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NON-TANGENTIAL LIMITS, FINE LIMITS AND THE DIRICHLET INTEGRAL

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Abstract. Let B denote the unit ball in $\mathbb{R}^n$. This paper characterizes the subsets E of B with the property that $\sup_E h = \sup_B h$ for all harmonic functions h on B which have finite Dirichlet integral. It also examines, in the spirit of a celebrated paper of Brelot and Doob, the associated question of the connection between non-tangential and fine cluster sets of functions on B at points of the boundary.

1. Introduction

Let $B(x, r)$ denote the open ball of centre x and radius r in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, and let $B = B(0, 1)$. If $\mathcal{A}$ is a collection of harmonic functions on B, then it is natural to ask which non-empty subsets E of B have the property that

$$\sup_E h = \sup_B h \quad \text{for all } h \in \mathcal{A}. \tag{1}$$

In the case where $\mathcal{A} = h^\infty$, the collection of all bounded harmonic functions on B, it is known (cf. [2]) that (1) holds if and only if $\sigma(E_{NT}) = \sigma(\partial B)$, where σ denotes surface area measure on ∂B and E_{NT} is the (Borel) set of points of ∂B which can be approached non-tangentially by a sequence in E. In the case where $\mathcal{A} = h^1$, the collection of differences of positive harmonic functions on B, it has been shown (see [10] and [8]) that (1) holds if and only if

$$\int_{E(1/2)} |x - y|^{-n}\, dx = +\infty \quad \text{for all } y \in \partial B,$$

where $E(1/2) = \bigcup_{x \in E} B(x, (1 - |x|)/2)$. Below we present the corresponding result when $\mathcal{A} = \mathcal{D}$, the collection of all harmonic functions h on B which have finite Dirichlet integral; that is, $\int_B |\nabla h(x)|^2\, dx < +\infty$. We will use $\mathcal{C}(\cdot)$ to denote Newtonian (if $n \geq 3$) or logarithmic (if $n = 2$) capacity on $\mathbb{R}^n$.

Theorem 1. *Let $\emptyset \neq E \subseteq B$ and $\mathcal{A} = \mathcal{D}$. Then* (1) *holds if and only if $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$.*

When $n = 2$, Theorem 1 is closely related to a recent result of Stray [13] concerning holomorphic functions in the Dirichlet space. However, our methods are completely different. If $\sigma(E_{NT}) = \sigma(\partial B)$, then it follows easily, by considering

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Poisson integrals in B of suitable potentials, that $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$. In fact, the capacitary condition is much weaker, as the following example shows.

Example 1. Let $n = 2$, let $\mathbb{R}^2$ be identified with $\mathbb{C}$ in the usual manner, and let

$$E = \left\{ \left(1 - 2^{-2j}\right) \exp\left(ik2^{1-j}\pi\right) : j \in \mathbb{N} \text{ and } k = 0, 1, ..., 2^j - 1 \right\}.$$

Then $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$, but $\sigma(E_{NT}) = 0$. (See §3.4 for details.)

Brelot and Doob, in their landmark paper [3], were able to relate classical and potential theoretic boundary limit theorems by establishing the relationship between non-tangential and minimal fine cluster sets of functions. Inspired by their work and Theorem 1 we will now provide corresponding results which describe the relationship between non-tangential and fine cluster sets of functions.

Recall that the *fine topology* on $\mathbb{R}^n$ is the coarsest topology for which all superharmonic functions are continuous. A set A is said to be *thin* at a point x if x is not a fine limit point of A. (For an account of these concepts see Chapter 1.XI of the book by Doob [7].) By Wiener's criterion, this is equivalent to the condition

$$\sum_k 2^{k(n-2)} \mathcal{C}^*(\{y \in A : 2^{-k-1} \leq |x - y| \leq 2^{-k}\}) < +\infty \quad (n \geq 3)$$

or

$$\sum_k \frac{k}{\log 1/\mathcal{C}^*(\{y \in A : 2^{-k-1} \leq |x - y| \leq 2^{-k}\})} < +\infty \quad (n = 2),$$

