

COMPLEMENTED SUBSPACES OF PRODUCTS OF BANACH SPACES

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ABSTRACT. We show that complemented subspaces of uncountable products of Banach spaces are products of complemented subspaces of countable subproducts.

1. INTRODUCTION

The following old unsolved problem (L. Nachbin [8]) of describing injective locally convex spaces is one of the general problems of the structure theory of locally convex spaces.

Problem 1. Is every injective locally convex space isomorphic to a product of Banach injective spaces?

In investigations related to this problem (see, for instance, [3], [1], [2], [4]) the following problem ([2, p. 71], [6, p. 147]) arose.

Problem 2. Is every complemented subspace of a product of a (countable) family of Banach spaces isomorphic to a product of Banach spaces?

G. Metafuno and V. B. Moscatelli [5, p. 251] conjectured that this is false in general. Later this conjecture has been confirmed by M. Ostrovskii [9] who showed that not all complemented subspaces of countable products of Banach spaces are isomorphic to products of Banach spaces.

Our main result shows that for uncountable products the situation is somewhat different.

Theorem. *A complemented subspace of an uncountable product of Banach spaces is a product of complemented subspaces of countable subproducts.*

The following immediate corollary of this result provides a partial solution to Problem 1.

Corollary. *Every injective locally convex space is isomorphic to a product of injective Fréchet spaces.*

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2. RESULTS

The following statement expresses a key fact used in the proof of Theorem 2.2.

Let us first of all set up notation which will be used in the proofs of Proposition 2.1 and Theorem 2.2. For a subset $S \subseteq T$, where T is an indexing set, let

$$B_S = \prod\{B_t: t \in S\} \text{ and } B = \prod\{B_t: t \in T\}.$$

Also, for $S \subseteq R \subseteq T$, let $\pi_S: B \rightarrow B_S$ and $\pi_S^R: B_R \rightarrow B_S$ denote canonical projections onto the corresponding subproducts.

Proposition 2.1. *Let $r: B \rightarrow B$ be a continuous linear map of an uncountable product of Banach spaces into itself. Also, let A be a countable subset of T . Then there exist a countable subset S of T and a continuous linear map $r_S: B_S \rightarrow B_S$ such that $A \subseteq S$ and $\pi_S \circ r = r_S \circ \pi_S$.*

Proof. Let $\text{exp}_\omega T$ denote the set of all countable subsets of the indexing set T . Consider the following relation:

$\mathcal{L} = \{(S, R) \in (\text{exp}_\omega T)^2 : S \subseteq R \text{ and there exists a continuous linear map}$

$$r_S^R: \prod\{B_t: t \in R\} \rightarrow \prod\{B_t: t \in S\} \text{ such that } \pi_S \circ r = r_S^R \circ \pi_R\}.$$

We need to verify the following three properties of the above defined relation.

(1) If $S \in \text{exp}_\omega T$, then there exists $R \in \text{exp}_\omega T$ such that $(S, R) \in \mathcal{L}$.

Proof. Since B_S is a Fréchet space it follows that $\ker(\pi_S \circ r)$ is an intersection of countably many kernels of continuous seminorms. Consequently, $\pi_S \circ r$ factorizes through the projection π_R for some countable R . Without loss of generality, we may assume that $S \subseteq R$. This shows that $(S, R) \in \mathcal{L}$.

(2) If $(S, R) \in \mathcal{L}$, $P \in \text{exp}_\omega T$ and $R \subseteq P$, then $(S, P) \in \mathcal{L}$.

Proof. This is trivial because π_R factorizes through π_P .

(3) Suppose that $(S_i, R) \in \mathcal{L}$ and $S_i \subseteq S_{i+1}$ for each $i \in \omega$. Then $(\bigcup\{S_i: i \in \omega\}, R) \in \mathcal{L}$.

Proof. Consider the following projective sequence:

$$\prod\{B_t: t \in S_0\} \xleftarrow{\pi_{S_0}^{S_1}} \cdots \xleftarrow{\pi_{S_i}^{S_{i+1}}} \prod\{B_t: t \in S_{i+1}\} \xleftarrow{\pi_{S_{i+1}}^{S_{i+2}}} \cdots,$$

the limit of which is isomorphic to the product $\prod\{B_t: t \in S\}$, where $S = \bigcup\{S_i: i \in \omega\}$.

