PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 11, Pages 3193–3200
S 0002-9939(01)05962-7
Article electronically published on May 21, 2001

CHARACTERIZATION OF COMPLETIONS OF REDUCED LOCAL RINGS

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. We find necessary and sufficient conditions for a complete local ring to be the completion of a reduced local ring. Explicitly, these conditions on a complete local ring T with maximal ideal $\mathfrak{m}$ are (i) $\mathfrak{m} = (0)$ or $\mathfrak{m} \notin \operatorname{Ass} T$, and (ii) for all $\mathfrak{p} \in \operatorname{Ass} T$, if $r \in \mathfrak{p}$ is an integer of T, then $\operatorname{Ann}_T(r) \not\subseteq \mathfrak{p}$.

1. INTRODUCTION

In this paper all rings are commutative with unity. Local rings are defined to be Noetherian, while quasi-local rings are not necessarily Noetherian. When we write (R, M) is a quasi-local ring, we mean that R is a quasi-local ring with maximal ideal M. In this case $\widehat{R}$ denotes the M-adic completion of R.

In 1986, Lech solved the problem of characterizing completions of local domains, proving that a complete local ring $(T, \mathfrak{m})$ is the completion of a local domain if and only if (1) $\mathfrak{m} = (0)$ or $\mathfrak{m} \notin \operatorname{Ass} T$ and (2) no nonzero integer of T is a zero-divisor [6]. Heitmann, in 1993, continued this work by finding all completions of local unique factorization domains [4]. To be precise, a complete local ring T is the completion of a local UFD if and only if it is a field, a discrete valuation ring, or a ring of depth at least two with no nonzero integer being a zero-divisor. Following this trend, we asked the corresponding question for reduced rings: given a complete local ring, when is it the completion of a reduced local ring? In this paper we prove a theorem that answers this question.

Theorem 1. *Let $(T, \mathfrak{m})$ be a complete local ring, and let R_0 be its prime subring. Then T is the completion of a reduced local ring if and only if the following two properties hold:*
(i) $\mathfrak{m} = (0)$ *or* $\mathfrak{m} \notin \operatorname{Ass} T$.
(ii) *For all* $\mathfrak{p} \in \operatorname{Ass} T$*, if* $r \in \mathfrak{p} \cap R_0$*, then* $\operatorname{Ann}_T(r) \not\subseteq \mathfrak{p}$.

To illustrate the applicability of this theorem, we include two examples of complete local rings and ask whether they are completions of reduced local rings. In both cases, we know that they are not completions of local domains by Lech's Theorem.

Received by the editors January 18, 2000 and, in revised form, March 27, 2000.
2000 *Mathematics Subject Classification.* Primary 13B35.
Key words and phrases. Reduced rings, completions.
This research was supported by NSF Grant DMS-9820570 and conducted as part of the Williams College Math REU under the guidance of advisor S. Loepp.

Example 1. Let p be a prime integer, and let $\widehat{\mathbb{Z}}_{(p)}$ denote the p-adic integers. Let $T_1 = \widehat{\mathbb{Z}}_{(p)}[\![x,y]\!]/(px,y^2)$. In this case, $\operatorname{Ass} T_1 = \{(x,y),(p,y)\}$, so (i) of Theorem 1 is satisfied. Moreover, the prime subring is $\mathbb{Z}$, and

$$\begin{aligned}(x,y)\cap\mathbb{Z} &= (0),\\ (p,y)\cap\mathbb{Z} &= p\mathbb{Z} \text{ and } xp=0 \text{ with } x\notin(p,y).\end{aligned}$$

Thus, by Theorem 1, T_1 is the completion of a reduced local ring.

Example 2. Let $T_2 = \widehat{\mathbb{Z}}_{(p)}[\![x,y]\!]/(px,x^2)$. In this case, $(p,x)\in\operatorname{Ass} T_2$ and the prime subring is $\mathbb{Z}$. Note that $p\in(p,x)\cap\mathbb{Z}$ and no element outside of (p,x) annihilates p, so condition (ii) of Theorem 1 is not satisfied. So T_2 is not the completion of a reduced local ring, although the prime subring is reduced.

