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SOME CLASSES OF TOPOLOGICAL QUASI $*$-ALGEBRAS

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Abstract. The completion $\overline{\mathcal{A}}[\tau]$ of a locally convex $*$-algebra $\mathcal{A}[\tau]$ with not jointly continuous multiplication is a $*$-vector space with partial multiplication xy defined only for x or $y \in \mathcal{A}_0$, and it is called a topological quasi $*$-algebra. In this paper two classes of topological quasi $*$-algebras called strict CQ*-algebras and HCQ*-algebras are studied. Roughly speaking, a strict CQ*-algebra (resp. HCQ*-algebra) is a Banach (resp. Hilbert) quasi $*$-algebra containing a C*-algebra endowed with another involution $\#$ and C*-norm $\| \|_\#$. HCQ*-algebras are closely related to left Hilbert algebras. We shall show that a Hilbert space is a HCQ*-algebra if and only if it contains a left Hilbert algebra with unit as a dense subspace. Further, we shall give a necessary and sufficient condition under which a strict CQ*-algebra is embedded in a HCQ*-algebra.

1. Introduction

Topological quasi $*$-algebras were first introduced by Lassner [6] for the mathematical description of some quantum physical models, and after that, they have been studied by Lassner [6, 7], Trapani [10] and Bagarello-Trapani [3, 4], etc. In this paper we shall study two classes of topological quasi $*$-algebras called strict CQ*-algebras and HCQ*-algebras from a mathematical point of view but also in the perspective of possible physical applications. Let $\mathcal{A}$ be a $*$-algebra with two involutions $*$ and $\#$ and two norms $\| \|$ and $\| \|_\#$ satisfying $\|x^*\| = \|x\|, \|x\| \leq \|x\|_\#$ and $\|x^\# x\|_\# = \|x\|_\#^2$ for each $x, y \in \mathcal{A}$. Then the completion $\overline{\mathcal{A}}[\| \|]$ of $\mathcal{A}[\| \|]$ is a topological quasi $*$-algebras containing (under natural assumptions) two C*-algebras $\mathcal{A}_\#[\| \|_\#]$ and $\mathcal{A}_\flat[\| \|_\flat]$ with different involutions $\#$ and $\flat$, respectively, which are connected by the isometric involution $J : x \to x^*$. This is called a *pseudo CQ*-algebra*. If $\|x\|_\# = \sup\{\|xy\|;\ \|y\| \leq 1\}$, then $\overline{\mathcal{A}}[\| \|]$ is a particular kind of CQ*-algebra as defined and studied in [1, 2], and it is called a *strict CQ*-algebra*, and denoted by $(\overline{\mathcal{A}}[\| \|], \#, \| \|_\#)$. Let $\overline{\mathcal{A}}[\| \|]$ be a topological quasi $*$-algebra with isometric involution $J : x \to x^*$ and Hilbertian norm $\| \|$. If $\mathcal{A}$ has another involution $\#$ satisfying $\|x\| \leq \|L_x\|$ and $L_x^* = L_{x^\#}$ for each $x \in \mathcal{A}$, where L_x is the bounded linear operator on the Hilbert space $\overline{\mathcal{A}}[\| \|]$ defined by $L_x y = xy, y \in \mathcal{A}$, then $\overline{\mathcal{A}}[\| \|]$ is a strict CQ*-algebra with involution $\#$ and C*-norm $\|x\|_\# \equiv \|L_x\|, x \in \mathcal{A}$, and it is called a *HCQ*-algebra* and denoted by $(\overline{\mathcal{A}}[\| \|], \#)$. HCQ*-algebras are closely related to left Hilbert algebras. Let $(\overline{\mathcal{A}}[\| \|], \#)$ be an HCQ*algebra. Then $\mathcal{A}$ is

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a left Hilbert algebra in the Hilbert space $\overline{\mathcal{A}}[\| \ \|]$ with involution $\#$, and the full left Hilbert algebra $\mathcal{A}''$ of $\mathcal{A}$ has unit. But, the isometric involution J does not necessarily coincide with the modular conjugation operator $J_{\mathcal{A}}$ of the left Hilbert algebra $\mathcal{A}$. If $J_{\mathcal{A}} = J$, then the HCQ*-algebra $(\overline{\mathcal{A}}[\| \ \|], \#)$ is said to be *standard*. Suppose that $(\overline{\mathcal{A}}[\| \ \|], \#)$ is standard. Then $\mathcal{A}$ is contained in the maximal Tomita algebra $(\mathcal{A}'')_0$ of $\mathcal{A}''$ and $(\overline{(\mathcal{A}'')_0}[\| \ \|], \#)$ is a standard HCQ*-algebra with the one-parameter group $\{\Delta_{\mathcal{A}}^{it}\}_{t\in\mathbb{R}}$ of $*$-automorphisms, where $\Delta_{\mathcal{A}}$ is the modular operator of $\mathcal{A}$. From these results, it is shown that a Hilbert space is a standard HCQ*-algebra if and only if it contains a left Hilbert algebra as dense subspace. Finally, we give a necessary and sufficient condition under which a strict CQ*-algebra is embedded into a standard HCQ*-algebra using the GNS-construction of positive sesquilinear form on the strict CQ*-algebra $\overline{\mathcal{A}}[\| \ \|]$.

