

DUNFORD–PETTIS SETS

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(Communicated by Dale Alspach)

ABSTRACT. Bibasic sequences are used to study relative weak compactness and relative norm compactness of Dunford–Pettis sets.

A Banach space X has the Dunford–Pettis property provided that every weakly compact operator with domain X and range an arbitrary Banach space Y maps weakly compact sets in X into norm compact sets in Y . Localizing this notion, a bounded subset A of X is said to be a Dunford–Pettis subset of X if $T(A)$ is relatively norm compact in Y whenever $T : X \rightarrow Y$ is a weakly compact operator. Consequently, a Banach space X has the Dunford–Pettis property if and only if each of its weakly compact sets is a Dunford–Pettis set. The survey article by Diestel [5] is an excellent source of information about classical results in Banach spaces which relate to the Dunford–Pettis property.

Kevin Andrews utilized Dunford–Pettis sets in a study of the Bochner integral in [1]. In Theorem 1 of [1], Andrews showed that a subset K of X is a Dunford–Pettis subset of X if and only if

$$\lim_n (\sup\{|x_n^*(x)| : x \in K\}) = 0$$

whenever (x_n^*) is a weakly null sequence in X^* ($=$ the continuous linear dual of X). In Corollary 4 of [1], Andrews used this characterization to show that if X has the Dunford–Pettis property and ℓ^1 does not embed in X , then the space $L^1(\mu, X)$ has the Dunford–Pettis property.

E. Bator showed in [2] that a dual space has the weak Radon–Nikodym property if and only if each Dunford–Pettis subset of X is relatively compact. In addition to reproducing Bator’s result, Emmanuele [8] established several other structure properties for Banach spaces in which all Dunford–Pettis sets are relatively compact. Since every bounded subset of a Banach space X whose dual space X^* has the Schur property is a Dunford–Pettis subset of X , it is clear that there are Dunford–Pettis sets which are not relatively weakly compact. However, we note that Odell [13, p. 377] showed that every sequence in a Dunford–Pettis set has a weakly Cauchy subsequence. In this paper we study Dunford–Pettis sets which fail to be relatively norm or weakly compact.

The following definitions and notation will be helpful. A sequence (x_n, f_n^*) in $X \times X^*$ is called bibasic [14, p. 85], [4] if (x_n) is a basic sequence in X , (f_n^*) is a basic sequence in X^* , and $f_i^*(x_j) = \delta_{ij}$. If (x_n, f_n^*) is a bibasic sequence, $X_0 = [x_n]$,

Received by the editors April 14, 1998 and, in revised form, March 15, 2000.

2000 *Mathematics Subject Classification*. Primary 46B20; Secondary 46B15, 46B45.

Key words and phrases. Bibasic sequences, Dunford–Pettis sets.

and $x_n^* = f_n^*|_{x_0}$ (i.e., (x_n^*) is the sequence of coefficient functionals corresponding to the basic sequence (x_n)), then certainly f_n^* is a continuous linear extension of x_n^* to all of X for each n . The bibasic sequence (x_n, f_n^*) is said to be seminormalized (or bounded) if there are positive numbers p and q so that $p \leq \|x_n\| \leq q$ and $p \leq \|f_n^*\| \leq q$ for all n . The unit vector basis of c_0 will be denoted by (e_n) , and the unit vector basis of ℓ^1 will be denoted by (e_n^*) . If K is a subset of X , then $K - K$ is defined to be $\{x - y : x, y \in K\}$. The reader is referred to Diestel [6] or Lindenstrauss and Tzafriri [9] for unexplained notation or terminology.

In Theorem 1 of [4], Davis, Dean, and Lin showed that every infinite dimensional Banach space contains a bounded bibasic sequence. Our first lemma will facilitate our discussion which shows that special bibasic sequences occur in $X \times X^*$ whenever X contains a Dunford–Pettis set which fails to be relatively norm or weakly compact. Although versions of (ii) of the lemma are well known, a proof has been included for the convenience of the reader.

Lemma 1. (i) *If K is a Dunford–Pettis subset of the Banach space X , (x_n, f_n^*) is a biorthogonal sequence in $K \times X^*$, and $\sup_n \|f_n^*\| < \infty$, then some subsequence of (f_n^*) is equivalent to the unit vector basis of ℓ^1 .*

(ii) *If (x_n) is an unconditional basis for X , (x_n^*) is the sequence of coefficient functionals corresponding to (x_n) , and the subsequence $(x_{n_i}^*)$ of (x_n^*) is equivalent to (e_i^*) , then $(x_{n_i}) \sim (e_i)$.*

Proof. (i) Suppose that (x_n, f_n^*) is as in the statement of (i) and that $(f_{n_i}^*)$ is a weakly Cauchy subsequence of (f_n^*) . Let

$$z_i^* = f_{n_i}^* - f_{n_{i+1}}^*$$

for each i . Therefore, $(z_i^*) \rightarrow 0$ weakly and

$$\lim_i (\sup\{|z_i^*(u)| : u \in K\}) = 0.$$

However, $z_i^*(x_{n_i}) = 1$ for each i . Thus no subsequence of (f_n^*) can be weakly Cauchy, and, by Rosenthal’s ℓ^1 -theorem [12], [6], some subsequence of (f_n^*) is equivalent to (e_i^*) .