To prove Theorem 1, we use a construction based on that of Heitmann [4]. Specifically, we use Proposition 1 in [5] which states that if $(A,\mathfrak{m}\cap A)$ is a quasi-local subring of a complete local ring $(T,\mathfrak{m})$, $A\to T/\mathfrak{m}^2$ is surjective, and $IT\cap A = I$ for every finitely generated ideal I of A, then A is Noetherian and the natural homomorphism $\widehat{A}\to T$ is an isomorphism. We begin with the prime subring of T and construct an increasing chain of rings whose union satisfies the conditions of the proposition. During this process, we construct each ring so that it is reduced. However, it is not possible to simply preserve reducedness at each step (see Example 2). We need to carry through a slightly stronger condition on each ring R: for all $\mathfrak{p}\in\operatorname{Ass} T$, if $r\in\mathfrak{p}\cap R$, then $\operatorname{Ann}_T(r)\not\subseteq\mathfrak{p}$.

2. Proof of Theorem 1

Proof of necessity. Assume that $T=\widehat{A}$ for some reduced local ring A. Then we claim that $\mathfrak{p}\cap A$ is a minimal prime of A for any $\mathfrak{p}\in\operatorname{Ass} T$. Note that $\mathfrak{p}\cap A$ is a set of elements in A which are zero-divisors in T, so $\mathfrak{p}\cap A$ is actually a set of zero-divisors in A. Since A is reduced, $\mathfrak{p}\cap A$ must be contained in the union of minimal primes of A; by prime avoidance and minimality, it follows that $\mathfrak{p}\cap A$ is a minimal prime. Now, we will check property (i). If $\mathfrak{m}\in\operatorname{Ass} T$, then by our claim, $\mathfrak{m}\cap A$ is a minimal prime of A. But $\mathfrak{m}\cap A$ is a maximal prime of A by faithful flatness, so $\mathfrak{m}\cap A$ is the nilradical of A which is (0). Thus $\mathfrak{m}=(\mathfrak{m}\cap A)T=(0)$. To check (ii), let $a\in\mathfrak{p}\cap A$ for some $\mathfrak{p}\in\operatorname{Ass} T$. Since $\mathfrak{p}\cap A$ is a minimal prime of a reduced ring A, $A_{\mathfrak{p}\cap A}$ is a field. Hence, a must be annihilated by an element in $A-(\mathfrak{p}\cap A)$. Since A contains R_0, (ii) follows. □

The remainder of this paper deals with the proof of sufficiency. We shall show that if $(T,\mathfrak{m})$ is a complete local ring with properties (i) and (ii) of Theorem 1, then we can construct a reduced local ring A such that $\widehat{A}=T$. Note first that if $\mathfrak{m}=(0)$, then T is a field, and so T is the completion of a reduced local ring, namely itself. Thus we now prove the theorem for the case $\mathfrak{m}\notin\operatorname{Ass} T$. In order to construct a reduced local ring that completes to T, we will need the following proposition, which is essentially a “completion-proving machine.” The proof of this proposition can be found in [5].

Proposition 2. *Let $(A,\mathfrak{m}\cap A)$ be a quasi-local subring of a complete local ring $(T,\mathfrak{m})$. Assume that A surjects onto $T/\mathfrak{m}^2$ and for every finitely generated ideal I in A, $IT\cap A=I$. Then A is Noetherian and the natural homomorphism $\widehat{A}\longrightarrow T$ is an isomorphism.*

Let $(T, \mathfrak{m})$ be a complete local ring, R a quasi-local subring of T, and C a subset of $\operatorname{Spec} T$. In our construction we will want to choose elements of T that are transcendental over $R/(P \cap R)$ as elements of T/P for all $P \in C$. The following two lemmas allow us to do this. These lemmas and their proofs can be found in [4] as Lemma 2 and Lemma 3.

