then C is the natural extension of B/A .

Proof. (i) \Rightarrow (ii): Assume (i). Then $\|Bu\|^2 \leq \alpha \|Au\|^2$, $u \in H$, for some constant $\alpha > 0$, so that $B^*B \leq \alpha A^*A$ and we have (ii).

(ii) \Rightarrow (iii): Assume (ii). Then by Lemma 2.1 there exists a bounded operator C_1 such that $B^* = A^*C_1$, or $B = C_1^*A$. Hence $B/A \subset C_1^* := C$.

(iii) \Rightarrow (i): If $B/A \subset C$, then $B = CA$, so that $\|Bu\| = \|CAu\| \leq \|C\| \|Au\|$, and $\|Bu\|/\|Au\|$ is bounded. The additional statement is obvious. This completes the proof.

For a positive bounded extension of a positive symmetric quotient B/A , that is, a quotient satisfying $A^*B = B^*A \geq 0$, we have

Theorem 3.2. *Let B/A be a positive symmetric quotient. Then the following conditions are equivalent:*

- (i) B/A admits a positive bounded extension (defined on H).
- (ii) There exists a constant $\alpha > 0$ such that $B^*B \leq \alpha B^*A$.
- (iii) $\ker(B^*A)^{1/2} \subset \ker B$ and $B/(B^*A)^{1/2}$ is bounded.

In this case, there exists a smallest positive bounded extension $(B/A)_N$ of B/A , which has norm equal to $\alpha_{(B^*A)^{1/2}, B}$ ($= \inf\{\alpha > 0 : B^*B \leq \alpha B^*A\}$); moreover, among positive bounded extensions with this norm, there exists a largest extension $(B/A)_F$. Namely, $(B/A)_N \leq D \leq (B/A)_F$ holds for any positive bounded extension D of B/A such that $\|D\| = \alpha_{(B^*A)^{1/2}, B}$.

Proof. (i) \Rightarrow (ii): Suppose that condition (i) holds, i.e., $B = CA$ for some positive bounded operator C . Then, since $B^*B = A^*CCA \leq \|C\|A^*CA = \|C\|B^*A$, condition (ii) holds.

(ii) \Rightarrow (iii): Suppose that condition (ii) holds. Then we easily see that the kernel condition $\ker(B^*A)^{1/2} \subset \ker B$ holds, and that $\|Bu\|^2/\|(B^*A)^{1/2}u\|^2 \leq \alpha$ for $u \in H$, $(B^*A)^{1/2}u \neq 0$, so that $B/(B^*A)^{1/2}$ is bounded.

(iii) \Rightarrow (i): Suppose that condition (iii) holds. Then there exists a natural extension Z of $B/(B^*A)^{1/2}$ (by Lemma 3.1). The operator Z satisfies the equation

$$B = Z(B^*A)^{1/2}, \quad \ker Z \supset \ker(B^*A)^{1/2}.$$

We want to show that ZZ^* (≥ 0) is a positive bounded extension of B/A . Since

$$(B^*A)^{1/2}\{(B^*A)^{1/2} - Z^*A\} = B^*A - (B^*A)^{1/2}Z^*A = B^*A - B^*A = 0$$

by $B = Z(B^*A)^{1/2}$, we have

$$\{(B^*A)^{1/2} - Z^*A\}H \subset \ker(B^*A)^{1/2}.$$

Further, by $\ker Z \supset \ker(B^*A)^{1/2}$,

$$\{(B^*A)^{1/2} - Z^*A\}H \subset (B^*A)^{1/2}H + Z^*H \subset \{\ker(B^*A)^{1/2}\}^\perp.$$

Consequently we have $(B^*A)^{1/2} - Z^*A = 0$, i.e., $(B^*A)^{1/2} = Z^*A$. Hence, we have

$$ZZ^*A = Z(B^*A)^{1/2} = B.$$

This means that ZZ^* is a positive bounded extension of B/A , and we have condition (i).