where $\mathcal{C}^*(\cdot)$ denotes outer (Newtonian or logarithmic) capacity. If, instead, the weaker condition

$$2^{k(n-2)} \mathcal{C}^*(\{y \in A : 2^{-k-1} \leq |x - y| \leq 2^{-k}\}) \to 0 \text{ as } k \to \infty \quad (n \geq 3)$$

or

$$\frac{k}{\log 1/\mathcal{C}^*(\{y \in A : 2^{-k-1} \leq |x - y| \leq 2^{-k}\})} \to 0 \text{ as } k \to \infty \quad (n = 2)$$

holds, then A is said to be *semi-thin* at x. Now let $f : S \to [-\infty, +\infty]$, where $S \subseteq \mathbb{R}^n$, let $x \in \overline{S}$ and $l \in [-\infty, +\infty]$. We say that l is a *fine* (respectively*, semi-fine*) *cluster value of* f *at* x if, for every neighbourhood N of l in $[-\infty, +\infty]$, the set $f^{-1}(N)$ is not thin (respectively, not semi-thin) at x. Finally, we define the non-tangential approach region

$$K(z, \delta, \varepsilon) = \{x : \varepsilon > 1 - |x| > \delta\, |x - z|\} \; (z \in \partial B; 0 < \delta < 1; 0 < \varepsilon < 1).$$

The following result is straightforward to prove, using ideas from [14].

Proposition 1. *Let* h *be a harmonic function on* $K(z, \delta, \varepsilon)$ *such that*

$$\int_{K(z,\delta,\varepsilon)} (1 - |x|)^{2-n} \left|\nabla h(x)\right|^2 dx < +\infty,$$

and let $\delta < \delta_1 < 1$. *If there is a sequence* (x_k) *of points in* $K(z, \delta_1, \varepsilon)$ *such that* $x_k \to z$ *and* $h(x_k) \to l$, *then* l *is a semi-fine (and hence a fine) cluster value of* h *at* z.

Less obvious is the next result, which goes in the opposite direction. If $f : B \to [-\infty, +\infty]$, then the *non-tangential* and *fine cluster sets* of f at a point $z \in \partial B$ are defined respectively by

$$C_{NT}(f,z) = \{l \in [-\infty,+\infty] : f(x_k) \to l \text{ for some sequence } (x_k) \text{ of points in } B \text{ which approaches } z \text{ non-tangentially}\}$$

and

$$C_F(f,z) = \{l \in [-\infty,+\infty] : l \text{ is a fine cluster value of } f \text{ at } z\}.$$

Theorem 2. *Let $f : B \to [-\infty, +\infty]$. Then there is a Euclidean-G_δ set $A \subseteq \partial B$ such that $\mathcal{C}(A) = \mathcal{C}(\partial B)$ and $C_F(f,z) \subseteq C_{NT}(f,z)$ whenever $z \in A$.*

Theorem 2 can be viewed as a fine topology analogue of a maximality theorem of Collingwood and Lohwater (see Theorem 4.10 in [5]). We note that Mizuta [12] has also considered the relationship between non-tangential, normal and fine cluster sets. His results, which are specific to harmonic functions satisfying a Dirichlet-type integral condition, are of a completely different nature. The sharpness of Theorem 2, even for harmonic functions, is demonstrated by the next result.

Theorem 3. *Let $A \subseteq \partial B$ be a Euclidean-G_δ set such that $\mathcal{C}(A) = \mathcal{C}(\partial B)$. Then there is a harmonic function h on B such that $C_F(h,z) = [-\infty,+\infty]$ and $C_{NT}(h,z) = \{0\}$ whenever $z \in \partial B \backslash A$.*

Proposition 1 and Theorems 1 and 2 will be proved in §§2 - 4 respectively. Theorem 3 will be proved in §5 using recent results concerning approximation by harmonic functions.