Lemma 3. *Let T be a complete local ring with maximal ideal $\mathfrak{m}$, C a countable subset of $\operatorname{Spec} T$ such that $\mathfrak{m} \notin C$, and D a countable set of elements of T. If I is an ideal of T which is contained in no single P in C, then $I \not\subseteq \bigcup\{P + r | P \in C, r \in D\}$.*

Lemma 4. *Let $(T, \mathfrak{m})$ be a local ring. Let $C \subseteq \operatorname{Spec} T$, $D \subseteq T$, and let I be an ideal of T such that $I \not\subseteq P$ for all $P \in C$. Suppose $|C \times D| < |T/\mathfrak{m}|$. Then $I \not\subseteq \bigcup\{P + r | P \in C, r \in D\}$.*

We will use the following definition in the construction of our reduced subrings.

Definition. Let $(R, \mathfrak{m} \cap R)$ be a quasi-local subring of a complete local ring $(T, \mathfrak{m})$. Then R is an **L-subring** of T iff:
(i) If $T/\mathfrak{m}$ is countable, then R is countable. Otherwise, $|R| < |T/\mathfrak{m}|$.
(ii) For all $\mathfrak{p} \in \operatorname{Ass} T$, if $r \in \mathfrak{p} \cap R$, then there exists $s \in T - \mathfrak{p}$ such that $sr = 0$.

At each stage of the construction we will want our constructed intermediate subrings to be L-subrings. In the following lemma, we prove that condition (ii) of L-subrings will ensure that all of these subrings will be reduced.

Lemma 5. *Let $R \subseteq T$ be rings. Assume that for all $\mathfrak{p} \in \operatorname{Ass} T$ and $r \in \mathfrak{p} \cap R$, there exists $s \in T - \mathfrak{p}$ such that $sr = 0$. Then R is reduced.*

Proof. Suppose r is a nonzero element of the nilradical of R. Let n be the smallest integer such that $r^n = 0$. So $n > 1$ and $r^{n-1} \neq 0$. Then $r \in \operatorname{Ann}_T(r^{n-1}) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} T$. So by assumption, there exists $s \notin \mathfrak{p}$ such that $sr = 0$. Then $s \in \operatorname{Ann}_T(r^{n-1}) \subseteq \mathfrak{p}$, which gives a contradiction. ☐

Definition. If R is an L-subring of a complete local ring T, then $S \supseteq R$ is called a **B-extension** of R if S is also an L-subring of T and $|S| \leq \max(\aleph_0, |R|)$.

Note that the B-extension relation is transitive.

The crucial lemma of this proof is the following one. We will use it to assist us in constructing rings that satisfy the finitely generated ideal condition of Proposition 2.

Lemma 6. *Let R be an L-subring of a complete local ring $(T, \mathfrak{m})$ with $\mathfrak{m} \notin \operatorname{Ass} T$. Let I be a finitely generated ideal of R, and let $c \in IT \cap R$. Then there exists a B-extension S of R with $c \in IS$.*

Proof. We proceed by induction on m, the number of generators of I. Let $c \in IT \cap R$. First consider the case $m = 1$ where $I = yR$ for some $y \in R$. We first partition $\operatorname{Ass} T = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_\mu\} \cup \{\mathfrak{q}_1, \ldots, \mathfrak{q}_\nu\}$ so that $y \in \mathfrak{p}_i$ for all i and $y \notin \mathfrak{q}_j$ for all j. Without loss of generality, take $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ to be the maximal elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_\mu\}$. Since R is an L-subring, by condition (ii), for each $i \leq k$, there exists $s_i \notin \mathfrak{p}_i$ such that $s_i y = 0$. For all $i, j \leq k$ with $i \neq j$, choose an element $r_{ij} \in \mathfrak{p}_j - \mathfrak{p}_i$. Define $u_i = s_i \prod_{j \neq i}^k r_{ij}$. Then $u_i \notin \mathfrak{p}_i$ and $u_i \in \mathfrak{p}_j$ for all $j \neq i$ while u_i annihilates y. Now define $s = \sum_{i=1}^k u_i$. Then $sy = 0$ while $s \notin \mathfrak{p}_i$ for all $i \leq k$, and consequently, $s \notin \mathfrak{p}_i$ for all $i \leq \mu$.