Next we show the existence of the smallest and the largest positive bounded extensions of B/A . First, we prove that the operator ZZ^* obtained above is the smallest positive bounded extension. Let D be any positive bounded extension of B/A , that is, a positive operator such that $B = DA$. Then we define a densely defined linear operator V as a mapping

$$V : (B^*A)^{1/2}u + v \rightarrow D^{1/2}Au, \quad u \in H, v \in \ker(B^*A)^{1/2}.$$

Then, by a simple computation, we can see that V is a bounded operator (precisely, a contraction). Thus, the domain of V is naturally extended to the whole space H . Note that, for $w = (B^*A)^{1/2}u + v$,

$$Zw = Z\{(B^*A)^{1/2}u + v\} = Z(B^*A)^{1/2}u = Bu = D^{1/2}D^{1/2}Au = D^{1/2}Vw.$$

Thus we obtain $Z = D^{1/2}V$ on H , so that

$$ZZ^* = D^{1/2}VV^*D^{1/2} \leq D^{1/2}D^{1/2} = D.$$

Hence, we see that ZZ^* is the smallest positive bounded extension $(B/A)_N$ of B/A . For the norm of $(B/A)_N$ we have

$$\|(B/A)_N\| = \|ZZ^*\| = \|Z\|^2 = \sup_{(B^*A)^{1/2}u \neq 0} \frac{\|Bu\|^2}{\|(B^*A)^{1/2}u\|^2} = \alpha_{(B^*A)^{1/2}, B}.$$

Next, we have to construct the largest positive bounded extension of B/A having norm $\hat{\alpha} := \|(B/A)_N\| (= \alpha_{(B^*A)^{1/2}, B})$. Then $\hat{\alpha} - B/A = (\hat{\alpha}A - B)/A$ is positive symmetric and bounded. In fact,

$$(\hat{\alpha}A - B)^*A = \hat{\alpha}A^*A - B^*A = \hat{\alpha}A^*A - A^*ZZ^*A = A^*(\hat{\alpha} - ZZ^*)A \geq 0.$$

Also, since

$$(\hat{\alpha}A - B)^*(\hat{\alpha}A - B) - \hat{\alpha}(\hat{\alpha}A - B)^*A = -(\hat{\alpha}A - B)^*B = -(\hat{\alpha}A^*B - B^*B) \leq 0,$$

we see that $\hat{\alpha} - B/A$ has the smallest positive extension $(\hat{\alpha} - B/A)_N$ by Theorem 3.2(ii). Now let D be positive bounded and an extension of B/A such that $\|D\| = \hat{\alpha}$. Then we have $\hat{\alpha}A - B = \hat{\alpha}A - DA = (\hat{\alpha} - D)A$, so that $\hat{\alpha} - D$ is a positive bounded extension of a quotient $\hat{\alpha} - B/A = (\hat{\alpha}A - B)/A$. Hence, $(\hat{\alpha} - B/A)_N \leq \hat{\alpha} - D$, i.e.,

$$D \leq \hat{\alpha} - (\hat{\alpha} - B/A)_N.$$

Note that $\hat{\alpha} - (\hat{\alpha} - B/A)_N$ is a positive bounded extension of B/A , and its norm is $\hat{\alpha}$. Hence we see that

$$(3.1) \quad (B/A)_F := \hat{\alpha} - (\hat{\alpha} - B/A)_N$$

is the largest positive extension of B/A with norm $\hat{\alpha}$. This completes the proof.

In general, the norm $(\alpha_{A,B})^{1/2}$ of B/A is not necessarily equal to the norm $\alpha_{(B^*A)^{1/2}, B}$ of $(B/A)_N$. When are they identical? The following proposition is an answer to this question.

Proposition 3.3. *Suppose that B/A admits a positive bounded extension. If $\ker A^* \subset \ker B^*$, then $\|B/A\| = \|(B/A)_N\| = \|(B/A)_F\|$, that is, B/A admits the norm preserving smallest and largest positive selfadjoint extensions.*

Proof. First, note that $(B/A)_N = ZZ^*$ where Z is the operator defined in the proof of Theorem 3.2 and is, at the same time, the natural extension of $B/(B^*A)^{1/2}$. Suppose that $\ker A^* \subset \ker B^*$. Then we see that ZZ^* is the natural extension of B/A . In fact, since $\overline{ZH} = \overline{BH}$, i.e., $\ker Z^* = \ker B^*$, we see that the condition $\ker A^* \subset \ker B^*$ implies $\ker A^* \subset \ker Z^* = \ker ZZ^*$, so that ZZ^* is the natural extension of B/A . Since a natural extension is a norm preserving extension, we arrive at our assertion. This completes the proof.

4. UNBOUNDED EXTENSIONS

In this section, we will construct positive selfadjoint extensions of a given (unbounded) positive symmetric quotient by using Theorem 3.2.

