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CONDITIONAL WEAK COMPACTNESS IN VECTOR-VALUED FUNCTION SPACES

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ABSTRACT. Let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ) and let E' be the Köthe dual of E with $\operatorname{supp} E' = \Omega$. Let $(X, \|\cdot\|_X)$ be a real Banach space, and X^* the topological dual of X. Let $E(X)$ be a subspace of the space $L^0(X)$ of equivalence classes of strongly measurable functions $f\colon \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. For a subset H of $E(X)$ for which the set $\{\|f(\cdot)\|_X : f \in H\}$ is $\sigma(E, E')$-bounded the following statement is equivalent to conditional $\sigma(E(X), E'(X^*))$-compactness: the set $\{\|f(\cdot)\|_X : f \in H\}$ is conditionally $\sigma(E, E')$-compact and $\{\int_A f(\omega) d\mu\colon f \in H\}$ is a conditionally weakly compact subset of X for each $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Applications to Orlicz-Bochner spaces are given.

1. Introduction and preliminaries

Given a dual pair $\langle L, K\rangle$, a subset A of L is said to be *conditionally* $\sigma(L, K)$-*compact* whenever each sequence in A contains a $\sigma(L, K)$-Cauchy subsequence (cf. [MN, p. 100]). The problem of characterizing relatively sequentially $\sigma(L^p(X), L^q(X^*))$-compact subsets of Lebesgue-Bochner spaces $L^p(X)$ (where $1 \leqslant p < \infty$ and q conjugate to p) over a finite measure space was considered by F. Bombal [B_1] and J. Batt and W. Hiermeyer [BH, Theorem 2.1]. Moreover, F. Bombal characterized relatively sequentially $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact subsets of Orlicz-Bochner spaces $L^\varphi(X)$ [B_2, Theorem 3]. C. Abott, E. Bator, R. Bilyeu and P. Lewis [ABBL] obtained the following characterization of conditionally $\sigma(L^1(X), L^\infty(X^*))$-compact subsets of $L^1(X)$.

Theorem 1.1 (cf. [ABBL, Theorem 2.5]). *Let (Ω, Σ, μ) be a finite measure space. Then for a norm bounded subset H of $L^1(X)$ the following statements are equivalent:*
(i) *H is conditionally $\sigma(L^1(X), L^\infty(X^*))$-compact.*
(ii) a) *The subset $\{\|f(\cdot)\|_X\colon f \in H\}$ of L^1 is uniformly integrable.*
b) *The set $\{\int_A f(\omega) d\mu\colon f \in H\}$ is conditionally weakly compact in X for each $A \in \Sigma$.*

In this paper, by making use of Theorem 1.1 we characterize conditionally $\sigma(E(X), E'(X^*))$-compact subsets of $E(X)$, where E is an ideal of L^0 over a σ-finite measure space.

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Now we establish notation and terminology (see [AB], [KA]).

Let (Ω, Σ, μ) be a complete σ-finite measure space and let L^0 denote the space of equivalence classes of all Σ-measurable functions defined and finite a.e. on Ω. Let χ_A stand for the characteristic function of a set A and let $\mathbb{N}$ denote the set of all natural numbers. Let E be an ideal of L^0 with $\operatorname{supp} E = \Omega$, and let E' stand for the Köthe dual of E, i.e.,

$$E' = \{v \in L^0\colon \int_\Omega |u(\omega)v(\omega)| d\mu < \infty \quad \text{for all} \ \ u \in E\}.$$

We assume that $\operatorname{supp} E' = \Omega$.

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let S_X and B_X denote the unit sphere and the closed unit ball in X, resp. Let X^* stand for the Banach dual of X. By $L^0(X)$ we denote the set of equivalence classes of all strongly Σ-measurable functions $f\colon \Omega \to X$. For $f \in L^0(X)$ let us set $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{f \in L^0(X)\colon \ \widetilde{f} \in E\}.$$

By $\sigma(E(X), E'(X^*))$ we will denote the weak topology on $E(X)$ with respect to the dual system $\langle E(X), E'(X^*)\rangle$ under the natural duality $\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle d\mu$ for $f \in E(X), \ g \in E'(X^*)$.