Since $c \in yT$, $c = yt$ for some $t \in T$. Note that $c = y(t + as)$ for any choice of $a \in T$. For each i, let D_i be a full set of coset representatives for those choices of $v \in T$ which make $\overline{t+v} \in T/\mathfrak{p}_i$ algebraic over $R/(\mathfrak{p}_i \cap R)$. Let $D = \bigcup D_i$ and $C = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_\mu\}$. Notice that R satisfies condition (i) of the L-subring, $\mathfrak{m} \notin C$, and $sT \not\subseteq \mathfrak{p}_i$ for all i. Therefore we may apply Lemma 3 when $T/\mathfrak{m}$ is countable or Lemma 4 otherwise, which gives us an element $as \in sT$ such that $as \notin \bigcup\{\mathfrak{p}_i + v | i \leq \mu, v \in D\}$. So $t' = t + as \notin \bigcup\{\mathfrak{p}_i + (t+v) | i \leq \mu, v \in D\}$. Therefore $\overline{t'} = \overline{t+as} \in T/\mathfrak{p}_i$ is transcendental over $R/(\mathfrak{p}_i \cap R)$ for all i.

We claim that $S = R[t']_{\mathfrak{m} \cap R[t']}$ is the desired extension. It is clear that $c \in IS$, $R \subseteq S \subseteq T$, S is quasi-local, and that $|S| \leq \max(\aleph_0, |R|)$. Therefore we only need to check condition (ii) of L-subrings. It suffices to check condition (ii) on $R[t']$. Pick $\mathfrak{P} \in \operatorname{Ass} T$.

Case 1 : $\mathfrak{P} = \mathfrak{p}_i$ for some i.
Let $f \in \mathfrak{p}_i \cap R[t']$. Write $f = r_n(t')^n + \cdots + r_0$ for some $r_j \in R$. Since $\overline{t'} \in T/\mathfrak{p}_i$ is transcendental over $R/(\mathfrak{p}_i \cap R)$, we see that each $r_j \in \mathfrak{p}_i \cap R$. By condition (ii) on R, for each j, there exists $b_j \notin \mathfrak{p}_i$ such that $b_j r_j = 0$. Thus $\prod_{j=1}^n b_j$ annihilates f but is not contained in $\mathfrak{p}_i$.

Case 2 : $\mathfrak{P} = \mathfrak{q}_i$ for some i.
Let $f \in \mathfrak{q}_i \cap R[t']$. Writing f as before, we see that $y^n f \in R$. So $y^n f \in \mathfrak{q}_i \cap R$. By condition (ii), there exists $b \notin \mathfrak{q}_i$ such that $by^n f = 0$. Notice $by^n \notin \mathfrak{q}_i$.

This completes the $m = 1$ step of the induction.

Now let $m > 1$. So $I = (y_1, \ldots, y_m)R$ for some $y_1, \ldots, y_m \in R$. Define $J = (y_1, \ldots, y_{m-1})R$. We know $c = t_1 y_1 + \cdots + t_m y_m$ for some $t_1, \ldots, t_m \in T$. Just as in the $m = 1$ case, we choose $s \in \operatorname{Ann}_T(y_m)$ such that $s \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} T$ containing y_m. Define $t' = t_m + a_0 s + a_1 y_1 + \cdots + a_{m-1} y_{m-1}$ where the a_i's are to be chosen later. Observe that $c = t' y_m + t'_1 y_1 + \cdots + t'_{m-1} y_{m-1}$ where $t'_i = t_i - a_i y_m$.