2. Strict CQ*-algebras and HCQ*-algebras

Let $\mathcal{A}[\| \ \|]$ be a normed $*$-algebra with isometric involution $*$ and separately (but not jointly) continuous multiplication. Then the completion, $\overline{\mathcal{A}}[\| \ \|]$, of $\mathcal{A}[\| \ \|]$ is a topological quasi $*$-algebra that we call, as is natural, a *Banach quasi $*$-algebra*. In particular, if $\| \ \|$ is a Hilbertian norm, then $\overline{\mathcal{A}}[\| \ \|]$ is called a *Hilbert quasi $*$-algebra*. For any $a \in \overline{\mathcal{A}}[\| \ \|]$ we put

$$L_a x = ax \text{ and } R_a x = xa, \qquad x \in \mathcal{A}.$$

Then L_a and R_a are linear maps of $\mathcal{A}$ into $\overline{\mathcal{A}}[\| \ \|]$. In particular, if $a \in A$, then L_a and R_a can be extended to bounded linear operators on the Banach space $\overline{\mathcal{A}}[\| \ \|]$ and they are denoted by the same symbols L_a and R_a.

Let $\overline{\mathcal{A}}[\| \ \|]$ be a Banach quasi $*$-algebra and assume that the $*$-algebra $\mathcal{A}$ has another norm $\| \ \|_\#$ and another involution $\#$ satisfying the following conditions:

(a.1) $\|x^\# x\|_\# = \|x\|_\#^2, \quad \forall x \in \mathcal{A}.$
(a.2) $\|x\| \le \|x\|_\#, \quad \forall x \in \mathcal{A}.$
(a.3) $\|xy\| \le \|x\|_\# \|y\|, \quad \forall x, y \in \mathcal{A}.$

Then by (a.2), the identity map $i : \mathcal{A}[\| \ \|_\#] \to \mathcal{A}[\| \ \|]$ has a continuous extension $\hat{i}$ from the completion $\mathcal{A}_\#$ of $\mathcal{A}[\| \ \|_\#]$ ($\mathcal{A}_\#$ is, of course, a C*-algebra) into $\overline{\mathcal{A}}[\| \ \|]$. If $\hat{i}$ is injective, then $\mathcal{A}_\#$ is (identified with) a dense subspace of $\overline{\mathcal{A}}$. This happens if, and only if,

(a.4) two norms $\| \ \|$ and $\| \ \|_\#$ are compatible in the following sense [5]: for any sequence $\{x_n\} \subset \mathcal{A}$ such that $\|x_n\| \to 0$ and $x_n \to x$ in $\mathcal{A}_\#[\| \ \|_\#]$, $x = 0$ results, i.e. if $\hat{i}^{-1} : \mathcal{A}[\| \ \|] \to \mathcal{A}_\#[\| \ \|_\#]$ is closable.

Definition 2.1. A Banach quasi $*$-algebra $\overline{\mathcal{A}}[\| \ \|]$ is said to be a *psuedo CQ*-algebra* if the $*$-algebra $\mathcal{A}$ has a another norm $\| \ \|_\#$ and another involution $\#$ satisfying the conditions (a.1)–(a.4) above. Furthermore, if $\|x\|_\# = \|L_x\| \equiv \sup\{\|xy\|; y \in \mathcal{A} \text{ s.t. } \|y\| \le 1\}$ for each $x \in \mathcal{A}$, then $\overline{\mathcal{A}}[\| \ \|]$ is said to be a *strict CQ*-algebra*.

A pseudo CQ*-algebra $\overline{\mathcal{A}}[\| \ \|]$ is fully determined by the involution $\#$ and the C*-norm $\| \ \|_\#$, and so it will often be denoted by $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$. On the other hand, a strict CQ*-algebra is fully determined when the new involution $\#$ is known; so it can be simply denoted as $(\overline{\mathcal{A}}[\| \ \|], \#)$, making lighter in this way the notation introduced by two of us in [1, 2]. Let $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ be a pseudo CQ*-algebra and, as above, let $\mathcal{A}_\#$ be the C*-algebra obtained by completing the $\#$-algebra $\mathcal{A}$ with respect to the C*-norm $\| \ \|_\#$. Let J be the involution $*$ of the Banach quasi

$*$-algebra $\overline{\mathcal{A}}[\| \ \|]$. Then $\mathcal{A}_\flat \equiv J\mathcal{A}_\#$ is a C*-algebra equipped with the operations $x^* + y^* \equiv (x+y)^*, \lambda x^* \equiv (\overline{\lambda} x)^*, x^* y^* \equiv (yx)^*$, the involution $(x^*)^\flat \equiv x^{\#*}$ and the C*-norm $\|x^*\|_\flat \equiv \|x\|_\#,\ \forall x, y \in \mathcal{A}_\flat$.