The following characterization of conditional $\sigma(E, E')$-compactness is needed.

Proposition 1.2 ([N$_2$, Theorem 1.1])**.** *For a $\sigma(E, E')$-bounded subset A of E the following statements are equivalent:*

(i) *A is conditionally $\sigma(E, E')$-compact.*

(ii) *For each $v \in E'$ the subset $\{uv\colon \ u \in A\}$ of L^1 is uniformly integrable.*

(iii) *The functional p_A on E' defined by $p_A(v) = \sup_{u \in A} \int_\Omega |u(\omega)v(\omega)| d\mu$ is an order continuous Riesz seminorm.*

2. Conditionally $\sigma(E(X), E'(X^*))$-compact sets in $E(X)$

Let $\operatorname{ca}(\Omega, \Sigma)$ stand for the Riesz space of countably additive set functions ν on Σ. For a sequence (A_n) in Σ we write $A_n \searrow_\mu \emptyset$ whenever $A_n \downarrow$ and $\mu\Big(\bigcap_{n=1}^\infty A_n\Big) = 0$ (that is, $A_n \downarrow$ and $\mu(A_n \cap A) \to 0$ for each $A \in \Sigma$ with $\mu(A) < \infty$).

The following well-known result characterizes uniformly μ-continuous sets in $\operatorname{ca}(\Omega, \Sigma)$.

Lemma 2.1. *For a subset $\mathcal{K}$ of $\operatorname{ca}(\Omega, \Sigma)^+$ the following statements are equivalent:*

(i) *$\mathcal{K}$ is uniformly μ-continuous (i.e., $\lim_n (\sup_{\nu \in \mathcal{K}} \nu(A_n)) = 0$ as $A_n \searrow_\mu \emptyset$).*

(ii) *For each $\eta > 0$ there exist $\delta > 0$ and $A_0 \in \Sigma$ with $\mu(A_0) < \infty$ such that $\nu(A) \leqslant \eta$ and $\nu(\Omega \setminus A_0) \leqslant \eta$ for all $A \in \Sigma$ with $\mu(A) \leqslant \delta$ and all $\nu \in \mathcal{K}$.*

We shall need the following technical result.

Proposition 2.2. *Let $\mathcal{K}$ be a subset of $\operatorname{ca}(\Omega, \Sigma)^+$ such that each $\nu \in \mathcal{K}$ is μ-continuous. Assume that $\mathcal{K}$ is not uniformly μ-continuous. Then there exist a pairwise disjoint sequence (B_n) in Σ, a number $\varepsilon_0 > 0$ and a sequence (ν_n) in $\mathcal{K}$ such that $\nu_n(B_n) > \varepsilon_0$ for all $n \in \mathbb{N}$.*

Proof. In view of Lemma 2.1 there exists $\varepsilon_0 > 0$ such that either there exist a sequence (A_n) in Σ and a sequence (ν_n^1) in $\mathcal{K}$ such that

$$\mu(A_n) \to 0 \quad \text{and} \quad \nu_n^1(A_n) > 2\varepsilon_0 \tag{1}$$

or there exists a sequence (ν_n^2) in $\mathcal{K}$ such that

$$\nu_n^2(\Omega \setminus \Omega_n) > 2\varepsilon_0 \tag{2}$$

whenever $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < \infty$ for $n \in \mathbb{N}$.

Assume that condition (1) holds. Then arguing as in [BL, p. 546] one can find a pairwise disjoint sequence (B_n) in Σ and a subsequence $(\nu_{k_n}^1)$ of (ν_n^1) such that $\nu_{k_n}^1(B_n) \geqslant \varepsilon_0$. Let $\nu_n = \nu_{k_n}^1$ for $n \in \mathbb{N}$.