2. Proof of Proposition 1

Let h be a harmonic function on $K(z,\delta,\varepsilon)$ such that

$$\int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} |\nabla h(x)|^2 \, dx < +\infty,$$

let $\delta < \delta_0 < \delta_1 < 1$ and $0 < \varepsilon_1 < \varepsilon_0 < \varepsilon$. Then there is a decreasing function $\phi : (0,+\infty) \to (0,+\infty)$ such that $\phi(t) \le 2\phi(2t)$ for all t and $\phi(t) \to +\infty$ as $t \to 0$, and such that $L < +\infty$, where

$$L = \int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} \phi(1-|x|) |\nabla h(x)|^2 \, dx.$$

Let $\eta_0 > 0$ be small enough so that $B(x, \eta_0 (1-|x|)) \subseteq K(z,\delta,\varepsilon)$ whenever $x \in K(z,\delta_0,\varepsilon_0)$. By the volume mean value inequality, applied to the subharmonic function $|\nabla h|^2$ and the ball $B(x,\eta_0(1-|x|))$, we see that there is a positive constant M, depending only on η_0 and n, such that

$$(1-|x|)^{2-n} \phi(1-|x|) |\nabla h(x)|^2 \le ML(1-|x|)^{-n} \quad (x \in K(z,\delta_0,\varepsilon_0)),$$

and hence

$$(1-|x|) |\nabla h(x)| \le \sqrt{\frac{ML}{\phi(1-|x|)}} \quad (x \in K(z,\delta_0,\varepsilon_0)). \tag{2}$$

Let $\eta_1 > 0$ be small enough so that $B(x,\eta_1(1-|x|)) \subseteq K(z,\delta_0,\varepsilon_0)$ whenever $x \in K(z,\delta_1,\varepsilon_1)$. Also, let (x_k) be a sequence of points in $K(z,\delta_1,\varepsilon_1)$ such that

$x_k \to z$ and $h(x_k) \to l$, and let $D = \bigcup_k B(x_k, \eta_1(1-|x_k|))$. It follows from (2) and the mean value theorem of differential calculus that

$$h(x) \to l \quad (x \to z; x \in D),$$

in view of the fact that $\phi(t) \to +\infty$ as $t \to 0$. Since $\mathcal{C}(B(x,r))$ is r^{n-2} $(n \geq 3)$ or r $(n = 2)$, it is clear that D is not semi-thin at z. Thus l is a semi-fine (and hence a fine) cluster value of h at z.

3. Proofs of Theorem 1 and Example 1

3.1. For the proofs of Theorems 1 - 3 we will assume that $n \geq 3$ and omit the minor modifications required to adapt our arguments to the plane. Before proving Theorem 1 we will assemble some preliminary observations. A function $f : S \to [-\infty, +\infty]$ is said to have a *fine limit* at a fine limit point x of S if $C_F(f,x)$ consists of only one point. If, further, $C_F(f,x) = \{f(x)\}$, then f is said to be *finely continuous* at x.

Lemma A. *Let $f : B \to [0, +\infty)$ be integrable on B and let $0 < \varepsilon < 1$. Then there is a set $Y \subset \partial B$, of zero $(n-2)$-dimensional Hausdorff measure, such that*

$$\int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} f(x)dx < +\infty \quad (z \in \partial B \backslash Y; 0 < \delta < 1).$$

Theorem A. *If $h \in \mathcal{D}$, then there is a finite-valued extension $\overline{h}$ of h to $\overline{B}$, and a polar set $Z \subset \partial B$, such that*

$$h(rz) \to \overline{h}(z) \quad (r \to 1-; \; z \in \partial B \backslash Z) \tag{3}$$

and $\overline{h}$ is finely continuous at each point of $\partial B \backslash Z$.

In proving Lemma A it is clearly enough to show that, for a fixed choice of δ, there is a set $Y_\delta \subset \partial B$, of zero $(n-2)$-dimensional Hausdorff measure, such that

$$\int_{K(z,\delta,\varepsilon)} (1-|x|)^{2-n} f(x)dx < +\infty \quad (z \in \partial B \backslash Y_\delta).$$

This can be done by imitating the proof of the analogous result for functions on half-spaces, which may be found in Lemma 5 of [14].