For all $0 < i < m$, define $C_i = \{\mathfrak{p} \in \operatorname{Ass} T | y_i \notin \mathfrak{p}\}$. Also let $C_0 = \{\mathfrak{p} \in \operatorname{Ass} T | s \notin \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Ass} T | y_m \in \mathfrak{p}\}$. Finally, define $C = C_0 \cup C_1 \cup \cdots \cup C_{m-1}$. As in the $m = 1$ case, we can use Lemmas 3 and 4 to choose a_0 such that $\overline{t_m + a_0 s} \in T/\mathfrak{p}$ is transcendental over $R/(\mathfrak{p} \cap R)$ for all $\mathfrak{p} \in C_0$. Once this is done, we can choose a_1 such that $\overline{t_m + a_0 s + a_1 y_1} \in T/\mathfrak{p}$ is transcendental over $R/(\mathfrak{p} \cap R)$ for all $\mathfrak{p} \in C_1$. We continue this process all the way to a_{m-1}. Observe that with these choices, $\overline{t'} \in T/\mathfrak{p}$ is transcendental over $R/(\mathfrak{p} \cap R)$ for all $\mathfrak{p} \in C$. We claim that $R' = R[t']_{\mathfrak{m} \cap R[t']}$ is a B-extension of R. R' is clearly quasi-local and satisfies $|R'| \leq \max(\aleph_0, |R|)$. Again, it suffices to check condition (ii) on $R[t']$.

Let $\mathfrak{p} \in \operatorname{Ass} T$.

Case 1 : $\mathfrak{p} \in C$.
Let $f \in \mathfrak{p} \cap R[t']$. The proof is the same as Case 1 when $m = 1$.

Case 2 : $\mathfrak{p} \notin C$.
This means that $y_m \notin \mathfrak{p}$ while $y_1, \ldots, y_{m-1} \in \mathfrak{p}$. So $JT \subseteq \mathfrak{p}$. Let $f \in \mathfrak{p} \cap R[t']$. Define $\xi_1 = -(t'_1 y_1 + \cdots + t'_{m-1} y_{m-1}) \in JT$. Then $y_m t' = c + \xi_1$. Therefore for any j, defining

$$\xi_j = \sum_{k=1}^{j} \binom{j}{k} c^{j-k} \xi_1^k,$$

we have that $\xi_j \in JT$ and $(y_m t')^j = c^j + \xi_j$. So if $f = r_n(t')^n + r_{n-1}(t')^{n-1} + \cdots + r_0$, then

$$\begin{aligned} y_m^n f &= r_n(y_m t')^n + r_{n-1} y_m (y_m t')^{n-1} + \cdots + r_0 y_m^n \\ &= (r_n c^n + r_{n-1} y_m c^{n-1} + \cdots + r_0 y_m^n) \\ &\qquad + (r_n \xi_n + r_{n-1} y_m \xi_{n-1} + \cdots + r_1 y_m^{n-1} \xi_1). \end{aligned}$$

Setting

$$\xi = r_n \xi_n + r_{n-1} y_m \xi_{n-1} + \cdots + r_1 y_m^{n-1} \xi_1 \in JT,$$

we get $y_m^n f - \xi \in R$ since $c \in R$. Then since $f \in \mathfrak{p}$ and $JT \subseteq \mathfrak{p}$, $y_m^n f - \xi \in \mathfrak{p} \cap R$. So by condition (ii) of L-subrings, there exists $u \in T - \mathfrak{p}$ such that $u(y_m^n f - \xi) = 0$. Meanwhile, for each $i < m$, $y_i \in \mathfrak{p} \cap R$, there exists $u_i \in T - \mathfrak{p}$ such that $u_i y_i = 0$. Then

$$\left(\prod_{i<m} u_i\right) u y_m^n f = \left(\prod_{i<m} u_i\right) u \xi = 0$$

since $\xi \in (y_1, \ldots, y_{m-1})T$. Notice that $y_m^n \left(\prod_{i<m} u_i\right) u \notin \mathfrak{p}$, satisfying condition (ii).