Now assume that condition (2) holds. Let $C_n = \Omega \setminus \Omega_n$ for $n \in \mathbb{N}$. Then $C_n \searrow_\mu \emptyset$, so $\nu(C_n) \to 0$ for each $\nu \in \mathcal{K}$. Let $l_1 = 1$ and choose $l_2 \in \mathbb{N}$ such that $l_2 > l_1$, $\nu_{l_1}^2(C_{l_2}) < \varepsilon_0$. Then choose $l_3 \in \mathbb{N}$ such that $l_3 > l_2$ and $\nu_{l_2}(C_{l_3}) < \varepsilon_0$. Continuing this process inductively we can find an increasing sequence (l_n) in $\mathbb{N}$ such that $\nu_{l_n}^2(C_{l_{n+1}}) < \varepsilon_0$. Let $B_n = \Omega_{l_{n+1}} \setminus \Omega_{l_n}$ for $n \in \mathbb{N}$. Then (B_n) is a disjoint sequence and since $B_n = C_{l_n} \setminus C_{l_{n+1}}$ for $n \in \mathbb{N}$, by making use of (2) we obtain that $\nu_{l_n}^2(B_n) = \nu_{l_n}^2(C_{l_n}) - \nu_{l_n}^2(C_{l_{n+1}}) > 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0$. Put $\nu_n = \nu_{l_n}^2$ for $n \in \mathbb{N}$. □

For a subset H of $E(X)$ let $\widetilde{H} = \{\widetilde{f}\colon f \in H\}$.

Now we are ready to state our main result.

Theorem 2.3. *Let H be a subset of $E(X)$ such that the subset $\widetilde{H}$ of E is $\sigma(E, E')$-bounded. Then the following statements are equivalent:*

(i) *H is conditionally $\sigma(E(X), E'(X^*))$-compact.*

(ii) a) *$\widetilde{H}$ is conditionally $\sigma(E, E')$-compact.*

b) *$\left\{\int_A f(\omega)d\mu\colon f \in H\right\}$ is a conditionally weakly compact subset of X for each $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$.*

Proof. (i) $\Rightarrow$ (ii) To prove that (a) holds, in view of Proposition 1.2 it is enough to show that for each $0 \leqslant v \in E'$ the subset $\{\widetilde{f}v\colon f \in H\}$ of L^1 is uniformly integrable. Assume on the contrary that there exists $0 \leqslant v_0 \in E'$ such that the set $\{\widetilde{f}v_0\colon f \in H\}$ is not uniformly integrable. For each $f \in H$ set $\nu_f(A) = \int_A \widetilde{f}(\omega)v_0(\omega)d\mu$ for $A \in \Sigma$. Then ν_f is a non-negative μ-continuous countably additive set function on Σ but the family $\{\nu_f\colon f \in H\}$ is not uniformly μ-continuous. Hence in view of Proposition 2.2 there exist a pairwise disjoint sequence (B_n) in Σ, a sequence (f_n) in H, and a number $\varepsilon_0 > 0$ such that $\nu_{f_n}(B_n) = \int_{B_n} \widetilde{f}_n(\omega)v_0(\omega)d\mu > \varepsilon_0$ for each $n \in \mathbb{N}$. Clearly $v_0 f_n \in L^1(X)$, so in view of [Bu, Theorem 1.1.(4)]

$$\begin{aligned}\nu_{f_n}(B_n) &= \|\chi_{B_n} v_0 \widetilde{f}_n\|_{L^1} = \|\chi_{B_n} v_0 f_n\|_{L^1(X)} \\ &= \sup\Big\{\Big|\int_{B_n} \langle v_0(\omega) f_n(\omega), g(\omega)\rangle d\mu\Big|\colon\ g \in L^\infty(X^*),\ \|g\|_{L^\infty(X^*)} \leqslant 1\Big\}.\end{aligned}$$

Hence one can produce a sequence (g_n) in $L^\infty(X^*)$ with $\|g_n\|_{L^\infty(X^*)} \leqslant 1, \chi_{\Omega \setminus B_n} g_n = 0$ and such that

$$\Big|\int_{B_n} \langle v_0(\omega) f_n(\omega), g_n(\omega)\rangle d\mu\Big| > \varepsilon_0. \tag{1}$$