Proposition 2.2. *A pseudo CQ*-algebra $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ contains two C*-algebras $\mathcal{A}_\#$ and $\mathcal{A}_\flat \equiv J\mathcal{A}_\#$ with different involutions $\#$ and $\flat$, respectively, as dense subalgebra. In particular, if $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ is a strict CQ*-algebra, then $L_{\mathcal{A}_\#}$ and $R_{\mathcal{A}_\flat}$ are C*-algebras, $L_x R_y = R_y L_x$ for each $x \in \mathcal{A}_\#$ and $y \in \mathcal{A}_\flat$ and $R_{\mathcal{A}_\flat} = J L_{\mathcal{A}_\#} J$.*

By Proposition 2.2, every strict CQ*-algebra is a CQ*-algebra in the sense of [1, 2] but the converse is not true in general ($(\mathcal{A}_\# \cap \mathcal{A}_\flat)$ is not required to be $\#$-invariant).

We summarize the situation with the following scheme:

$$\begin{array}{ccccc} & \subset & \mathcal{A}_\# & \subset & \\ \mathcal{A}[\| \ \|] & & \updownarrow J & & \overline{\mathcal{A}}[\| \ \|] \\ & \subset & \mathcal{A}_\flat & \subset & \\ \text{normed } *\text{-algebra} & & \text{C}^*\text{-algebras} & & \text{CQ}^*\text{-algebra,} \end{array}$$

which summarizes the situation: the $*$-algebra $\mathcal{A}[\| \ \|]$ is contained in its closures, $\mathcal{A}_\# = \overline{\mathcal{A}}[\| \ \|_\#]$ and $\mathcal{A}_\flat = \overline{\mathcal{A}}[\| \ \|_\flat] = J\mathcal{A}_\#$. These C*-algebras, moreover, are both contained in $\overline{\mathcal{A}}[\| \ \|]$.

Definition 2.3. A Hilbert quasi $*$-algebra $\overline{\mathcal{A}}[\| \ \|]$ is said to be a *HCQ*-algebra* if there is another involution $\#$ of $\mathcal{A}$ such that $L_x^* = L_{x^\#}$ and $\|x\| \leq \|L_x\|$ for each $x \in \mathcal{A}$. Here we denote it by $(\overline{\mathcal{A}}[\| \ \|], \#)$.

HCQ*-algebras are closely related to left Hilbert algebras. Before going forth, for the reader's convenience, we briefly review the definitions and the basic properties of left Hilbert algebras. A $*$-algebra $\mathfrak{A}$ with involution $\#$ is said to be a *left Hilbert algebra* if it is a dense subspace in a Hilbert space $\mathcal{H}$ with inner product $(\ | \ )$ satisfying the following conditions:

(i) For any $x \in \mathfrak{A}$ the map $y \in \mathfrak{A} \to xy \in \mathfrak{A}$ is continuous.

(ii) $(xy|z) = (y|x^\# z),\ \forall x, y, z \in \mathfrak{A}$.

(iii) $\mathfrak{A}^2 \equiv \{xy; x, y \in \mathcal{A}\}$ is total in $\mathcal{H}$.

(iv) The involution $x \to x^\#$ is closable in $\mathcal{H}$.

By (i), for any $x \in \mathfrak{A}$ we denote by $\pi_\mathfrak{A}(x)$ the unique continuous linear extension to $\mathcal{H}$ of the map $y \in \mathfrak{A} \to xy \in \mathfrak{A}$; then $\pi_\mathfrak{A}$ is a $*$-representation of $\mathfrak{A}$ on $\mathcal{H}$. We denote by $S_\mathfrak{A}$ the closure of the involution $\#$. Let $S_\mathfrak{A} = J_\mathfrak{A} \Delta_\mathfrak{A}^{\frac{1}{2}}$ be the polar decomposition of $S_\mathfrak{A}$. Then $J_\mathfrak{A}$ is an isometric involution on $\mathcal{H}$ and $\Delta_\mathfrak{A}$ is a non-singular positive self-adjoint operator in $\mathcal{H}$ such that $S_\mathfrak{A} = J_\mathfrak{A} \Delta_\mathfrak{A}^{\frac{1}{2}} = \Delta_\mathfrak{A}^{-\frac{1}{2}} J_\mathfrak{A}$ and $S_\mathfrak{A}^* = J_\mathfrak{A} \Delta_\mathfrak{A}^{-\frac{1}{2}} = \Delta_\mathfrak{A}^{\frac{1}{2}} J_\mathfrak{A}$, and $J_\mathfrak{A}$ is called the *modular conjugation operator* of $\mathfrak{A}$ and $\Delta_\mathfrak{A}$ is called the *modular operator* of $\mathfrak{A}$. We define the commutant $\mathfrak{A}'$ of $\mathfrak{A}$ as follows: For any $y \in \mathcal{D}(S_\mathfrak{A}^*)$ we put $\pi'_\mathfrak{A}(y)x = \pi_\mathfrak{A}(x)y, x \in \mathfrak{A}$ and put $\mathfrak{A}' = \{y \in \mathcal{D}(S_\mathfrak{A}^*); \pi'_\mathfrak{A}(y) \text{ is bounded}\}$. Then $\mathfrak{A}'$ is a left Hilbert algebra in $\mathcal{H}$ with involution $S_\mathfrak{A}^*$ and multiplication $y_1 y_2 \equiv \pi'_\mathfrak{A}(y_2) y_1$. Similarly, the commutant $\mathfrak{A}''$ of $\mathfrak{A}'$ is defined by $\mathfrak{A}'' = \{x \in \mathcal{D}(S_\mathfrak{A}); y \in \mathfrak{A}' \to xy \text{ is continuous}\}$. For any $x \in \mathfrak{A}''$ we denote by $\pi_\mathfrak{A}(x)$ the unique continuous linear operator on $\mathcal{H}$ such that $\pi_\mathfrak{A}(x)y = \pi'_\mathfrak{A}(y)x, y \in \mathfrak{A}'$. Then $\mathfrak{A}''$ is a left Hilbert algebra in $\mathcal{H}$ with involution $S_\mathfrak{A}$ and multiplication $x_1 x_2 \equiv \pi_\mathfrak{A}(x_1) x_2$ containing $\mathfrak{A}$. A left Hilbert algebra $\mathfrak{A}$ is said to be *full* if $\mathfrak{A} = \mathfrak{A}''$. It is well-known as the Tomita fundamental theorem that

