

## ROTATION NUMBERS IN THE INFINITE ANNULUS

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ABSTRACT. Using the notion of free transverse triangulation we prove that the rotation number of a given probability measure invariant by a homeomorphism of the open annulus depends continuously on the homeomorphism under some boundedness conditions.

### 0. NOTATION

We give the same name to the projections  $p_1 : (x, y) \mapsto x$  and  $p_2 : (x, y) \mapsto y$  defined on the infinite annulus  $\mathbf{A} = \mathbf{T}^1 \times \mathbf{R}$  or on the plane  $\mathbf{R}^2$ . We consider the projection

$$\begin{aligned}\pi : \mathbf{R}^2 &\rightarrow \mathbf{A} \\ (x, y) &\mapsto (x + \mathbf{Z}, y)\end{aligned}$$

of the universal covering space  $\mathbf{R}^2$  and the translation

$$\begin{aligned}T : \mathbf{R}^2 &\rightarrow \mathbf{R}^2 \\ (x, y) &\mapsto (x + 1, y).\end{aligned}$$

We will write  $\mathbf{A}_q = \mathbf{R}^2/T^q$  for the finite covering space of order  $q \geq 2$  of  $\mathbf{A}$  and  $\pi_q : \mathbf{A}_q \rightarrow \mathbf{A}$  for the projection.

We consider the set  $H(\mathbf{A})$  of homeomorphisms of  $\mathbf{A}$  homotopic to the identity and the set  $h(\mathbf{A})$  of lifts to  $\mathbf{R}^2$  of such maps. We will put on both sets the compact-open topology. If  $\mu$  is a Borelian probability measure on  $\mathbf{A}$ , the set  $H_\mu(\mathbf{A})$  of homeomorphisms  $F \in H(\mathbf{A})$  which preserve  $\mu$  is closed as is the set  $h_\mu(\mathbf{A})$  of lifts.

If  $E$  is a topological space, we will write respectively  $\bar{X}$ ,  $\text{int}(X)$ ,  $\partial X$  for the closure, the interior and the frontier of  $X \subset E$ .

### 1. INTRODUCTION

If  $F$  is a homeomorphism of the compact annulus  $\mathbf{T}^1 \times [0, 1]$  homotopic to the identity and  $f$  is a lift of  $F$  to  $\mathbf{R} \times [0, 1]$ , the *rotation number* of a point  $z \in \mathbf{T}^1 \times [0, 1]$  is defined when it exists by the relation

$$\rho(z) = \lim_{n \rightarrow \pm\infty} \frac{p_1 \circ f^n(\tilde{z})}{n}$$

where  $\tilde{z} \in \pi^{-1}(\{z\})$ .

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If  $F$  preserves a probability measure  $\mu$ , it is a direct consequence of the Birkhoff Ergodic Theorem that the rotation number is well defined  $\mu$ -a.e., that the measurable function  $\rho$  obtained in this way is integrable and that  $\int \rho d\mu = \int r d\mu$ , where the map  $r : \mathbf{A} \rightarrow \mathbf{R}$  is lifted to  $\mathbf{R} \times [0, 1]$  by  $p_1 \circ f - p_1$ . This integral is called the *rotation number* of  $\mu$ . The rotation number is well defined for a homeomorphism  $F$  of an abstract compact annulus if we choose a generator of the first homology group. Indeed, if  $G$  is a homeomorphism of  $\mathbf{T}^1 \times [0, 1]$  which induces the identity on the first homology group and  $g$  is a lift of  $G$  to  $\mathbf{R} \times [0, 1]$ , it is easy to see that a point  $z$  has a rotation number for  $f$  iff  $G(z)$  has a rotation number for  $g \circ f \circ g^{-1}$  and that these numbers are equal. Moreover, if  $\mu$  is an invariant measure on  $\mathbf{A}$ , the rotation number of  $\mu$  for  $f$  is equal to the rotation number of  $G_*(\mu)$  for  $g \circ f \circ g^{-1}$ . The notion of rotation number is very classical and was introduced in the case of a general compact manifold by S. Schwartzman [S] as an element of the first group of homology.