Let S be a positive bounded operator such that $0 \leq S \leq 1$ and $(1 - S)H$ is dense in H . Then we see that $S/(1 - S)$ is a positive selfadjoint quotient, that is, $S/(1 - S) = \{S/(1 - S)\}^*$. To see this, from Lemma 2.5(iii), it suffices to show $S^{-1}((1 - S)H) = (1 - S)H$. If $u \in S^{-1}((1 - S)H)$, then $Su \in (1 - S)H$, so that $u = (1 - S)u + Su \in (1 - S)H$. Conversely, if $u \in (1 - S)H$ and $u = (1 - S)v$, $v \in H$, then $Su = S(1 - S)v = (1 - S)Sv \in (1 - S)H$. Hence, we have $S^{-1}((1 - S)H) = (1 - S)H$.

Conversely, is a positive selfadjoint operator T represented by $S/(1 - S)$ using S such that $0 < S < 1$ and $(1 - S)H$ is dense in H ? We give the following proposition to answer this question.

Proposition 4.1. *Let T be a densely defined positive selfadjoint operator. Then T is represented by $S/(1 - S)$ using a unique positive bounded operator S such that $0 \leq S \leq 1$ and $(1 - S)H$ is dense in H .*

Proof. Let T be a positive selfadjoint operator. Then since $T^{1/2}$ is also a positive selfadjoint operator, there exists a unique positive pure contraction B (i.e. $\|B\| \leq 1$, $\ker(1 - B^2) = \{0\}$) such that $T^{1/2} = B/(1 - B^2)^{1/2}$ by Kaufman's theorem ([7, Theorem 2]). By Lemma 2.7,

$$T = T^{1/2}T^{1/2} = (B/(1 - B^2)^{1/2})(B/(1 - B^2)^{1/2}) = BM/(1 - B^2)^{1/2}N,$$

where M and N are bounded operators such that $(1 - B^2)^{1/2}M = BN$ and $NH = B^{-1}((1 - B^2)^{1/2}H)$. We can put $M = B$ and $N = (1 - B^2)^{1/2}$, so that we have

$$T = B^2/(1 - B^2) = S/(1 - S) \quad (S = B^2).$$

This completes the proof.

Let B/A be a positive symmetric quotient. Then we have

$$B^*B \leq B^*A + B^*B = B^*(A + B) = B^*A_1 \quad (A_1 = A + B).$$

Thus B/A_1 admits a positive bounded extension with $\alpha_{(B^*A_1)^{1/2}, B} = 1$ by Theorem 3.2(ii). Let S_1 be the smallest extension of B/A_1 such that $0 \leq S_1 \leq 1$, $B = S_1(A + B)$, or $S_1A = (1 - S_1)B$. If $(1 - S_1)H$ is dense in H , then

$$(4.1) \quad B/A \subset \{S_1/(1 - S_1)\}^* = S_1/(1 - S_1),$$

by Lemma 2.5(i), and we obtain a positive selfadjoint extension $S_1/(1 - S_1)$ of B/A . So, when is $(1 - S_1)H$ dense in H ? To answer this question, we give the following theorem.

Theorem 4.2. *Let B/A be a positive symmetric quotient and let S_1 be the smallest positive bounded extension of B/A_1 , i.e., $S_1 = (B/A)_N$ ($A_1 = A + B$). Then the following conditions are equivalent:*

- (i) $(1 - S_1)H$ is dense in H .
- (ii) $\ker(B^*A)^{1/2} \subset \ker B$ and the quotient $B/(B^*A)^{1/2}$ is closable.
- (iii) $(B^*A)^{1/2}x_n \rightarrow 0$ and $Bx_n \rightarrow y$ for a sequence $\{x_n\}$ in H imply $y = 0$.

In particular, if AH is dense in H , then every condition of (i)–(iii) is satisfied.

Proof. (i) \Leftrightarrow (ii): Recall that $S_1 = Z_1Z_1^*$, where Z_1 is the unique solution of $B = Z_1(B^*A_1)^{1/2}$, or $B^* = (B^*A_1)^{1/2}Z_1^*$ with the restriction $\ker Z_1 \supset \ker(B^*A_1)^{1/2}$. On the other hand, since $A_1 = A + B$, putting $\tilde{A} = (B^*A)^{1/2}$, we see that Z_1^* is identical to the unique solution Y of the equation

$$B^* = (\tilde{A}^*\tilde{A} + B^*B)^{1/2}Y, \quad \ker Y^* \supset \ker(\tilde{A}^*\tilde{A} + B^*B)^{1/2}.$$

Hence by Lemma 2.6,

$$(1 - S_1)^{1/2}H = (1 - Z_1Z_1^*)^{1/2}H = (1 - Y^*Y)^{1/2}H$$

is dense in H if and only if $B/\tilde{A} = B/(B^*A)^{1/2}$ is closable.