Theorem A is taken from Deny ([6], Chap. IV, Théorème 3), who proved the result for the more general class of Beppo Levi functions h on B. Actually, Deny's result asserts only that $\overline{h}$ has a fine limit at each point of $\partial B \backslash Z$, but the proof makes it clear that $\overline{h}$ is actually finely continuous at each point of a set which has this form.

Lemma 1. *The unit sphere ∂B, with the topology induced on it by the fine topology on $\mathbb{R}^n$, is a Baire space.*

To prove Lemma 1, we note that the fine topology on $\mathbb{R}^n$ has a base which consists of (Euclidean) compact sets. The same is therefore also true of the topology it induces on ∂B. We can now adopt the argument given on pp. 167, 168 of [7] to see that this space is Baire.

If A is a bounded set in $\mathbb{R}^n$, then we use $\widehat{R}_1^A$ to denote the capacitary potential of A. Thus the Riesz measure ν_A associated with $\widehat{R}_1^A$ satisfies $\nu_A(\mathbb{R}^n) = \mathcal{C}^*(A)$.

Lemma 2. *Let $A \subseteq \partial B$ and $z \in \partial B$. If A is thin at z, then $\widehat{R}_1^A(z) < 1$.*

To see this we note that, if $z \in \overline{A}$, then there is a positive superharmonic function u on $\mathbb{R}^n$ such that

$$\liminf_{x \to z, x \in A} u(x) > u(z).$$

If we define $v(x) = u(x) + a\,|x+z|^{2-n}$, where a is a suitably chosen positive number, then v is a positive superharmonic function on $\mathbb{R}^n$ such that

$$\inf_{A \backslash \{z\}} v > v(z).$$

It follows easily that $\widehat{R}_1^A(z) < 1$, and this inequality is obviously also true when $z \notin \overline{A}$.

3.2. We now turn to the "if" part of Theorem 1. Suppose that $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$ and suppose further, for the sake of contradiction, that E_{NT} is thin at some point z of ∂B. Then $\widehat{R}_1^{E_{NT}}(z) < 1$, by Lemma 2, and this leads to the contradictory conclusion that $\mathcal{C}(E_{NT}) = \widehat{R}_1^{E_{NT}}(0) < 1 = \mathcal{C}(\partial B)$. Hence the fine closure of E_{NT} is all of ∂B.

Now let $h \in \mathcal{D}$. By Theorem A, there exist a function $\overline{h} : \bar{B} \to \mathbb{R}$ and a polar set $Z \subset \partial B$ such that $\overline{h}\,|_B = h$ and (3) holds, and such that $\overline{h}$ is finely continuous at all points of $\partial B \backslash Z$. Let $M = \sup_E h$ and suppose, to avoid triviality, that $M < +\infty$. By Proposition 1 and Lemma A (with $f = |\nabla h|^2$), there is a polar subset Y of ∂B such that $\overline{h} \leq M$ on $E_{NT} \backslash (Y \cup Z)$. (Here we are using the fact that a bounded set of finite $(n-2)$-dimensional Hausdorff measure is polar; see [4], Chap. IV, Theorem 1). The fine closure of $E_{NT} \backslash (Y \cup Z)$ is also ∂B, so $\overline{h} \leq M$ on $\partial B \backslash Z$. We note that the subharmonic function h^2 has a harmonic majorant on B because

$$\int_B (1 - |x|)\,(\Delta\left(h^2\right))(x)dx \leq 2 \int_B |\nabla h(x)|^2\,dx < +\infty.$$

Hence h is equal to the Poisson integral of a function g on ∂B (see [7], 1.II.14). Further, from (3) and Fatou's boundary limit theorem, $g = \overline{h}\,|_{\partial B}$ almost everywhere (σ) on ∂B, and so $h \leq M$ on B. Thus (1) holds and we have now established the "if" part of Theorem 1.