Since $(S_i, R) \in \mathcal{L}$, there exists a continuous linear map $r_{S_i}^R: \prod\{B_t: t \in R\} \rightarrow \prod\{B_t: t \in S_i\}$ such that $\pi_{S_i} \circ r = r_{S_i}^R \circ \pi_R$, $i \in \omega$. Note that $\pi_{S_i}^{S_{i+1}} \circ r_{S_{i+1}}^R = r_{S_i}^R$ for each $i \in \omega$. Indeed, let $x \in \prod\{B_t: t \in R\}$ and consider any point $y \in \prod\{B_t: t \in T\}$ such that $x = \pi_R(y)$. Since $(S_i, R), (S_{i+1}, R) \in \mathcal{L}$ we have

$$\begin{aligned} \pi_{S_i}^{S_{i+1}} \left(r_{S_{i+1}}^R(x) \right) &= \pi_{S_i}^{S_{i+1}} \left(r_{S_{i+1}}^R(\pi_R(y)) \right) = \pi_{S_i}^{S_{i+1}} \left(\pi_{S_{i+1}}(r(y)) \right) \\ &= \pi_{S_i}(r(y)) = r_{S_i}^R(\pi_R(y)) = \pi_{S_i}^R(x). \end{aligned}$$

In this situation the collection $\left\{ \pi_{S_i}^R: \prod\{B_t: t \in R\} \rightarrow \prod\{B_t: t \in S_i\}: i \in \omega \right\}$ uniquely defines a continuous linear map $r_S^R: \prod\{B_t: t \in R\} \rightarrow \prod\{B_t: t \in S\}$ such that $\pi_{S_i}^S \circ r_S^R = r_{S_i}^R$ for each $i \in \omega$ (r_S^R is simply the diagonal product of $r_{S_i}^R$'s).

It only remains to note that $\pi_S \circ r = r_S^R \circ \pi_R$ which completes the proof of the fact that $(S, R) \in \mathcal{L}$.

If $A \in \exp_\omega T$, then based on properties (1)–(3) we can inductively find an increasing sequence $\{S_i : i \in \omega\} \subseteq \exp_\omega T$ such that $A = S_0$ and $(S_i, S_{i+1}) \in \mathcal{L}$ for each $i \in \omega$. Clearly $S = \bigcup\{S_i : i \in \omega\}$ satisfies $(S, S) \in \mathcal{L}$. The proof is completed. □

Theorem 2.2. *A complemented subspace of a product of an uncountable family of Banach spaces is isomorphic to a product of Fréchet spaces. More formally, if X is a complemented subspace of the product $\prod\{B_t : t \in T\}$ of Banach spaces $B_t, t \in T$, then X is isomorphic to the product $\prod\{F_j : j \in J\}$, where F_j is a complemented subspace of the product $\prod\{B_t : t \in T_j\}$ with $|T_j| = \omega$ for each $j \in J$.*

Proof. Let X be a complemented subspace of the product $B = \prod\{B_t : t \in T\}$, where $|T| = \tau > \omega$, and let $r : B \rightarrow X$ be a continuous projection. Let us agree that a subset $S \subseteq T$ is called r -admissible if $\pi_S(r(z)) = \pi_S(z)$ for each point $z \in \pi_S^{-1}(\pi_S(X))$.

Claim 1. The union of an arbitrary family of r -admissible sets is r -admissible.

Let $\{S_j : j \in J\}$ be a collection of r -admissible sets and $S = \bigcup\{S_j : j \in J\}$. Let $z \in \pi_S^{-1}(\pi_S(X))$. Clearly $z \in \pi_{S_j}^{-1}(\pi_{S_j}(X))$ for each $j \in J$ and consequently $\pi_{S_j}(r(z)) = \pi_{S_j}(z)$ for each $j \in J$. Assuming that there is a point $z_0 \in \pi_S^{-1}(\pi_S(X))$ such that $\pi_S(r(z_0)) \neq \pi_S(z_0)$, we conclude that there exists an index $s \in S$ such that $\pi_{\{s\}}^S(\pi_S(r(z_0))) \neq \pi_{\{s\}}^S(\pi_S(z_0))$. Since $S = \bigcup\{S_j : j \in J\}$ it follows that there exists an index $j \in J$ such that $s \in S_j$. Then we have $\pi_{S_j}^S(\pi_S(r(z_0))) \neq \pi_{S_j}^S(\pi_S(z_0))$. But this is impossible:

$$\pi_{S_j}^S(\pi_S(r(z_0))) = \pi_{S_j}(r(z_0)) = \pi_{S_j}(z_0) = \pi_{S_j}^S(\pi_S(z_0)).$$

This contradiction proves the claim.

Claim 2. If $S \subseteq T$ is r -admissible, then $\pi_S(X)$ is a complemented subspace of $B_S = \prod\{B_t : t \in S\}$.

Indeed, let $i_S : B_S \rightarrow B$ be the canonical section of π_S (this means that $i_S = \text{id}_{B_S} \Delta \mathbf{0} : B_S \rightarrow B_S \times B_{T-S} = B$). Consider a continuous linear map $r_S = \pi_S \circ r \circ i_S : B_S \rightarrow \pi_S(X)$. Obviously, $i_S(y) \in \pi_S^{-1}(\pi_S(X))$ for any point $y \in \pi_S(X)$. Since S is r -admissible the latter implies that

$$y = \pi_S(i_S(y)) = \pi_S(r(i_S(y))) = r_S(y).$$

This shows that $\pi_S(X)$ is a complemented subspace of B_S .