Set $g_0 = \sum_{n=1}^{\infty} g_n$. Then $g_0 \in L^0(X^*)$ and $\|g_0\|_{L^\infty(X^*)} \leqslant 1$. Clearly $v_0 g_0 \in E'(X^*)$, so $\chi_A v_0 g_0 \in E'(X^*)$ for each $A \in \Sigma$. In view of the assumption (i) there exists a $\sigma(E(X), E'(X^*))$-Cauchy subsequence (f_{k_n}) of (f_n) so for each $A \in \Sigma$, $\lim_n \int_A \langle f_{k_n}(\omega), v_0(\omega) g_0(\omega)\rangle d\mu$ exists. Setting $\mu_n(A) = \int_A \langle f_{k_n}(\omega), v_0(\omega) g_0(\omega)\rangle d\mu$ for $A \in \Sigma$, in view of Nikodym's convergence theorem (see [D, Chap. 7]), $\{\mu_n \colon n \in \mathbb{N}\}$ is uniformly countably additive on Σ. Hence there exists $m_0 \in \mathbb{N}$ such that for $m \geqslant m_0$, $\sup_n |\mu_n(B_m)| \leqslant \varepsilon_0$ (see [D, Chap. 7, Theorem 10]). Hence for each $m \geqslant m_0$ we get

$$|\mu_m(B_{k_m})| = \Big| \int_{B_{k_m}} \langle f_{k_m}(\omega), v_0(\omega) g_{k_m}(\omega)\rangle d\mu \Big| = \Big| \int_{B_{k_m}} \langle v_0(\omega) f_{k_m}(\omega), g_{k_m}(\omega)\rangle d\mu \Big| \leqslant \varepsilon_0$$

which contradicts (1). This contradiction establishes that (a) holds.

To show that (b) holds, take $A \in \Sigma$ with $\chi_A \in E'$, and let (f_n) be a sequence in H. Set $g = \chi_A x^*$ where $x^* \in S_{X^*}$. Then $g \in E'(X^*)$ and by assumption (i) there exists a subsequence (f_{k_n}) of (f_n) such that $\lim_n \int_\Omega \langle f_{k_n}(\omega), g(\omega)\rangle d\mu$ exists. Since $\int_\Omega \langle f_{k_n}(\omega), g(\omega)\rangle d\mu = x^*\Big(\int_A f_{k_n}(\omega) d\mu\Big)$, the set $\Big\{\int_A f(\omega) d\mu \colon f \in H\Big\}$ is conditionally weakly compact in X.

(ii) $\Rightarrow$ (i) Let (f_n) be a sequence in H. Since supp $E' = \Omega$ there exists a sequence (Ω_m) in Σ such that $\Omega_m \uparrow \Omega$ and $\mu(\Omega_m) < \infty$, $\chi_{\Omega_m} \in E'$ for $m \in \mathbb{N}$ (see [Z, Theorem 86.2]). Setting $A_m = \Omega \setminus \Omega_m$ for $m \in \mathbb{N}$ we see that $A_m \searrow_\mu \emptyset$. Given $m \in \mathbb{N}$ we have $\sup_n \int_{\Omega_m} \widetilde{f}_n(\omega) d\mu = c_m < \infty$, because $\chi_{\Omega_m} \in E'$ and $\widetilde{H}$ is $\sigma(E, E')$-bounded. Hence $\{\chi_{\Omega_m} f_n \colon n \in \mathbb{N}\} \subset L^1_{\Omega_m}(X)$, and by assumption (a), $\{\chi_{\Omega_m} \widetilde{f}_n \colon n \in \mathbb{N}\}$ is a uniformly integrable subset of $L^1_{\Omega_m}$. Combining this observation with (b), in view of Theorem 1.1 we see that $\{\chi_{\Omega_m} f_n \colon n \in \mathbb{N}\}$ is a conditionally $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-compact subset of $L^1_{\Omega_m}(X)$.

In view of the above observation there exists a $\sigma(L^1_{\Omega_1}(X), L^\infty_{\Omega_1}(X^*))$-Cauchy subsequence $(\chi_{\Omega_1} f_{k_n^1})$ of $(\chi_{\Omega_1} f_n)$. Next, there exists a $\sigma(L^1_{\Omega_2}(X), L^\infty_{\Omega_2}(X^*))$-Cauchy subsequence $(\chi_{\Omega_2} f_{k_n^2})$ of $(\chi_{\Omega_1} f_{k_n^1})$. It follows that the diagonal sequence $(f_{k_n^n})$ has the property that for each $m \in \mathbb{N}$ $(\chi_{\Omega_m} f_{k_n^n})$ is a $\sigma(L^1_{\Omega_m}(X), L^\infty_{\Omega_m}(X^*))$-Cauchy sequence. Put $h_n = f_{k_n^n}$ for $n \in \mathbb{N}$.