$J_{\mathfrak{A}}\pi_{\mathfrak{A}}(\mathfrak{A})'' J_{\mathfrak{A}} = \pi_{\mathfrak{A}}(\mathfrak{A})'$ and $\Delta_{\mathfrak{A}}^{it}\pi_{\mathfrak{A}}(\mathfrak{A})''\Delta_{\mathfrak{A}}^{-it} = \pi_{\mathfrak{A}}(\mathfrak{A})''$, $\forall t \in \mathbb{R}$. Let $\mathfrak{A}$ be a full left Hilbert algebra in $\mathcal{H}$, and $\mathfrak{A}_0 \equiv \{x \in \bigcap_{\alpha\in\mathbb{C}} \mathcal{D}(\Delta_{\mathfrak{A}}^{\alpha}); \Delta_{\mathfrak{A}}^{\alpha} x \in \mathfrak{A},\ \forall \alpha \in \mathbb{C}\}$. Then $\mathfrak{A}_0$ is a left Hilbert subalgebra in $\mathcal{H}$ such that $\mathfrak{A}_0'' = \mathfrak{A}$, $J_{\mathfrak{A}}\mathfrak{A}_0 = \mathfrak{A}_0$ and $\{\Delta_{\mathfrak{A}}^{\alpha}; \alpha \in \mathbb{C}\}$ is a complex one-parameter group of automorphisms of $\mathfrak{A}_0$ such that $(\Delta_{\mathfrak{A}}^{\alpha} x)^{\#} = \Delta_{\mathfrak{A}}^{-\overline{\alpha}} x^{\#}$ and $(\Delta_{\mathfrak{A}}^{\alpha} x)^{*} = \Delta_{\mathfrak{A}}^{-\overline{\alpha}} x^{*}$ for each $\alpha \in \mathbb{C}$ and $x \in \mathfrak{A}_0$. This $\mathfrak{A}_0$ is called the *maximal Tomita algebra* of $\mathfrak{A}$. For more details, see [8, 9, 11].

Proposition 2.4. *Suppose that $(\overline{\mathcal{A}}[\|\ \|], \#)$ is a HCQ*-algebra. Then the following statements hold:*

(i) *$(\overline{\mathcal{A}}[\|\ \|], \#)$ is a strict CQ*-algebra with the C*-norm $\|x\|_{\#} = \|L_x\|, x \in \mathcal{A}$.*

(ii) *$\mathcal{A}$ is a left Hilbert algebra in the Hilbert space $\mathcal{H} \equiv \overline{\mathcal{A}}[\|\ \|]$ whose full left Hilbert algebra $\mathcal{A}''$ has a unit u.*

Proof. (i) The proof is mostly trivial. We prove only that the condition (a.4) is satisfied in this case. Indeed, if $\{x_n\} \subset \mathcal{A}$ is a sequence such that $\|x_n\| \to 0$ and $x_n \to x$ in $\mathcal{A}_{\#}[\|\ \|_{\#}]$, then by the assumption $L_{x_n} \to L_x$ with respect to the operator norm. The continuity of the multiplication in $\mathcal{A}[\|\ \|]$ easily implies that $L_x = 0$; thus $\|x\|_{\#} = 0$ and $x = 0$.

(ii) We first show that $\mathcal{A}$ is a left Hilbert algebra in $\mathcal{H}$ with involution $\#$. Since the C*-algebra $\mathcal{A}_{\#}$ has an approximate identity $\{u_\alpha\}, \mathcal{A}$ is dense in the C*-algebra $\mathcal{A}_{\#}$ and $\|x\| \leq \|x\|_{\#}$ for each $x \in \mathcal{A}$, and then it follows that $\mathcal{A}^2$ is total in $\mathcal{A}[\|\ \|]$. The assumption $L_x^* = L_{x^{\#}} (\forall x \in \mathcal{A})$ implies that $(xy|z) = (y|x^{\#}z)$ for each $x, y, z \in \mathcal{A}$, where $(\ |\ )$ is the inner product defined by the Hilbertian norm $\|\ \|$. Further, we have $\pi_{\mathcal{A}}(x) = L_x$, $\forall x \in \mathcal{A}$ and $\pi_{\mathcal{A}}(x)$ is bounded. Take any sequence $\{x_n\}$ in $\mathcal{A}$ such that $\lim_{n\to\infty} \|x_n\| = 0$ and $\lim_{n\to\infty} \|x_n{}^{\#} - y\| = 0$. Then it follows that $(y|x_1x_2^{\flat}) = \lim_{n\to\infty} (x_n^{\#}|x_1x_2^{\flat}) = \lim_{n\to\infty} (x_2x_1^{\flat}|x_n) = 0$ for each $x_1, x_2 \in \mathcal{A}$, which implies that $x \in \mathcal{A} \mapsto x^{\#} \in \mathcal{A}$ is closable. Thus $\mathcal{A}$ is a left Hilbert algebra in $\mathcal{H}$ with the involution $\#$. We next show that the full left Hilbert algebra $\mathcal{A}''$ has a unit u. For any $\varepsilon > 0$ and for any finite subsets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ of $\mathcal{A}$, we define the set