3.3. Conversely, suppose that (1) holds and let

$$F(r) = \partial B \cap \left(\bigcup_{x \in E \backslash B(0,r)} B(x, 2(1-|x|)) \right) \quad (0 < r < 1).$$

Clearly (1) implies that $\bar{E} \cap \partial B \neq \emptyset$, so $F(r) \neq \emptyset$ for all r. Suppose further, for the sake of contradiction, that there exists r such that the fine closure of $F(r)$ is a proper subset of ∂B. Then the function $h = 1 - \widehat{R}_1^{F(r)}$, which is harmonic on B, is strictly positive there, in view of Lemma 2. We note (see [7], 1.IV.5) that there is an increasing sequence (u_k) of C^∞ Newtonian potentials, with associated measures (μ_k), such that $u_k \uparrow \widehat{R}_1^{F(r)}$ and u_k is harmonic outside $\overline{B(0,1+1/k)} \backslash B(0,1-1/k)$ and that

$$\int_{\mathbb{R}^n} |\nabla u_k(x)|^2\,dx = a_n \int u_k d\mu_k$$

by Green's first identity, where a_n is a positive constant depending only on n. Since $|\nabla u_k| \to |\nabla h|$ locally uniformly on B, it follows from the reciprocity law that

$$\int_B |\nabla h(x)|^2 \, dx \leq a_n \int \widehat{R}_1^{F(r)} d\nu_{F(r)} \leq a_n \mathcal{C}(F(r)) < +\infty;$$

that is, $h\,|_B \in \mathcal{D}$.

Let $M = \sup_B h$ and let v_r denote the harmonic measure of $\partial B \backslash F(r)$ in B. The definition of $F(r)$ ensures that there is a constant $c \in (0,1)$, independent of E, such that $v_r \leq c$ on $E \backslash B(0,r)$. Since $h = 1 - \widehat{R}_1^{F(r)} = 0$ on $F(r)$, it follows that $h \leq cM$ on $E \backslash B(0,r)$, and so

$$\sup_E h \leq \max \left\{ cM, \sup_{B(0,r)} h \right\} < M = \sup_B h.$$

This contradicts (1). Hence, for every $r \in (0,1)$, the fine closure of $F(r)$ is all of ∂B.

Let

$$F = \bigcap_{k=1}^{\infty} F\left(\frac{k}{k+1}\right).$$

Then, by Lemma 1, the fine closure of F is also all of ∂B. Finally, it is clear that $F \subseteq E_{NT}$, so the fine closure of E_{NT} is also ∂B. Thus $\hat{R}_1^{E_{NT}} = \hat{R}_1^{\partial B}$, and hence $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$, as required.

3.4. We now present the details of Example 1. Let E be the set defined there. Then

$$E_{NT} = \bigcup_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} \bigcup_{k=0}^{2^j-1} F_{j,k,l},$$

where

$$F_{j,k,l} = \left\{ e^{i\theta} : \left| \theta - k2^{1-j}\pi \right| < l2^{-2j} \right\}.$$

The set $F_{j,k,l}$ is an open arc of the unit circle and $\mathcal{C}(F_{j,k,l}) \geq l2^{-2j-2}$ (see [11], p. 173, (2.4.4)). It follows easily from Wiener's criterion that the set

$$A_{l,m} = \bigcup_{j=m}^{\infty} \bigcup_{k=0}^{2^j-1} F_{j,k,l}$$

is finely dense in the circle. Hence, for any $l \in \mathbb{N}$, the set $\bigcap_{m=1}^{\infty} A_{l,m}$ is finely dense in the circle, by Lemma 1. It follows that $\mathcal{C}(E_{NT}) = \mathcal{C}(\partial B)$. However,

$$\sigma\left(A_{l,m}\right) \leq \sum_{j=m}^{\infty} \sum_{k=0}^{2^j-1} l2^{1-2j} = l2^{2-m},$$

so $\sigma\left(\bigcap_{m=1}^{\infty} A_{l,m}\right) = 0$ and hence $\sigma(E_{NT}) = 0$, as claimed.