This completes the proof that R' is a B-extension of R.

Now consider $c' = c - t' y_m$. Recall that $c = t' y_m + t'_1 y_1 + \cdots + t'_{m-1} y_{m-1}$ so that $c' \in JT \cap R[t']$. Since JR' is generated by $m-1$ elements in R', which is an L-subring, our induction hypothesis shows that there exists a B-extension S of R' with $c' \in JS$. Then $c \in IS$, so S is the desired ring. □

The following lemma helps us satisfy the surjectivity condition of Proposition 2.

Lemma 7. *Let R be an L-subring of a complete local ring $(T, \mathfrak{m})$ with $\mathfrak{m} \notin \operatorname{Ass} T$, and let $u \in T$. Then there exists a B-extension S of R and an element $v \in S$ such that $u - v \in \mathfrak{m}^2$.*

Proof. Because $\mathfrak{m} \notin \operatorname{Ass} T$, $\mathfrak{m}^2$ is not contained in any associated prime. For each $\mathfrak{p} \in \operatorname{Ass} T$, let $D_\mathfrak{p}$ be a full set of coset representatives for those choices of t which make $\overline{u+t} \in T/\mathfrak{p}$ algebraic over $R/(\mathfrak{p} \cap R)$. Let $D = \bigcup_{\mathfrak{p} \in \operatorname{Ass} T} D_\mathfrak{p}$. As before, we use Lemma 3 when $T/\mathfrak{m}$ is countable or Lemma 4 otherwise to obtain $t \in \mathfrak{m}^2$ such that $\overline{u+t} \in T/\mathfrak{p}$ is transcendental over $R/(\mathfrak{p} \cap R)$ for all $\mathfrak{p} \in \operatorname{Ass} T$. Let $v = u + t$ and $S = R[v]_{\mathfrak{m} \cap R[v]}$. Then S satisfies condition (ii) of L-subrings by the same argument as in Case 1 of the $m = 1$ case of the proof of Lemma 6. The cardinality condition $|S| \le \max(\aleph_0, |R|)$ is obvious. □

Definition. For Ω a well-ordered set and $\alpha \in \Omega$, define $\gamma(\alpha) = \sup\{\beta \in \Omega | \beta < \alpha\}$.

The next lemma shows that condition (ii) of L-subrings is preserved when we take the union of a chain of B-extensions.

Lemma 8. *Let Ω be a well-ordered set with least element α_0, $(T, \mathfrak{m})$ a complete local ring, and R_{α_0} an L-subring of T. Suppose $\{R_\alpha\}_{\alpha \in \Omega}$ is an increasing chain of L-subrings such that if $\gamma(\alpha) < \alpha$, then R_α is a B-extension of $R_{\gamma(\alpha)}$, and otherwise $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$. Then $S = \bigcup_{\alpha \in \Omega} R_\alpha$ satisfies condition* (ii) *of L-subrings, and also $|S| \le \max(\aleph_0, |R_{\alpha_0}|, |\Omega|)$.*

Proof. If $\mathfrak{p} \in \operatorname{Ass} T$ and $r \in \mathfrak{p} \cap S$, then $r \in \mathfrak{p} \cap R_\alpha$ for some $\alpha \in \Omega$. Then since each R_α is an L-subring, it is clear that condition (ii) of L-subrings holds for S. We prove by transfinite induction that for all $\alpha \in \Omega$,

$$|R_\alpha| \leq \max(\aleph_0, |R_{\alpha_0}|, |\Omega|).$$

Assume that $|R_\beta| \leq \max(\aleph_0, |R_{\alpha_0}|, |\Omega|)$ has been proved for all $\beta < \alpha$. If $\gamma(\alpha) < \alpha$, then the inequality is clear by the induction hypothesis. Otherwise, we have