Let $g \in E'(X^*)$. For $n \in \mathbb{N}$ let us put

$$g_n(\omega) = \begin{cases} g(\omega) & \text{if } \omega \in \Omega_n \text{ and } \|g(\omega)\|_{X^*} \leqslant n, \\ 0 & \text{elsewhere.} \end{cases}$$

Given $\varepsilon > 0$ there exist $m_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\text{(2)} \qquad \sup_n \int_{\Omega \setminus \Omega_{m_0}} \widetilde{f}_n(\omega)\, \widetilde{g}\,(\omega) d\mu \leqslant \frac{\varepsilon}{4} \quad \text{and} \quad \sup_n \int_A \widetilde{f}_n(\omega)\, \widetilde{g}\,(\omega) d\mu \leqslant \frac{\varepsilon}{4}$$

for each $A \in \Sigma$ with $\mu(A) \leqslant \delta$. For $\eta = \frac{\varepsilon}{4c_{m_0}}$ let

$$B_n = \{\omega \in \Omega_{m_0} \colon \, \|g(\omega) - g_n(\omega)\|_{X^*} \geqslant \eta\}.$$

It is easy to observe that $B_n \downarrow \emptyset$, so $\mu(B_n) \to 0$. Choose $n_0 \in \mathbb{N}$ with $n_0 \geqslant m_0$ such that $\mu(B_{n_0}) \leqslant \delta$. Then by (2) we get

$$\sup_n \int\limits_{B_{n_0}} \widetilde{h}_n\,(\omega)\,\widetilde{g}\,(\omega) d\mu \leqslant \frac{\varepsilon}{4}. \tag{3}$$

Hence, by (3) we have

$$\begin{aligned}
&\Big| \int\limits_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega)\rangle d\mu \Big| \leqslant \int\limits_{\Omega_{m_0}} \widetilde{h}_n\,(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu \\
&\leqslant \int\limits_{B_{n_0}} \widetilde{h}_n\,(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu + \int\limits_{\Omega_{m_0} \setminus B_{n_0}} \widetilde{h}_n\,(\omega) \|g(\omega) - g_{n_0}(\omega)\|_{X^*} d\mu \qquad (4) \\
&\leqslant \int\limits_{B_{n_0}} \widetilde{h}_n\,(\omega)\,\widetilde{g}\,(\omega) d\mu + \eta \int\limits_{\Omega_{m_0}} \widetilde{h}_n\,(\omega) d\mu \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4c_{m_0}} \cdot c_{m_0} = \frac{\varepsilon}{2}.
\end{aligned}$$

Since $\int\limits_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega)\rangle d\mu \xrightarrow[n]{} a$ for some $a \in \mathbb{R}$, we can choose $n_1 \in \mathbb{N}$ such that for $n \geqslant n_1$

$$\Big| \int\limits_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega)\rangle d\mu - a \Big| \leqslant \frac{\varepsilon}{4}. \tag{5}$$

Thus by (2), (4) and (5) for $n \geqslant n_1$ we get

$$\begin{aligned}
&\Big| \int\limits_{\Omega} \langle h_n(\omega), g(\omega)\rangle d\mu - a \Big| \\
&\leqslant \Big| \int\limits_{\Omega \setminus \Omega_{m_0}} \langle h_n(\omega), g(\omega)\rangle d\mu \Big| + \Big| \int\limits_{\Omega_{m_0}} \langle h_n(\omega), g(\omega) - g_{n_0}(\omega)\rangle d\mu \Big| \\
&\quad + \Big| \int\limits_{\Omega_{m_0}} \langle h_n(\omega), g_{n_0}(\omega)\rangle d\mu - a \Big| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

This shows that (h_n) is a $\sigma(E(X), E'(X^*))$-Cauchy subsequence of (f_n), so H is conditionally $\sigma(E(X), E'(X^*))$-compact. □

Corollary 2.4. *Assume that a Banach space X contains no isomorphic copy of ℓ^1, and let H be a subset of $E(X)$ such that $\widetilde{H}$ is $\sigma(E, E')$-bounded. Then the following statements are equivalent:*

(i) *H is conditionally $\sigma(E(X), E'(X^*))$-compact.*

(ii) *$\widetilde{H}$ is conditionally $\sigma(E, E')$-compact.*

Proof. (i) ⇒ (ii) Obvious.