(ii) Now suppose that X , (x_n) , and $(x_{n_i}^*)$ are as in the statement of (ii). Let M be the unconditional basis constant for (x_n) , and let J be a positive integer so that

$$(1/J) \|\sum s_i x_{n_i}^*\| \leq \sum |s_i| \leq J \|\sum s_i x_{n_i}^*\|$$

for each finite sequence (s_1, \dots, s_n) of scalars. If $x \in X$ and $x^* \in X^*$, then

$$\begin{aligned} x^*(x) &= \lim_m \sum_{i=1}^m x_i^*(x) x^*(x_i) \\ &= \lim \langle \sum_1^m x^*(x_i) x_i^*, x \rangle \\ &= \lim_m \langle P_m^*(x^*), x \rangle, \end{aligned}$$

where $(P_m)_{m=1}^\infty$ is the usual sequence of projection operators generated by a basic sequence. Thus

$$\|x\| \leq \sup\{|\langle \sum_{n=1}^k s_n x_n^*, x \rangle| : k \in \mathbf{N}, \|\sum s_n x_n^*\| \leq M\}.$$

If $x = \sum_{i=1}^p \alpha_i x_{n_i}$, then

$$\begin{aligned} (1/J)(\alpha_1, \dots, \alpha_p) \|_\infty &= \sup\{ |(1/J)x_{n_i}^*(x)| : 1 \leq i \leq p \} \leq \|x\| \\ &\leq \sup\{ |\langle \sum_{n=1}^k s_n x_n^*, \sum_{i=1}^p \alpha_i x_{n_i} \rangle| : k \in \mathbf{N}, \|\sum_{n=1}^k s_n x_n^*\| \leq M \} \\ &\leq \sup\{ |\langle \sum_{u=1}^k t_u x_{n_u}^*, \sum_{i=1}^p \alpha_i x_{n_i} \rangle| : k \in \mathbf{N}, \|\sum_{u=1}^k t_u x_{n_u}^*\| \leq M^2 \} \\ &\leq JM^2 \|(\alpha_1, \dots, \alpha_p)\|_\infty. \end{aligned}$$

Hence $(x_{n_i}) \sim (e_i)$. □

Theorem 2. *Suppose that K is a subset of the Banach space X .*

- (i) *If K is a nonrelatively norm compact Dunford–Pettis set, then there is a seminormalized bibasic sequence (x_n, f_n^*) in $(K - K) \times X^*$ such that (f_n^*) is equivalent to (e_n^*) .*
- (ii) *If (x_n, f_n^*) is a biorthogonal sequence in $(K - K) \times X^*$ and $\sup_n \|f_n^*\| < \infty$, then K is not relatively norm compact.*
- (iii) *If K is a nonrelatively weakly compact Dunford–Pettis set, then there is a bibasic sequence (x_n, f_n^*) in $K \times X^*$ and an element $x^* \in X^*$ so that (f_n^*) is equivalent to (e_n^*) and $\lim_n x^*(x_n) > 0$.*
- (iv) *If (x_n, f_n^*) is a biorthogonal sequence in $K \times X^*$, (x_n) is basic, and $x^* \in X^*$ so that $\lim x^*(x_n) > 0$, then K is not relatively weakly compact.*

Proof. (i) Suppose that K is a nonrelatively norm compact Dunford–Pettis set. Let (y_n) be a sequence in K and ϵ be a positive number so that if

$$x_n = y_n - y_{n+1}$$

for each n , then $\|x_n\| > \epsilon$ for each n and $(x_n) \rightarrow 0$ weakly. Apply a classical construction of Bessaga and Pelczynski [3], [9, p. 5], and suppose (without loss of generality) that (x_n) is a seminormalized basic sequence. Let (x_n^*) be the associated sequence of coefficient functionals, and for each n let f_n^* be a Hahn–Banach extension of x_n^* to all of X . Apply (i) of Lemma 1 to the Dunford–Pettis set $K - K$ to complete the proof of (i).

(ii) If the hypotheses of (ii) are satisfied, then there is an $\epsilon > 0$ so that $\|x_n - x_m\| > \epsilon$ for $n \neq m$. Thus $K - K$ is not relatively norm compact; hence K is not relatively norm compact.