$$\begin{aligned} |R_\alpha| &= \left|\bigcup_{\beta<\alpha} R_\beta\right| \\ &\leq \sum_{\beta<\alpha} |R_\beta| \\ &\leq |\{\beta \in \Omega | \beta < \alpha\}| \cdot \max(\aleph_0, |R_{\alpha_0}|, |\Omega|) \qquad \text{by the induction hypothesis} \\ &\leq \max(\aleph_0, |R_{\alpha_0}|, |\Omega|). \end{aligned}$$

Using this,

$$|S| \leq |\Omega| \cdot \max(\aleph_0, |R_{\alpha_0}|, |\Omega|) = \max(\aleph_0, |R_{\alpha_0}|, |\Omega|).$$

□

Lemma 9. *Let R be an L-subring of a complete local ring $(T, \mathfrak{m})$ with $\mathfrak{m} \notin \operatorname{Ass} T$, and let $u \in T$. Then there exists a B-extension S of R such that*

(1) *S contains an element v of T such that $\overline{v} = \overline{u} \in T/\mathfrak{m}^2$.*

(2) *For every finitely generated ideal I of S, $IT \cap S = I$.*

Proof. By Lemma 7, there exists a B-extension $S^{(0)}$ of R which contains some v with $\overline{v} = \overline{u} \in T/\mathfrak{m}^2$. Let

$$\Omega = \{(I, c) | I \text{ finitely generated ideal of } S^{(0)} \text{ and } c \in IT \cap S^{(0)}\}.$$

Note that $|\Omega| \leq \max(\aleph_0, |S^{(0)}|) = \max(\aleph_0, |R|)$. When Ω is finite, by transitivity of the B-extension relation, we can simply apply Lemma 6 $|\Omega|$ times to construct a B-extension $S^{(1)}$ of $S^{(0)}$ with $IT \cap S^{(0)} \subseteq IS^{(1)}$ for any finitely generated ideal I of $S^{(0)}$. In the case when Ω is infinite, we well-order Ω in such a way that there is no maximal element. Let α_0 be the initial element. We shall inductively define the family of L-subrings $\{S_\alpha\}_{\alpha \in \Omega}$, starting from $S_{\alpha_0} = S^{(0)}$. Let us assume that S_β has been defined for all $\beta < \alpha$. If $\gamma(\alpha) \neq \alpha$ and $\gamma(\alpha) = (I, c)$, we use Lemma 6 to define S_α to be a B-extension of $S_{\gamma(\alpha)}$ such that $c \in IS_\alpha$. If $\gamma(\alpha) = \alpha$, define $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$; by Lemma 8, this is an L-subring. So by induction, $\{S_\alpha\}_{\alpha \in \Omega}$ is a family satisfying the hypotheses of Lemma 8. Now, define $S^{(1)} = \bigcup_{\alpha \in \Omega} S_\alpha$. By Lemma 8, this is a B-extension of $S^{(0)}$. Now let I be a finitely generated ideal of $S^{(0)}$ and let $c \in IT \cap S^{(0)}$. If α is the successor of (I, c) in Ω, then we have $(I, c) = \gamma(\alpha) < \alpha$. So by construction, $c \in IS_\alpha \subseteq IS^{(1)}$. Thus $IT \cap S^{(0)} \subseteq IS^{(1)}$ for any finitely generated ideal I of $S^{(0)}$.