$$\begin{aligned} K(\varepsilon; \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\}) \\ = \{a \in \mathcal{H}; \|a\| \leq 1, |(ax_k - x_k|y_k)| \leq \varepsilon \\ \text{and } |(x_k a - x_k|y_k)| \leq \varepsilon, k = 1, \ldots, m\}. \end{aligned}$$

Since the C*-algebra $\mathcal{A}_{\#}$ has an approximate identity and $\|x\| \leq \|x\|_{\#}$ for each $x \in \mathcal{A}_{\#}$, it follows that $K(\varepsilon; \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\}) \neq \phi$. Now let $\mathcal{K}$ be the family of all subsets $K(\varepsilon; \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\})$ where $\varepsilon > 0$ and $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_m\}$ are finite subsets. Then $\mathcal{K}$ is a family of non-empty weakly closed subsets of the weakly compact set $\mathcal{H}_1 \equiv \{a \in \mathcal{H}; \|a\| \leq 1\}$. Hence, the intersection of all the sets in $\mathcal{K}$ is non-empty. Hence, an element u of this intersection is such that u is a *quasi-unit* of the topological quasi $*$-algebra $\overline{\mathcal{A}}[\|\ \|]$, that is, $u \in \overline{\mathcal{A}}[\|\ \|]$ and $ux = xu = x$ for each $x \in \mathcal{A}$. Since

$$(S_{\mathcal{A}}x|u) = (x^{\#}|u) = (u|L_x u) = (u|x)$$

for each $x \in \mathcal{A}$, it follows that $u \in \mathcal{D}(S_{\mathcal{A}}^*)$ and $\pi'_{\mathcal{A}}(u) = I$. Hence, $u \in \mathcal{A}'$ and $S_{\mathcal{A}}^* u = u$, which implies that

$$(S_{\mathcal{A}}^* y|u) = (\pi'_{\mathcal{A}}(S_{\mathcal{A}}^* y)u|u) = (u|\pi'_{\mathcal{A}}(y)u) = (u|y)$$

for each $y \in \mathcal{A}'$. Hence, we have $u \in \mathcal{A}''$ and $S_{\mathcal{A}}u = u$. This completes the proof. □

By Proposition 2.4, the situation of HCQ*-algebras can be schematized with the following diagram:

$$\begin{array}{ccccccccc} & \subset & \mathcal{A}_{\#} & \subset & \mathcal{A}'' = L''_{\mathcal{A}}u & \subset & \mathcal{D}(S_{\mathcal{A}}) & \subset & \\ \mathcal{A} & & \updownarrow J & & \updownarrow J_{\mathcal{A}} & & \updownarrow J_{\mathcal{A}} & & \overline{\mathcal{A}}[\| \ \|] \\ & \subset & \mathcal{A}_{\flat} & \subset & \mathcal{A}' = L_{\mathcal{A}}{}'u & \subset & \mathcal{D}(S^*_{\mathcal{A}}) & \subset & \end{array}$$

We now look for conditions under which $J = J_{\mathcal{A}}$.

Lemma 2.5. *Let $(\overline{\mathcal{A}}[\| \ \|], \#)$ be a HCQ*-algebra. Then the following statements are equivalent:*
(i) $J = J_{\mathcal{A}}$.
(ii) $(x^{\#}|x^*) \geq 0$ *for each* $x \in \mathcal{A}$.

Proof. (i) $\Rightarrow$ (ii) This follows from

$$(x^{\#}|x^*) = (J_{\mathcal{A}}\Delta^{\frac{1}{2}}_{\mathcal{A}}x|J_{\mathcal{A}}x) = (x|\Delta^{\frac{1}{2}}_{\mathcal{A}}x) \geq 0, \ \forall x \in \mathcal{A}.$$

(ii) $\Rightarrow$ (i) By the assumption (ii) we have $S_{\mathcal{A}} = J(JJ_{\mathcal{A}}\Delta^{\frac{1}{2}}_{\mathcal{A}})$ and $JJ_{\mathcal{A}}\Delta^{\frac{1}{2}}_{\mathcal{A}} \geq 0$. The uniqueness of the polar decomposition of $S_{\mathcal{A}}$ implies $J = J_{\mathcal{A}}$. □

If any one of the two equivalent statements of Lemma 2.5 holds, we say that the HCQ*-algebra $(\overline{\mathcal{A}}[\| \ \|], \#)$ is *standard*.