In the case of a homeomorphism  $F$  of the open annulus, the situation is more complicated. First of all, the existence of a rotation number for a point is not stable by conjugacy. Indeed, if  $z$  has a rotation number (in the sense given above) and if the sequence  $(u_n)_{n \geq 0}$  defined by  $u_n = p_2 \circ F^n(z)$  is unbounded, we can extract a strictly monotone unbounded subsequence  $(u_{n_k})_{k \geq 0}$ . If we consider the homeomorphism

$$\begin{aligned} g : \mathbf{R}^2 &\rightarrow \mathbf{R}^2 \\ (x, y) &\mapsto (x + a(y), y) \end{aligned}$$

where  $a : \mathbf{R} \rightarrow \mathbf{R}$  is continuous and satisfies

$$a(u_{n_k}) = -p_1 \circ f^{n_k}(\tilde{z}) + p_1(\tilde{z}) + kn_k, \quad \tilde{z} \in \pi^{-1}(\{z\}),$$

and the element  $G \in H(\mathbf{A})$  lifted by  $g$ , then the point  $G(z)$  has no rotation number for  $g \circ f \circ g^{-1}$ . A natural way to define the rotation number of a point in that case is to limit ourselves to the recurrent points of  $F$ . We will say that a recurrent point  $z$  has a rotation number  $\rho(z)$  if for every subsequence  $(F^{n_k}(z))_{k \geq 0}$  of  $(F^n(z))_{n \geq 0}$  which converges to  $z$  we have

$$\lim_{k \rightarrow +\infty} \frac{p_1 \circ f^{n_k}(\tilde{z})}{n_k} = \rho(z)$$

for every  $\tilde{z} \in \pi^{-1}(z)$  and if we have a similar result for the subsequences of  $(F^n(z))_{n \leq 0}$  converging to  $z$ . For this definition it becomes clear that this notion is stable by conjugacy.

If  $F$  preserves a probability measure  $\mu$  and if  $r$  is  $\mu$ -integrable, we will get the same situation as in the compact case:  $\mu$ -a.e. point has a rotation number (in the strong sense) and we will get the rotation number of the measure by integrating this function. It may happen that after a conjugacy the map  $r'$  lifted by  $p_1 \circ g \circ f \circ g^{-1} - p_1$  is no longer integrable. However, if it is the case, the rotation number of the measure  $G_*(\mu)$  for  $g \circ f \circ g^{-1}$  is equal to  $\int r' d\mu$ . Indeed  $\mu$ -a.e. point is recurrent for  $F$  and the rotation number in the weak sense is invariant by conjugacy and coincides with the rotation number in the strong sense. So it is natural to say that a measure  $\mu$  has a rotation number if  $\mu$ -a.e. recurrent point has a rotation number in the weak sense and if the measurable function  $\rho$  obtained is integrable: the rotation number will be  $\int \rho d\mu$ . We have the following immediate properties:

i) If  $f' = T^k \circ f$  is another lift of  $F$ , the rotation number of a point (or an invariant measure) for  $f'$  is obtained by adding  $k$  to the rotation number for  $f$ .

ii) Fix  $q \geq 2$  and consider the homeomorphism  $F_q$  of  $\mathbf{A}_q$  lifted by  $f$ . If  $z$  is a recurrent point of  $F_q$  and if its image in  $\mathbf{A}$ , which is a recurrent point of  $F$ , has a rotation number  $\rho$ , then  $z$  has a rotation number equal to  $\rho/q$  (we use the natural generator of  $H_1(\mathbf{A}_q, \mathbf{R})$  defined by the translation  $T^q$ ). So if an invariant measure  $\mu$  of  $F$  has a rotation number  $\rho$  and if  $\mu_q$  is the natural probability measure defined on  $\mathbf{A}_q$  by  $\mu$ , there exists a set of full measure of  $\mathbf{A}_q$  which consists of recurrent points of  $F_q$  whose image in  $\mathbf{A}$  has a rotation number. We deduce that  $\mu_q$  has a rotation number which is equal to  $\rho/q$ .