We repeat this process for each $m \in \mathbb{N}$ to obtain B-extensions $S^{(m+1)}$ of $S^{(m)}$ such that for every finitely generated ideal I of $S^{(m)}$, $IT \cap S^{(m)} \subseteq IS^{(m+1)}$. Define $S = \bigcup_{n=0}^{\infty} S^{(n)}$. By Lemma 8, S is a B-extension of $S^{(0)}$, so by transitivity, S is a B-extension of R. Let $I = (z_1, \ldots, z_k)$ be a finitely generated ideal of S, and let

$c \in IT \cap S$. Then there exists n such that c and all z_i belong to $S^{(n)}$. Then by construction,

$$c \in \big((z_1, \dots, z_k)S^{(n)}\big)T \cap S^{(n)} \subseteq (z_1, \dots, z_k)S^{(n+1)} \subseteq IS = I.$$

Hence, we have $IT \cap S = I$ for any finitely generated ideal I of S. □

We now have enough tools to prove sufficiency of the conditions in Theorem 1.

Theorem 10. *Let $(T, \mathfrak{m})$ be a complete local ring with $\mathfrak{m} \notin \operatorname{Ass} T$, and let R_0 be the prime subring of T. Assume that for all $\mathfrak{p} \in \operatorname{Ass} T$, $r \in \mathfrak{p} \cap R_0$ implies $\operatorname{Ann}_T(r) \not\subseteq \mathfrak{p}$. Then T is the completion of a reduced local ring.*

Proof. Note that our condition on R_0 implies that R_0 localized at $\mathfrak{m} \cap R_0$ is an L-subring. Call this localized ring S_0.

Claim. $|T/\mathfrak{m}^2| \leq |T/\mathfrak{m}|^{N+1}$ where N is the number of generators of $\mathfrak{m}$.

Proof. $\mathfrak{m}/\mathfrak{m}^2$ is generated over $T/\mathfrak{m}$ by N elements. So $|\mathfrak{m}/\mathfrak{m}^2| \leq |T/\mathfrak{m}|^N$. Because $\frac{T/\mathfrak{m}^2}{\mathfrak{m}/\mathfrak{m}^2} \cong T/\mathfrak{m}$, it follows that $|T/\mathfrak{m}^2| = |\mathfrak{m}/\mathfrak{m}^2| \cdot |T/\mathfrak{m}| \leq |T/\mathfrak{m}|^{N+1}$. □

When $|T/\mathfrak{m}^2|$ is finite, we can simply apply Lemma 9 $|T/\mathfrak{m}^2|$ times to obtain an L-subring A satisfying the conditions of Proposition 2, completing the proof. When $T/\mathfrak{m}^2$ is infinite, then the claim above shows that $|T/\mathfrak{m}| = |T/\mathfrak{m}^2|$. In this case, let Ω be a full set of coset representatives of $T/\mathfrak{m}^2$, and well-order it in such a way that 0 is the initial element and every element has strictly fewer than $|\Omega| = |T/\mathfrak{m}|$ predecessors. We shall define a family of L-subrings $\{S_\alpha\}_{\alpha \in \Omega}$ inductively, starting from S_0. Assume that S_β has been defined for all $\beta < \alpha$. If $\gamma(\alpha) \neq \alpha$, define S_α to be the B-extension of $S_{\gamma(\alpha)}$ we obtain by applying Lemma 9, using $S_{\gamma(\alpha)}$ and the coset represented by $\gamma(\alpha)$. If $\gamma(\alpha) = \alpha$, define $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$. By the way Ω is ordered and by Lemma 8, S_α is an L-subring. Now define $A = \bigcup_{\alpha \in \Omega} S_\alpha$. This is not necessarily an L-subring, but it still satisfies property (ii), so by Lemma 5, A is reduced. By the way we ordered Ω, every element of Ω appears as $\gamma(\alpha) < \alpha$ for some $\alpha \in \Omega$, so it follows that the natural map $A \longrightarrow T/\mathfrak{m}^2$ is surjective. Moreover, by a similar argument as in the proof of Lemma 9, $IT \cap A = I$ for any finitely generated ideal I of A. So A satisfies all the hypotheses of Proposition 2. Thus we conclude that A is a reduced local ring which completes to T. □

Acknowledgements

We would like to thank S. Loepp for many helpful conversations, suggestions, and daily encouragement.

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