(ii) ⇒ (i) In view of Theorem 2.3 it is enough to show that $\Big\{\int\limits_A f(\omega) d\mu \colon \, f \in H\Big\}$ is a conditionally weakly compact subset of X for each $A \in \Sigma$ such that $\chi_A \in E'$. In fact, let $A \in \Sigma$, $\mu(A) < \infty$ with $\chi_A \in E'$. Hence $\sup\limits_{f \in H} \| \int\limits_A f(\omega) d\mu \| \leqslant$

$\sup_{f\in H}\int_A \widetilde{f}(\omega)d\mu < \infty$, so in view of the Rosenthal's ℓ^1-theorem [R] $\Big\{\int_A f(\omega)d\mu\colon f \in H\Big\}$ is a conditionally weakly compact subset of X, as desired. □

Assume now that $(E, \|\cdot\|_E)$ is a Banach function space. Then the space $E(X)$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is usually called a Köthe-Bochner space. The associated norm $\|\cdot\|_{E'}$ on the Köthe dual E' can be defined as follows:

$$\|v\|_{E'} = \sup\Big\{\Big|\int_\Omega u(\omega)v(\omega)d\mu\Big|\colon\ u \in E,\ \|u\|_E \leqslant 1\Big\}.$$

Clearly, for a subset H of $E(X)$ the set $\widetilde{H}$ is $\sigma(E, E')$-bounded whenever

$$\sup_{f\in H}\|f\|_{E(X)} < \infty.$$

Combining Corollary 2.4 and Proposition 1.2 we get:

Corollary 2.5. *Let $(E, \|\cdot\|_E)$ be a Banach function space, and assume that X contains no isomorphic copy of ℓ^1. Then the following statements are equivalent:*
(i) *The associated norm $\|\cdot\|_{E'}$ on E' is order continuous.*
(ii) *Every norm bounded set in $E(X)$ is conditionally $\sigma(E(X), E'(X^*))$-compact.*

We now apply the previous results to Orlicz spaces (see [KR], [L] for more details).

By a Young function we mean a mapping $\varphi\colon [0,\infty) \to [0,\infty)$ that is convex, vanishes only at 0 and $\lim_{t\to 0}\frac{\varphi(t)}{t} = 0$, $\lim_{t\to\infty}\frac{\varphi(t)}{t} = \infty$. Let L^φ be the Orlicz space associated with φ and provided with the Luxemburg norm $\|u\|_\varphi := \inf\{\lambda > 0\colon \int_\Omega \varphi(|u(\omega)|/\lambda)d\mu \leqslant 1\}$. Then $(L^\varphi)' = L^{\varphi^*}$, where φ^* denotes the complementary Young function.

We say that a Young function ψ *increases more rapidly than another* φ, in symbols $\varphi \prec\!\!\prec \psi$, if for each $c > 0$ there is $d > 1$ such that $c\varphi(t) \leqslant \frac{1}{d}\psi(dt)$ for all $t \geqslant 0$ (see [N_1]). Note that φ satisfies the ∇_2-condition iff $\varphi \prec\!\!\prec \varphi$.

As a consequence of Corollary 2.4 and [N_1, Theorem 2.5] we get:

Corollary 2.6. *Assume that X contains no isomorphic copy of ℓ^1. Then for a norm bounded subset H of the Orlicz-Bochner space $L^\varphi(X)$ the following statements are equivalent:*
(i) *H is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact.*
(ii) *There is a Young function ψ with $\varphi \prec\!\!\prec \psi$ and such that $H \subset L^\psi(X)$ and $\sup_{f\in H}\|f\|_{L^\psi(X)} < \infty$.*

Corollary 2.7. *Assume that X contains no isomorphic copy of ℓ^1 and φ satisfies the ∇_2-condition. Then every norm bounded subset of $L^\varphi(X)$ is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact.*

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