There are examples of probability measures without rotation number. The homeomorphism  $F \in H(\mathbf{A})$  lifted by

$$f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$$

$$(x, y) \mapsto (x + a(y), y)$$

preserves the volume form  $\frac{1}{\pi(1+y^2)} dx \wedge dy$  if  $a$  is  $C^1$ . If  $a$  is equal to  $(-1)^k k^2$  on any interval  $[k, k + 1/2]$ , the measure defined by the 2-form has no rotation number.

If  $z$  is a fixed point of  $F$ , there exists an integer  $p \in \mathbf{Z}$  such that  $f(z) = T^p(z)$ . This integer is the rotation number of the point  $z$  and the rotation number of the Dirac measure concentrated on  $z$ . In the previous example there were a lot of fixed points with rotation numbers arbitrarily large. In fact J. Franks [Fr2] proved that this is always the case when the measure has no rotation number and a support equal to  $\mathbf{A}$ . We will recall the proof of this result (Theorem 1), expressing it in terms of *free transverse triangulation* (notion introduced by M. Flucher [Fl]). This will permit us to show a continuity result (Theorem 2) for the rotation number of a measure. It is easy to see that it is optimal because the compact-open topology permits us to perturb a given map by changing it arbitrarily outside of a compact set. We denote by  $\text{Fix}(F)$  the set of fixed points of  $F$ , by  $R(f)$  the smallest interval of  $\mathbf{Z}$  which contains all the rotation numbers of the fixed points of  $F$  and by  $|R(f)|$  the cardinality of  $R(f)$ .

**Theorem 1.** *Let  $\mu$  be a probability measure,  $F$  an element of  $H_\mu(\mathbf{A})$  and  $f \in h_\mu(\mathbf{A})$  a lift of  $F$ . If  $R(f)$  is finite, then  $\mu$ -a.e. recurrent point of  $F$  has a rotation number. Moreover if the support of  $\mu$  is  $\mathbf{A}$ , the rotation numbers are bounded and the measure  $\mu$  has a rotation number.*

**Theorem 2.** *Let  $\mu$  be a probability measure whose support is  $\mathbf{A}$ . If  $(F_n)_{n \geq 0}$  is a sequence in  $H_\mu(\mathbf{A})$  converging to  $F \in H_\mu(\mathbf{A})$ , if  $(f_n)_{n \geq 0}$  is a sequence of lifts converging to a lift  $f$  of  $F$  and if the sequence  $(|R(f_n)|)_{n \geq 0}$  is uniformly bounded, then  $R(f)$  is finite and the sequence of rotation numbers of  $\mu$  for  $f_n$  converges to the rotation number of  $\mu$  for  $f$ .*

## 2. FREE DISK CHAINS

A *free disk* of a homeomorphism  $F$  of a surface  $S$  is a set homeomorphic to an open disk which doesn't meet its image by  $F$ . A *free disk chain*  $C$  is given by a family  $(U_i)_{1 \leq i \leq n}$  of free disks and a family  $(p_i)_{1 \leq i \leq n-1}$  of positive integers such that:

- i) two disks  $U_i$  and  $U_j$  are equal or disjoint;
- ii) for every  $i \in \{1, \dots, n - 1\}$  the set  $F^{p_i}(U_i)$  meets  $U_{i+1}$ .

We say that the chain *goes from*  $U_1$  to  $U_n$  and will define the *length of the chain*  $C$  to be the integer  $l(C) = \sum_{i=1}^{n-1} p_i$ ; we say that the chain is *closed* if  $U_1 = U_n$ . If  $\mathcal{D}$  is a family of disjoint free disks of  $F$ , a  $\mathcal{D}$ -*chain* is a chain of disks chosen in  $\mathcal{D}$ . We say that  $\mathcal{D}$  is *transitive* if for every pair of disks  $U_1$  and  $U_2$  in  $\mathcal{D}$  we can find a  $\mathcal{D}$ -chain from  $U_1$  to  $U_2$ .