(iii) Suppose that K is a nonrelatively weakly compact Dunford–Pettis set, and let (x_n) be a sequence in K with no weakly convergent subsequence. By Pelczynski’s version of the Eberlein–Smulian theorem [10], [6, p. 41], we may suppose that (x_n) is basic and $\lim_n x^*(x_n) > 0$ for some $x^* \in X^*$. Note that (x_n) is a seminormalized sequence, and let (x_n^*) and (f_n^*) be defined as in (i) above. Again apply Lemma 1 to finish the proof of (iii).

(iv) If (x_n, f_n^*) is a biorthogonal sequence in $K \times X^*$, (x_n) is basic, and $\lim x^*(x_n) > 0$, then (x_n) is a sequence in K without a weakly convergent subsequence. (The fact that $(x_i^*(x_n))_{n=1}^\infty \rightarrow 0$ for each i certainly implies that the only candidate for a weak limit is 0.) Therefore, K is not relatively weakly compact. □

Using a fundamental result of Pelczynski and Singer [11] dealing with the existence of *conditional* basic sequences, Davis, Dean, and Lin [4, Proposition 1] showed

that if X is an infinite dimensional space, then there is a bounded bibasic sequence (x_n, f_n^*) in $X \times X^*$ so that $(f_n^*) \not\sim (x_n^*)$. In the following result, we see that Dunford–Pettis sets which are not relatively weakly compact naturally generate such sequences.

Theorem 3. *If K is a nonrelatively weakly compact Dunford–Pettis subset of X , then there is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (f_n^*) is not equivalent to (x_n^*) . Further, if (x_n) is unconditional, then there is no sequence (g_n^*) in X^* so that $g_n^*|_{[x_j]} = x_n^*$ for each n and $(g_n^*) \sim (x_n^*)$.*

Proof. Suppose that K is a Dunford–Pettis subset of X and K fails to be relatively weakly compact. Use Theorem 2, and let (x_n, f_n^*) be a bounded bibasic sequence so that $\{x_n\} \subseteq K$, no subsequence of (x_n) converges weakly to a point of X , and $(f_n^*) \sim (e_n^*)$. Suppose that $(f_n^*) \sim (x_n^*)$. Consequently, $(x_n) \sim (e_n)$, and (x_n) converges weakly to 0. This contradiction finishes the proof of the first assertion.

Now suppose that (x_n) is unconditional and (g_n^*) is a sequence in X^* so that $g_n^*|_{[x_j]} = x_n^*$ for each n and $(g_n^*) \sim (x_n^*)$. Use Lemma 1(i) again, and let $(g_{n_i}^*)$ be a subsequence so that $(g_{n_i}^*) \sim (e_i^*)$. Therefore, $(x_{n_i}^*) \sim (e_i^*)$, and Lemma 1(ii) shows that $(x_{n_i}) \sim (e_i)$. Thus $(x_{n_i}) \rightarrow 0$ weakly, and we have a contradiction. \square

The following result studies the structure of Dunford–Pettis sets which contain unconditional basic sequences (x_n) so that $(x_n^*) \sim (f_n^*)$. As the theorem demonstrates, these are the sets which also contain weakly null basic sequences which are hereditarily Dunford–Pettis.

Theorem 4. *If K is a Dunford–Pettis subset of the Banach space X , then the following are equivalent:*

- (i) *There is an isomorphic embedding $T : c_0 \rightarrow X$ so that $\{T(e_n)\} \subseteq K$.*
- (ii) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional and $(x_n^*) \sim (f_n^*)$.*
- (iii) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional, $(f_n^*) \sim (x_n^*)$, and $[f_n^*]$ is complemented in X^* .*
- (iv) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is weakly null and $\{y_k\}$ is Dunford–Pettis in $[y_k]$ for each subsequence (y_k) of (x_n) .*
- (v) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional and $\{y_k\}$ is a Dunford–Pettis subset of $[y_k]$ for each subsequence (y_k) of (x_n) .*
- (vi) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is shrinking and $\{y_k\}$ is a Dunford–Pettis subset of $[y_k]$ for each subsequence (y_k) of (x_n) .*
- (vii) *There is a seminormalized bibasic sequence (x_n, f_n^*) in $K \times X^*$ so that (x_n) is unconditional and $\sum f_n^*(x_n)f_n^*$ converges for all $f^* \in X^*$.*

Proof. Suppose that (i) holds, and let $x_n = T(e_n)$ for each n . Let (x_n^*) be the sequence of coefficient functionals corresponding to (x_n) , and for each n let f_n^* be a Hahn–Banach extension of x_n^* to all of X . Let A be a positive number so that

$$A \sum |a_i| \leq \left\| \sum a_i x_i^* \right\|$$

for each finite sequence (a_1, \dots, a_m) of real numbers. Therefore,

$$(*) \quad A \sum |a_i| \leq \left\| \sum a_i f_i^*|_{[x_n]} \right\| \leq \left\| \sum a_i f_i^* \right\| \leq C \sum |a_i|,$$

where C is a bound for $(\|f_i^*\|)$. Then $(x_n^*) \sim (e_n^*) \sim (f_n^*)$, and (x_n, f_n^*) satisfies the conclusion of (ii).