4. Proof of Theorem 2

Let $f : B \to [-\infty, +\infty]$ and

$$D = \{z \in \partial B : C_F(f,z) \backslash C_K(f,z) \neq \emptyset\},$$

where, for each $z \in \partial B$,

$$\begin{aligned} C_K(f,z) &= \{l \in [-\infty,+\infty] : f(x_k) \to l \text{ for some sequence } (x_k) \text{ of} \\ &\qquad \text{points in } K(z,1/2,1/2) \text{ such that } x_k \to z\}. \end{aligned}$$

Further, let $\mathcal{I}$ denote the collection of closed intervals of $[-\infty,+\infty]$ with endpoints in $\mathbb{Q} \cup \{-\infty,+\infty\}$. Suppose that $z \in D$. Since $C_K(f,z)$ is a compact subset of $[-\infty,+\infty]$, we can find $I \in \mathcal{I}$, a finite union J of intervals from $\mathcal{I}$, and $\varepsilon \in \mathbb{Q} \cap (0,1)$ such that

$$I \cap C_F(f,z) \neq \emptyset, \quad I \cap J = \emptyset \tag{4}$$

and

$$f\left(K\left(z, \frac{1}{2}, \varepsilon\right)\right) \subseteq J. \tag{5}$$

If I, J and ε are as above, then we say that $z \in D(I,J,\varepsilon)$. Thus

$$D \subseteq \bigcup_{I,J,\varepsilon} D(I,J,\varepsilon), \tag{6}$$

where the union is over all possible choices of I, J, ε.

Now suppose that one of the sets in this union, $D_0 = D(I_0, J_0, \varepsilon_0)$ say, has the property that its (Euclidean) closure has non-empty interior U with respect to the fine-induced topology on ∂B, let

$$E = B \setminus \left(\bigcup_{z \in D_0} K\left(z, \frac{1}{2}, \varepsilon_0\right)\right) \tag{7}$$

and $h = 1 - \widehat{R}_1^{\partial B \backslash U}$. Then $h = 0$ on $\partial B \backslash U$ since the latter set is finely perfect. Also, $\overline{E} \cap \partial B \neq \emptyset$ by (4) and (5), and

$$B(x, 2(1-|x|)) \cap U = \emptyset \qquad (x \in E \backslash \overline{B(0, 1-\varepsilon_0)})$$

by the definition of E. Reasoning as in §3.3, we now see that $h > 0$ on B, and on U by Lemma 2, but $\sup_E h < \sup_B h$. Let $\sup_E h < M < \sup_B h$ and $V = \{x \in \mathbb{R}^n : h(x) > M\}$, and choose $z_1 \in U$ such that $h(z_1) > M$. Then V is a fine neighbourhood of z_1 and $V \cap E = \emptyset$. It now follows from (5) and (7) that $C_F(f, z_1) \subseteq J_0$, but this contradicts (4). Thus $U = \emptyset$.

Let

$$A = \bigcap_{I,J,\varepsilon} \left(\partial B \backslash \overline{D(I,J,\varepsilon)}\right). \tag{8}$$

Thus A is a (Euclidean) G_δ-set and each of the sets $\partial B \backslash \overline{D(I,J,\varepsilon)}$ is open and dense in the fine-induced topology on ∂B. By Lemma 1, the fine closure of A is ∂B and hence $\mathcal{C}(A) = \mathcal{C}(\partial B)$. Since $C_K(f,z) \subseteq C_{NT}(f,z)$, it follows that

$$C_F(f,z) \subseteq C_{NT}(f,z) \quad (z \in \partial B \backslash D).$$

From (6) and (8) we see that $A \subseteq \partial B \backslash D$, and so Theorem 2 is established.

5. Proof of Theorem 3

Let $A \subseteq \partial B$ be a Euclidean-G_δ set such that $\mathcal{C}(A) = \mathcal{C}(\partial B)$. As we argued at the beginning of §3.2, it follows that A is finely dense in ∂B. Thus there is an increasing sequence (F_k) of compact sets such that $\partial B \backslash A = \bigcup_k F_k$ and such that $\partial B \backslash F_k$ is finely dense in ∂B for each k. To avoid trivialities we may assume that the sets $\partial B \backslash A$ and F_1 are non-empty.