Remark 2.6. Let $(\overline{\mathcal{A}}[\| \ \|], \#)$ be a HCQ*-algebra. If it is standard, then $R'_{\mathcal{A}} = L''_{\mathcal{A}}$. Conversely, if $R'_{\mathcal{A}} = L''_{\mathcal{A}}$, then $JJ_{\mathcal{A}} = J_{\mathcal{A}}J$, but we don't know whether $J = J_{\mathcal{A}}$.

Since two HCQ*-algebras $(\overline{\mathcal{A}}[\| \ \|], \#)$ with $(\overline{\mathcal{B}}[\| \ \|], \#), \overline{\mathcal{A}}[\| \ \|] = \overline{\mathcal{B}}[\| \ \|]$ as Hilbert spaces, need not coincide as HCQ*-algebras, we introduce the following notion:

Definition 2.7. A HCQ*-algebra $\overline{\mathcal{A}}[\| \ \|]$ is said to be an *extension* of a HCQ*-algebra $\overline{\mathcal{B}}[\| \ \|]$ if $\mathcal{B}$ is a dense $*$-subalgebra of $\mathcal{A}$ and $S_{\mathcal{A}} = S_{\mathcal{B}}$.

Proposition 2.8. *Let $(\overline{\mathcal{A}}[\| \ \|], \#)$ be a standard HCQ*-algebra, and $\mathcal{B} \equiv (\mathcal{A}'')_0$ the maximal Tomita algebra of the full left Hilbert algebra $\mathcal{A}''$. Then $(\overline{\mathcal{B}}[\| \ \|], S_{\mathcal{A}})$ is a standard HCQ*-algebra and it is an extension of $(\overline{\mathcal{A}}[\| \ \|], S_{\mathcal{A}})$. Further, $\{\Delta^{it}_{\mathcal{A}}\}_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms of the Hilbert quasi $*$-algebra $\overline{\mathcal{B}}[\| \ \|]$, that is, $\Delta^{it}_{\mathfrak{A}}\mathcal{B} = \mathcal{B}$, $(\Delta^{it}_{\mathcal{A}}a)^* = \Delta^{it}_{\mathcal{A}}a^*$, $\Delta^{it}_{\mathcal{A}}(ax) = (\Delta^{it}_{\mathcal{A}}a)(\Delta^{it}_{\mathcal{A}}x)$ and $\Delta^{it}_{\mathcal{A}}(xa) = (\Delta^{it}_{\mathcal{A}}x)(\Delta^{it}_{\mathcal{A}}a)$ for all $a \in \overline{\mathcal{B}}[\| \ \|], x \in \mathcal{B}$ and $t \in \mathbb{R}$.*

Proof. It is almost clear that $\overline{\mathcal{B}}(\| \ \|)$ is a Hilbert quasi $*$-algebra with the involution $J_{\mathcal{A}} = J_{\mathcal{B}}$ and further $(\overline{\mathcal{B}}[\| \ \|], S_{\mathcal{A}})$ is a standard HCQ*-algebra. Since $\{\Delta^{it}_{\mathcal{A}}\}_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms of the Tomita algebra $\mathcal{B}$, it follows that $\{\Delta^{it}_{\mathcal{A}}\}_{t\in\mathbb{R}}$ is also a one-parameter group of $*$-automorphisms of the Hilbert quasi $*$-algebra $\overline{\mathcal{B}}[\| \ \|]$. □

Finally, we consider the question of when a Hilbert space can be regarded as a standard HCQ*-algebra. By Proposition 2.4, 2.8 and [9], Theorem 13.1, we have the following:

Theorem 2.9. *Let $\mathcal{H}$ be a Hilbert space. The following statements are equivalent:*
(i) *$\mathcal{H}$ is a standard HCQ*-algebra.*
(ii) *$\mathcal{H}$ contains a left Hilbert algebra with unit as dense subspace.*
(iii) *There exists a von Neumann algebra on $\mathcal{H}$ with a cyclic and separating vector.*

It is worth noticing, in particular, that the implication (iii) $\Rightarrow$ (i) *shows that the class of standard HCQ*-algebras is rather rich.*

3. The structure of strict CQ*-algebras

In this section we study when a strict CQ*-algebra is embedded in a standard HCQ*-algebra. For that, we need a GNS-like construction for a class of positive sesquilinear forms on strict CQ*-algebras $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$. A sesquilinear form φ on $\overline{\mathcal{A}}[\| \ \|] \times \overline{\mathcal{A}}[\| \ \|]$ is said to be *positive* if $\varphi(a,a) \geq 0$ for all $a \in \overline{\mathcal{A}}[\| \ \|]$, and φ is said to be *faithful* if $\varphi(a,a) = 0, a \in \overline{\mathcal{A}}[\| \ \|]$, implies $a = 0$. Further, we need the following notion:

Definition 3.1. Let $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ and $(\overline{\mathcal{B}}[\| \ \|_1], \#_1, \| \ \|_{\#_1})$ be strict CQ*-algebras. A linear map $\Phi : \overline{\mathcal{A}}[\| \ \|] \to \overline{\mathcal{B}}[\| \ \|_1]$ is said to be a $*$*-homomorphism* of $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ into $(\overline{\mathcal{B}}[\| \ \|_1], \#_1, \| \ \|_{\#_1})$ if (i) Φ is a $*$-homomorphism of the quasi $*$-algebra $\overline{\mathcal{A}}[\| \ \|]$ into the quasi $*$-algebra $\overline{\mathcal{B}}[\| \ \|_1]$, that is, $\Phi(\mathcal{A}) \subset \mathcal{B}$ and $\Phi(a)^* = \Phi(a^*), \Phi(ax) = \Phi(a)\Phi(x)$ and $\Phi(xa) = \Phi(x)\Phi(a)$ for all $a \in \overline{\mathcal{A}}[\| \ \|]$ and $x \in \mathcal{A}$; (ii) $\Phi\lceil\mathcal{A}_\#$ is a $*$-homomorphism of the C*-algebra $\mathcal{A}_\#$ into the C*-algebra $\mathcal{B}_{\#_1}$. A bijective (resp. injective) $*$-homomorphism Φ such that $\Phi(\mathcal{A}) = \mathcal{B}$ and $\Phi(\mathcal{A}_\#) = \mathcal{B}_{\#_1}$ is called a $*$*-isomorphism* of $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ onto (resp. into) $(\overline{\mathcal{B}}[\| \ \|_1], \#_1, \| \ \|_{\#_1})$. A $*$-homomorphism Φ is said to be *contractive* if $\|\Phi(a)\|_1 \leq \|a\|$ for all $a \in \overline{\mathcal{A}}[\| \ \|]$. A contractive $*$-isomorphism whose inverse is also contractive is called an *isometric* $*$*-isomorphism*.

Theorem 3.2. *Let $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ be a strict CQ*-algebra with quasi-unit u. Then the following statements are equivalent:*

(i) *There exists a contractive $*$-homomorphism (resp. $*$-isomorphism) of the strict CQ*-algebra $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ into a HCQ*-algebra $(\overline{\mathcal{B}}[\| \ \|_1], \#_1)$.*

(ii) *There exists a (resp. faithful) positive sesquilinear form φ on $\overline{\mathcal{A}}[\| \ \|] \times \overline{\mathcal{A}}[\| \ \|]$ satisfying*
  (ii)$_1$ $\varphi(x,y) = \varphi(u, x^\# y), \ \forall x, y \in \mathcal{A}$;
  (ii)$_2$ $|\varphi(x,y)| \leq \|x\| \|y\|, \ \forall x, y \in \mathcal{A}$;
  (ii)$_3$ $\varphi(x,y) = \varphi(y^*, x^*), \ \forall x, y \in \mathcal{A}$;
  Further, $(\overline{\mathcal{B}}[\| \ \|_1], \#_1)$ is standard if and only if
  (ii)$_4$ $\varphi(x^*, x^\#) \geq 0, \ \forall x \in \mathcal{A}$.

Proof. (i) $\Rightarrow$ (ii) We put

$$\varphi(a,b) = (\Phi(a)|\Phi(b)), \qquad a, b \in \overline{\mathcal{A}}[\| \ \|],$$

where $(\ | \ )$ is the inner product defined by the Hilbertian norm $\| \ \|_1$ on $\overline{\mathcal{B}}[\| \ \|]$. Then it is easily shown that φ is a positive sesquilinear form on $\overline{\mathcal{A}}[\| \ \|] \times \overline{\mathcal{A}}[\| \ \|]$ satisfying the condition (ii)$_1$ $\sim$ (ii)$_3$. If $(\overline{\mathcal{B}}[\| \ \|_1], \#_1)$ is standard, then (ii)$_4$ follows from Lemma 2.5.

(ii) $\Rightarrow$ (i) We put $\mathcal{N}_\varphi = \{a \in \overline{\mathcal{A}}[\| \ \|]; \varphi(a,a) = 0\}$. Then $\mathcal{N}_\varphi$ is a subspace of $\overline{\mathcal{A}}[\| \ \|]$ and, due to the positivity of φ, which implies $\varphi(a,b) = \overline{\varphi(b,a)}$ for each $a, b \in \overline{\mathcal{A}}[\| \ \|]$, it follows that the quotient space $\lambda_\varphi(\overline{\mathcal{A}}[\| \ \|]) \equiv \overline{\mathcal{A}}[\| \ \|]/\mathcal{N}_\varphi = \{\lambda_\varphi(a) \equiv a + \mathcal{N}_\varphi; a \in \overline{\mathcal{A}}[\| \ \|]\}$ is a pre-Hilbert space with inner product $(\lambda_\varphi(a)|\lambda_\varphi(b))_\varphi = \varphi(a,b), a, b \in \overline{\mathcal{A}}[\| \ \|]$. We denote by $\| \ \|_\varphi$ the norm defined by the inner product $(\ | \ )_\varphi$ and by $\mathcal{H}_\varphi$ the completion of $\lambda_\varphi(\overline{\mathcal{A}}[\| \ \|])[\| \ \|_\varphi]$. Since $\mathcal{A}$ is $\| \ \|$-dense in $\overline{\mathcal{A}}[\| \ \|]$, it follows that

(ii)$'_2$ $|\varphi(a,b)| \leq \|a\|\|b\|, \ \forall a, b \in \overline{\mathcal{A}}[\| \ \|]$;

(ii)$'_3$ $\varphi(a,b) = \varphi(b^*, a^*), \ \forall a, b \in \overline{\mathcal{A}}[\| \ \|]$,

and since (ii)$'$ and $\|x\| \leq \|x\|_\#, \ \forall x \in \mathcal{A}$, it follows that

(ii)$'_1$ $\varphi(x,y) = \varphi(u, x^\# y), \ \forall x, y \in \mathcal{A}_\#$.