Now let (x_n, f_n^*) satisfy the conclusions of (ii). Use Lemma 1, and let $(f_{n_i}^*)$ be a subsequence of (f_n^*) so that $(f_{n_i}^*) \sim (e_i^*)$. Since the restriction of an isomorphism is an isomorphism, $(x_{n_i}^*) \sim (e_i^*)$. Therefore, $(x_{n_i}) \sim (e_i)$ by Lemma 1(ii), and $\sum x_{n_i}$ is weakly unconditionally convergent. Thus

$$\{f^* \in X^* : \sum f^*(x_{n_i})f_{n_i}^* \text{ converges}\}$$

is all of X^* . Consequently, by [14, Cor. 12b, p. 93], the space $[f_{n_i}^*]$ is complemented in X^* . The bibasic sequence $(x_{n_i}, f_{n_i}^*)$ satisfies the requirements of (iii).

The preceding paragraph shows that (iii) implies (i). Therefore, (i), (ii), and (iii) are equivalent. Certainly a seminormalized shrinking basic sequence is weakly null, and (vi) implies (iv). Also, (i) and the inequality $(*)$ above guarantee the existence of a bibasic sequence which satisfies (iv), (v), (vi), and (vii). We complete the proof by showing that (vii) implies (ii), (iv) implies (i) and (v) implies (iv).

If (x_n, f_n^*) is a bibasic sequence which satisfies (vii), then, by Proposition 1.14 on p. 91 of [14], $[x_n]^\perp + [f_n^*] = X^*$. An application of Lemma 1.3 of [14] or Proposition 3 of [4] ensures that $(x_n^*) \sim (f_n^*)$, and (vii) implies (ii).

Now suppose that (iv) holds. Set $(y_n) = ((1/\|x_n\|)x_n)$, $(g_n^*) = (\|x_n\|f_n^*)$, and note that (y_n, g_n^*) is bibasic and (y_n) is weakly null. Following the lead of Diestel [5, p. 28] and Elton [7], we suppose that no subsequence of (y_n) is equivalent to (e_n) , and directly apply a result of Elton [7], [5, p. 27] to select a subsequence (z_n) of (y_n) so that if (w_n) is any subsequence of (z_n) , then

$$\sup_n \left\| \sum_{i=1}^n t_i w_i \right\| = \infty$$

if $(t_i) \notin c_0$. If (P_n) is the sequence of projections associated with (z_n) and M is the basis constant, then

$$\|P_n^{**}(z^{**})\| = \left\| \sum_{i=1}^n z^{**}(z_i^*)z_i \right\| \leq M \|z^{**}\|$$

for $z^{**} \in [z_n]^{**}$. Therefore (z_i^*) is weakly null in $[z_n]^*$. Since $\{z_i\}$ is Dunford-Pettis in $[z_i]$, $z_i^*(z_i) \rightarrow 0$. This clear contradiction ensures that some subsequence of (y_n) is equivalent to (e_n) . Thus some subsequence of (x_n) is equivalent to (e_n) , and (iv) implies (i).

Next we suppose that (x_n, f_n^*) satisfies the conditions of (v). We claim that (x_n) has a weakly null subsequence. If not, there is an $x^* \in X^*$, a subsequence (y_k) of (x_n) , and an $\epsilon > 0$ so that $x^*(y_k) > \epsilon$ for each k . Since (y_k) is seminormalized and unconditional, $(y_k) \sim (e_k^*)$. However, this is not possible since $\{e_k^*\}$ is certainly not a Dunford-Pettis subset of ℓ^1 . Therefore, (x_n) has a weakly null subsequence, and (v) implies (iv). □

We note that Problem 2.iii on p. 52 of [6] asserts that if (x_n) is a normalized basic sequence, $\epsilon > 0$, and x^* is an element of X^* so that $x^*(x_n) > \epsilon$ for each n , then (x_n) is equivalent to (e_n^*) . The sequence $(\sigma_n) = (\sum_{i=1}^n e_i)$ in c_0 clearly demonstrates that

unconditionality (or some other property) has been omitted from the description of the basic sequence in the problem.

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