We have the following fundamental result, due to J. Franks [Fr1], which is a consequence of the Brouwer Lemma on Arcs of Translation.

**Proposition 1.** *A fixed point free and orientation preserving homeomorphism of  $\mathbf{R}^2$  has no closed free disk chain.*

Let  $F$  be a fixed point free homeomorphism of a surface  $S$  and consider a triangulation of  $S$ . We can construct a subtriangulation sufficiently small such that the closure of any 2-cell is free. Such a triangulation will be called a *free triangulation*. If we perturb slightly a free triangulation we still get a free triangulation. We can do it in such a way that we have the following property: if  $\gamma$  and  $\gamma'$  are two edges such that  $F(\gamma)$  (resp.  $F^{-1}(\gamma)$ ) meets  $\gamma'$ , then  $F(\gamma)$  (resp.  $F^{-1}(\gamma)$ ) meets the two 2-cells adjacent to  $\gamma'$ . Such a triangulation will be called a *free transverse triangulation*. If  $\mu$  is a probability measure, we can perturb slightly a free transverse triangulation to get another free transverse triangulation such that each edge of the triangulation is a null-set. Such a triangulation will be called a *full free transverse triangulation* (for  $\mu$ ). We will denote by the same letter a triangulation  $\mathcal{D}$  and the family of 2-cells of this triangulation. We have the following result (see [F1] or [LS] for more details).

**Proposition 2.** *If  $F$  is a fixed point free homeomorphism of a connected surface  $S$  which preserves a probability measure  $\mu$  whose support is  $S$ , then the family of 2-cells of a free transverse triangulation is transitive.*

*Proof.* Let  $\mathcal{D} = (V_\alpha)_{\alpha \in A}$  be the family of 2-cells of this triangulation. Fix  $\alpha_0 \in A$  and consider the set  $A'$  of elements  $\alpha \in A$  such that there exists a  $\mathcal{D}$ -chain from  $V_{\alpha_0}$  to  $V_\alpha$ . It is easy to prove that the set  $W = \text{int}\left(\overline{\bigcup_{\alpha \in A'} V_\alpha}\right)$  is positively invariant and by the transversality condition that it satisfies  $F(\partial W) \subset W$ . From the fact that the support of the measure is  $S$  we deduce that  $\partial W = \emptyset$ , and from the connectedness of  $S$  we deduce that  $W = S$  and that  $A' = A$ .  $\square$

Consider  $F \in H(\mathbf{A})$ , a lift  $f \in h(\mathbf{A})$  and a family  $\mathcal{D}$  of disjoint free disks of  $F$ , and write  $\tilde{\mathcal{D}} = (V_\alpha)_{\alpha \in A}$  the lifted family (i.e. the family of connected components of the preimages by  $\pi$  of the disks) which consists of free disks of  $f$ . For  $\alpha \in A$  and  $k \in \mathbf{Z}$  we will write  $\beta = \alpha + k$  if  $V_\beta = T^k(V_\alpha)$ .

For every  $\alpha \in A$ , the set  $Z(\alpha)$  of integers  $k \in \mathbf{Z}$  such that there exists a  $\tilde{\mathcal{D}}$ -chain of  $f$  from  $V_\alpha$  to  $V_{\alpha+k}$  is stable by addition. Indeed, if  $k$  and  $k'$  belong to  $Z(\alpha)$  we can concatenate a  $\tilde{\mathcal{D}}$ -chain from  $V_\alpha$  to  $V_{\alpha+k}$  and a  $\tilde{\mathcal{D}}$ -chain, translated by  $T^k$ , from  $V_\alpha$  to  $V_{\alpha+k'}$  to get a  $\tilde{\mathcal{D}}$ -chain from  $V_\alpha$  to  $V_{\alpha+k+k'}$ . We write