Claim 3. Let S and R be r -admissible subsets of $T, S \subseteq R \subseteq T$. Then there exists an operator $g_S^R : X_S \rightarrow X_R$ such that $\pi_S^R|_{X_R} \circ g_S^R = \text{id}_{X_S}$.

Define $g_S^R = \pi_R \circ r \circ i_S|_X$. By the proof of Claim 2, for $z \in \pi_S(X)$ we get $\pi_S^R \circ g_S^R(z) = \pi_S \circ r \circ i_S(z) = z$.

Claim 4. Every countable subset of T is contained in a countable r -admissible subset of T .

Let A be a countable subset of T . Our goal is to find a countable r -admissible subset S such that $A \subseteq S$. By Proposition 2.1, there exist a countable subset S of T and a continuous homomorphism $r_S: B_S \rightarrow B_S$ such that $A \subseteq S$ and $\pi_S \circ r = r_S \circ \pi_S$. Consider a point $y \in \pi_S(X)$. Also pick a point $x \in X$ such that $\pi_S(x) = y$. Then

$$y = \pi_S(x) = \pi_S(r(x)) = r_S(\pi_S(x)) = r_S(y).$$

This shows that $r_S|_{\pi_S(X)} = \text{id}_{\pi_S(X)}$ (this shows, in fact, that $\pi_S(X)$ is complemented in B_S).

In order to show that S is r -admissible let us consider a point $z \in \pi_S^{-1}(\pi_S(X))$. By the observation made above, $r_S(\pi_S(z)) = \pi_S(z)$. Finally,

$$\pi_S(z) = r_S(\pi_S(z)) = \pi_S(r(x))$$

which implies that S is r -admissible.

We now use the above listed properties of r -admissible subsets and proceed as follows. By Claim 4, each element $t_\alpha \in T$ is contained in a countable r -admissible subset $S_\alpha \subseteq T$. According to Claim 1, the set $T_\alpha = \bigcup\{S_\beta: \beta \leq \alpha\}$ is r -admissible for each $\alpha < \tau$. Consider the projective system

$$\mathcal{S}_X = \{X_\alpha, p_\alpha^{\alpha+1}, \tau\},$$

where

$$X_\alpha = \pi_{T_\alpha}(X) \text{ and } p_\alpha^{\alpha+1} = \pi_{T_\alpha}^{\alpha+1}|_{\pi_{T_{\alpha+1}}(X)}: X_{\alpha+1} \rightarrow X_\alpha \text{ for each } \alpha < \tau.$$

Since $T = \bigcup\{T_\alpha: \alpha < \tau\}$, it follows that $X = \text{projlim } \mathcal{S}$. Obvious transfinite induction based on Claim 3 shows that

$$X = \text{projlim } \mathcal{S} = X_0 \times \prod\{\ker(p_\alpha^{\alpha+1}) : \alpha < \tau\}.$$

Since, by the construction, S_α is a countable r -admissible subset of T , it follows from Claim 2 that X_0 and $\ker(p_\alpha^{\alpha+1})$, $\alpha < \tau$, being complemented subspaces of countable products of Banach spaces, are Fréchet spaces. This finishes the proof of Theorem 2.2. \square

Recall that an object X of the category \mathcal{LSC} of locally convex spaces and their continuous linear maps is *injective* if any continuous linear map $f: A \rightarrow X$, defined on a linear subspace of a space B , admits a continuous linear extension $g: B \rightarrow X$ (i.e. $g|_A = f$).

The following statement is related to Problem 1 stated in the Introduction.

Corollary 2.3. *The following conditions are equivalent for a locally convex topological vector space X :*

- (1) X is an injective object of the category \mathcal{LSC} .
- (2) X is isomorphic to the product $\prod\{F_t: t \in T\}$, where each F_t , $t \in T$, is a complemented subspace of a product $\prod\{\ell_\infty(J_{t_n}): n \in \omega\}$.

Proof. (2) \implies (1). By [3, Lemma 0] and [7, p.105], $\ell_\infty(J)$ is an injective object of the category \mathcal{LCS} for any set J . Obviously (see, for instance, [3, Lemma 1.9]), the product of an arbitrary collection of injective objects of the category \mathcal{LCS} is also an injective object of this category. Consequently, the Fréchet space F_t , $t \in T$, as a complemented subspace of $\prod\{\ell_\infty(J_{t_n}): n \in \omega\}$, is injective. Finally, the space X , as a product of injectives, is an injective object of the category \mathcal{LSC} .

(1) \implies (2). Clearly X can be embedded into $\prod\{\ell_\infty(J_t) : t \in T\}$ for suitable J_t and T . The required conclusion follows from Theorem 2.2. \square

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