By (ii)$'_2$ $\mathcal{A}_\varphi \equiv \lambda_\varphi(\mathcal{A})$ is a dense subspace of the Hilbert space $\mathcal{H}_\varphi$ and further, it is a $*$-algebra equipped with the multiplication $\lambda_\varphi(x)\lambda_\varphi(y) = L_{\lambda_\varphi(x)}\lambda_\varphi(y) \equiv \lambda_\varphi(xy)$ and the involution $\lambda_\varphi(x)^* \equiv \lambda_\varphi(x^*)$. By (ii)$'_3$ the involution $\lambda_\varphi(x) \to \lambda_\varphi(x)^*$ can be extended to the isometric involution J_φ on $\mathcal{H}_\varphi$. By (ii)$'_1$ the linear functional on the C*-algebra $\mathcal{A}_\# : x \to \varphi(x,u)$ is positive, and so $\varphi(y^\#(x^\# x)y, u) \leq \|x\|^2_\# \varphi(y,y)$ for each $x, y \in \mathcal{A}$. Hence it follows from (ii)$_1$ that

$$\|\lambda_\varphi(x)\lambda_\varphi(y)\|^2_\varphi = \varphi(xy, xy) = \varphi(y^\# x^\# xy, u) \leq \|x\|^2_\# \|\lambda_\varphi(y)\|^2_\varphi$$

for each $x, y \in \mathcal{A}$, so that $L_{\lambda_\varphi(x)}$ is bounded and $\|L_{\lambda_\varphi(x)}\| \leq \|x\|_\#$ for each $x \in \mathcal{A}$. Thus $\mathcal{H}_\varphi = \overline{\mathcal{A}_\varphi}[\| \ \|_\varphi]$ is a Hilbert quasi $*$-algebra. Further, the map $\lambda_\varphi(x) \to \lambda_\varphi(x)^\# \equiv \lambda_\varphi(x^\#)$ is an involution of $\mathcal{A}_\varphi$ and by (ii)$_1$ $L^*_{\lambda_\varphi(x)} = L_{\lambda_\varphi(x)^{\#_1}}$ for each $x \in \mathcal{A}$. Hence, $(\overline{\mathcal{A}_\varphi}[\| \ \|_\varphi], \#_1)$ is a HCQ*-algebra. Here we put $\Phi(a) = \lambda_\varphi(a), a \in \overline{\mathcal{A}}[\| \ \|]$. Then it is easily shown that Φ is a $*$-homomorphism of the strict CQ*-algebra into the HCQ*-algebra $(\overline{\mathcal{A}_\varphi}[\| \ \|_\varphi], \#_1)$ satisfying $\Phi(\mathcal{A}_0) = \mathcal{A}_\varphi$, and by (ii)$'_2$ it is contractive. Suppose that φ is faithful. Then the $*$-representation of the C*-algebra $\mathcal{A}_\#$ on $\mathcal{H}_\varphi$ defined by $x \to L_{\lambda_\varphi(x)}, x \in \mathcal{A}_\#$ is faithful, which implies that $\|L_{\lambda_\varphi(x)}\| = \|x\|_\#$ for each $x \in \mathcal{A}_\#$. Further, since $\Phi(\mathcal{A}_0) = \mathcal{A}_\varphi$, it follows that $\Phi(\mathcal{A}_\#) = (\mathcal{A}_\varphi)_{\#_1}$ and $\Phi\lceil\mathcal{A}_\#$ is a $*$-isormorphism of the C*-algebra $\mathcal{A}_\#$ onto the C*-algebra $(\mathcal{A}_\varphi)_{\#_1}$. Hence Φ is a $*$-isomorphism of $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ into $(\overline{\mathcal{A}_\varphi}[\| \ \|_\varphi], \#_1)$. By Lemma 2.5, the HCQ*-algebra $(\overline{\mathcal{A}_\varphi}[\| \ \|_\varphi], \#_1)$ is standard if and only if (ii)$_4$ holds. This completes the proof. □

Now the question arises as to whether positive sesquilinear forms as described in (ii) do really exist. The answer is certainly positive due to the existence of standard HCQ*-algebras stated in Theorem 2.9. Indeed, the inner product $\langle , \rangle$ of a left Hilbert algebra satisfies conditions (ii)$_1$–(ii)$_4$.

Furthermore, Theorem 3.2 answers the main question in this section: any form φ over a strict CQ*-algebra $(\overline{\mathcal{A}}[\| \ \|], \#, \| \ \|_\#)$ with quasi-unit, can be used to construct a HCQ*-algebra where $\overline{\mathcal{A}}$ is contractively embedded.

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