$$Z(\mathcal{D}) = \bigcup_{\alpha \in A} Z(\alpha).$$

For every  $\tilde{\mathcal{D}}$ -chain  $C$  from  $V_\alpha$  to a translated disk  $V_{\alpha+k}$ , consider the rational number  $k/l(C)$ , where  $l(C)$  is the length of  $C$ . Write  $Q(\alpha)$  for the set of rational numbers obtained in this way. If we change the lift  $f$  by the lift  $T^s \circ f$  we change

the set  $Q(\alpha)$  by  $Q(\alpha) + s$ . We define

$$Q(\mathcal{D}) = \bigcup_{\alpha \in A} Q(\alpha).$$

If  $f$  is fixed point free, the set  $Z(\alpha)$  doesn't contain 0 by Proposition 1: we deduce that all the integers  $k \in Z(\alpha)$  have the same sign using the additive property. If  $F$  is fixed point free, for every  $k \in \mathbf{Z}$ , all the rational numbers in  $Q(\alpha) + k$  have the same sign: we deduce that there exists an integer  $m$  such that  $Q(\alpha) \subset ]m, m + 1[$ .

Suppose now that  $\mathcal{D}$  is transitive and fix  $\alpha_1$  and  $\alpha_2$  in  $A$ . By the transitivity of  $\mathcal{D}$  we know that there exist two integers  $l_1$  and  $l_2$ , a  $\tilde{\mathcal{D}}$ -chain from  $V_{\alpha_1}$  to  $V_{\alpha_2+l_2}$  and a  $\tilde{\mathcal{D}}$ -chain from  $V_{\alpha_2}$  to  $V_{\alpha_1+l_1}$ . Using concatenations and translations we deduce that for every  $n \geq 0$  and for every  $k_1 \in Z(\alpha_1)$  the number  $l_2 + nk_1 + l_1$  belongs to  $Z(\alpha_2)$ . If  $f$  is fixed point free we deduce that all the integers  $k \in Z(\mathcal{D})$  have the same sign, and if  $F$  is fixed point free that there exists an integer  $m$  such that  $Q(\mathcal{D}) \subset ]m, m + 1[$ .

**Proposition 3.** *Fix  $F \in H_\mu(\mathbf{A})$  and a lift  $f \in h_\mu(\mathbf{A})$ , where  $\mu$  is a probability measure whose support is  $\mathbf{A}$ . Consider a free triangulation  $\mathcal{D}$  of  $\mathbf{A} \setminus \text{Fix}(F)$  and a finite set  $X \subset Q(\mathcal{D})$ . If  $f' \in h_\mu(\mathbf{A})$  is sufficiently close to  $f$ , there exists a free transverse triangulation  $\mathcal{D}'$  of  $\mathbf{A} \setminus \text{Fix}(F')$ , where  $F' \in H_\mu(\mathbf{A})$  is lifted by  $f'$ , such that  $X \subset Q(\mathcal{D}')$ . Moreover if  $X$  contains a nonnegative number and a nonpositive number,  $f'$  has a fixed point.*

*Proof.* There exists a finite subset  $\mathcal{D}_1$  of  $\mathcal{D}$  such that  $X \subset Q(\mathcal{D}_1)$ . If  $f'$  is sufficiently close to  $f$ , the closure of every disk of  $\mathcal{D}_1$  is free (for  $F'$ ) and we have the same relation  $X \subset Q(\mathcal{D}_1)$  for  $f'$ . We can complete this finite family to construct a free triangulation of  $\mathbf{A} \setminus \text{Fix}(F')$ . If we perturb it slightly we will get a free transverse triangulation  $\mathcal{D}'$  of  $\mathbf{A} \setminus \text{Fix}(F')$  such that  $X \subset Q(\mathcal{D}')$ .