(ii) \Rightarrow (iii): Immediately from Lemma 2.6.

(iii) \Rightarrow (ii): Assume (iii). Then it is sufficient to show $\ker(B^*A)^{1/2} \subset \ker B$. If $(B^*A)^{1/2}u = 0$, then, putting $x_n = u$, we have $Bx_n \rightarrow y = Bu$, so that $Bu = 0$. Hence, $\ker(B^*A)^{1/2} \subset \ker B$. This completes the proof.

Remark. In the above theorem, condition (iii) corresponds to the condition “positively closable” in [1] of T. Ando and K. Nishio.

Let S_1 be the positive bounded operator given as above. As in the preceding argument, if $(1 - S_1)H$ is dense in H , then we see B/A admits a positive selfadjoint extension $S_1/(1 - S_1)$.

Conversely, if B/A admits a positive selfadjoint extension, then it follows from Proposition 4.1 that the extension is of the form $W/(1 - W)$ such that $0 \leq W \leq 1$ and $(1 - W)H$ is dense in H . Now,

$$\begin{aligned} B/A &\subset \{W/(1 - W)\}^* = W/(1 - W) \\ &\Leftrightarrow WA = (1 - W)B, \text{ i.e., } B = W(A + B) \end{aligned}$$

by Lemma 2.5(i). This means that W is a positive bounded extension of $B/(A + B)$, so that we have $S_1 \leq W$ by minimality of S_1 . Hence, since $(1 - W)^{1/2}H \subset (1 - S_1)^{1/2}H$, we see $(1 - S_1)H$ is dense in H . From this, we obtain the following theorem.

Theorem 4.3. *Let B/A be a positive symmetric quotient. Then B/A admits a positive selfadjoint extension if and only if every condition of (i)–(iii) in Theorem 4.2 is satisfied.*

Now, we introduce an order among positive selfadjoint quotients. Let us assume $W_1/(1 - W_1)$ and $W_2/(1 - W_2)$ are positive selfadjoint quotients (both $(1 - W_1)H$ and $(1 - W_2)H$ are assumed to be dense in H). Then we define

$$W_1/(1 - W_1) \leq W_2/(1 - W_2) \quad \text{if } W_1 \leq W_2.$$

We remark that this order coincides with the usual one when the selfadjoint quotients of both sides are bounded. By definition, we see that the positive selfadjoint

extension $S_1/(1-S_1)$ obtained in (4.1) ($S_1 = (B/A_1)_N$, $A_1 = A+B$) is the smallest extension with respect to the order. So, when does there exist the largest positive selfadjoint extension? We shall show that the largest positive extension exists if AH is dense in H .

Let B/A be a densely defined positive symmetric quotient and let $T_1 = (B/A_1)_F$ ($= 1 - (1 - B/A_1)_N$). Then the quotient $T_1/(1-T_1)$ is the largest positive selfadjoint extension of B/A . In fact, since $B = T_1(A+B)$, or $A = (1-T_1)(A+B)$, we easily see that $(1-T_1)H$ is dense in H , and from $(1-T_1)B = T_1A$ and Lemma 2.5(iii), we obtain

$$B/A \subset \{T_1/(1-T_1)\}^* = T_1/(1-T_1),$$

so that $T_1/(1-T_1)$ is the positive selfadjoint extension of B/A . Moreover, for any positive selfadjoint extension $W/(1-W)$ of B/A ($0 \leq W \leq 1$, $(1-W)H$ is dense in H), we have $A = (1-W)(A+B)$. Since $T_1 = 1 - [1 - B/(A+B)]_N = 1 - [A/(A+B)]_N$ by (3.1), we then have $1 - T_1 \leq 1 - W$, or $W \leq T_1$. Hence $T_1/(1-T_1)$ is the largest positive selfadjoint extension of B/A .

Theorem 4.4. *Let B/A be a positive symmetric quotient. If AH is dense in H , then B/A admits not only the smallest positive selfadjoint extension $S_1/(1-S_1)$ but also the largest one $T_1/(1-T_1)$, where $S_1 = [B/(A+B)]_N$ and $T_1 = [B/(A+B)]_F$. Namely, $S_1 \leq W \leq T_1$ holds for any positive selfadjoint extension $W/(1-W)$ of B/A .*

Remark. If B/A admits a positive bounded extension and $B^*B \leq \alpha B^*A$ for some $\alpha > 0$, then the smallest positive selfadjoint extension $S_1/(1-S_1)$ obtained in (4.1) coincides with $(B/A)_N$.

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