Let $g_k : \partial B \to [0,1)$ be defined by

$$g_k(z) = \frac{\{\operatorname{dist}(z, F_k)\}^2}{5} \quad (z \in \partial B; k \in \mathbb{N}).$$

Then the sets

$$E_k = \left\{ rz : z \in \partial B \backslash F_k \text{ and } 1 - r = \frac{g_k(z)}{3k-1} \right\} \quad (k \in \mathbb{N}),$$

$$D_1 = B \cap \{ rz : z \in \partial B \text{ and } 1 - r \geq g_1(z) \},$$

and

$$D_{k+1} = B \cap \left\{ rz : z \in \partial B \text{ and } \frac{g_k(z)}{3k} \geq 1 - r \geq \frac{g_{k+1}(z)}{3k+1} \right\} \quad (k \in \mathbb{N})$$

are all closed relative to B and are pairwise disjoint. Let

$$E = \left(\bigcup_k E_k \right) \cup \left(\bigcup_k D_k \right) \tag{9}$$

and let B^* denote the Alexandroff (one-point) compactification of B. Since, for a given value of r in $(0,1)$, only finitely many sets in the union in (9) meet $B(0,r)$, the set E is also closed relative to B.

If we define $u : E \to \mathbb{R}$ by

$$u(x) = \begin{cases} 0 & \text{if} \quad x \in \bigcup_k D_k, \\ q_k & \text{if} \quad x \in E_k; k \geq 1, \end{cases} \tag{10}$$

where (q_k) is an enumeration of $\mathbb{Q}$, then u extends to a locally constant (and hence harmonic) function on an open set which contains E. It is easy to check that $B^* \backslash E$ is connected and locally connected (see §3.2 of [9] for a discussion of local connectedness in this context). We note that, if $z \in \partial B \backslash A$ and $0 < \delta < 1$, then there exists a (smallest) number k_0 such that $z \in F_{k_0}$ and a number $\varepsilon_{z,\delta}$ in $(0,1)$ such that

$$K(z, \delta, \varepsilon_{z,\delta}) \subseteq D_{k_0} \subseteq \bigcup_k D_k \quad (0 < \delta < 1). \tag{11}$$

We also claim that, if $z \in \partial B \backslash A$, then E_k is non-thin at z for all sufficiently large k. To see this, we choose k_0 such that $z \in F_{k_0}$ and suppose that E_k is thin at z for some $k \geq k_0$. Since the radial projection map from E_k to $\partial B \backslash F_k$ is a Lipschitz map with Lipschitz constant 2, we have

$$\begin{aligned} &\mathcal{C}\left(\left\{ y \in \partial B \backslash F_k : 2^{-j-1} \leq |y - z| \leq 2^{-j} \right\}\right) \\ \leq \quad &2^{n-2} \mathcal{C}\left(\left\{ y \in E_k : 2^{-j-2} \leq |y - z| \leq 2^{-j+1} \right\}\right) \end{aligned}$$

for each $j \in \mathbb{N}$, by standard contraction and dilation properties of Newtonian capacity (see [11], Chap. 2, §3). Hence, by Wiener's criterion, we obtain the contradictory conclusion that $\partial B \backslash F_k$ is also thin at z. Thus our claim is verified.

We now apply a recent harmonic approximation result (see [1], or Corollary 3.10 in [9]) to observe that there is a harmonic function h on B such that

$$|h(x) - u(x)| < 1 - |x| \quad (x \in E).$$

It follows from (10) and (11) that $C_{NT}(h,z) = \{0\}$ whenever $z \in \partial B \backslash A$. For such z, it also follows from (10) and the claim verified in the previous paragraph that $C_F(h,z)$ contains all but a finite number of the rationals, and so $C_F(h,z) = [-\infty, +\infty]$. The proof of Theorem 3 is now complete.

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