For any integer  $q \geq 1$ , denote by  $\text{Fix}_{\geq q} F$  the set of fixed points of  $F$  whose rotation number has an absolute value larger than  $q$ . Consider  $M > 0$  such that the disks of  $\mathcal{D}_1$  are contained in  $\mathbf{T}^1 \times ]-M, M[$  and such that this annulus meets its image by  $F$ . Find an integer  $q \geq 1$  which bounds strictly the rotation numbers of the fixed points of  $F$  which are in  $\mathbf{T}^1 \times [-M, M]$ . If  $f' \in h_\mu(\mathbf{A})$  is sufficiently close to  $f$  and if  $F' \in H_\mu(\mathbf{A})$  is lifted by  $f'$ , the set  $\mathbf{T}^1 \times ]-M, M[$  meets its image by  $F'$  and  $q$  bounds strictly the rotation numbers of the fixed points of  $F'$  which are in  $\mathbf{T}^1 \times [-M, M]$ . If  $f'$  has no fixed point, the disks of the family  $\mathcal{D}_1^q$  obtained by lifting  $\mathcal{D}_1$  to  $\mathbf{A}_q$  are contained in the same connected component  $W'$  of  $\mathbf{A}_q \setminus \pi_q^{-1}(\text{Fix}(F'_{\geq q}))$  and this component is invariant by the lift of  $F'$ . Moreover the set  $Q(\mathcal{D}_1^q)$  contains a nonnegative and a nonpositive number. Completing  $\mathcal{D}_1^q$  and perturbing it we can find a free transverse triangulation  $\mathcal{D}'$  of  $W'$  such that  $Q(\mathcal{D}')$  has the same property. Since the triangulation  $\mathcal{D}'$  is transitive by Proposition 2, the map  $f'$  must have a fixed point. □

**Corollary 1.** *Fix  $F \in H_\mu(\mathbf{A})$  and a lift  $f \in h_\mu(\mathbf{A})$ , where  $\mu$  is a probability measure whose support is  $\mathbf{A}$ . If there exist two recurrent points  $z_1$  and  $z_2$  with rotation numbers  $\rho(z_1)$  and  $\rho(z_2)$  such that  $\rho(z_1) < 0 < \rho(z_2)$ , then  $f$  has a fixed point. Moreover every  $f' \in h_\mu(\mathbf{A})$  sufficiently close to  $f$  has a fixed point.*

*Proof.* If these points are not fixed by  $F$  we can find a free transverse triangulation  $\mathcal{D}$  of  $\mathbf{A} \setminus \text{Fix}(F)$  such that  $z_1$  and  $z_2$  belong to 2-cells. The hypothesis tells us that  $Q(\mathcal{D})$  contains a strictly negative and a strictly positive number and

we can apply Proposition 3. If at least one of them is fixed we can take an integer  $q > \max(|\rho(z_1)|, |\rho(z_2)|)$  and consider a free transverse triangulation  $\mathcal{D}$  of  $\mathbf{A}_q \setminus \pi_q^{-1}(\text{Fix}(F_{\geq q}))$  such that every preimage of  $z_1$  or  $z_2$  belongs to a disk of  $\mathcal{D}$ . We will get the same situation as above.  $\square$

**Corollary 2.** *Let  $(F_n)_{n \geq 0}$  be a sequence in  $H_\mu(\mathbf{A})$  which converges to  $F \in H_\mu(\mathbf{A})$  and  $(f_n)_{n \geq 0}$  a sequence of lifts in  $h_\mu(\mathbf{A})$  which converges to  $f \in h_\mu(\mathbf{A})$ , where  $\mu$  is a probability measure whose support is  $\mathbf{A}$ . If the integers  $|R(f_n)|$  are uniformly bounded, then  $R(f)$  is finite and there exists a uniform bound to the rotation numbers of the recurrent points of the  $F_n$ .*

*Proof.* Corollary 1, applied to a translation of  $f$  or  $f_n$ , tells us that  $R(f)$  and  $R(f_n)$  are intervals of  $\mathbf{Z}$ . Moreover, if  $\rho^- \in R(f)$  and  $\rho^+ \in R(f)$  satisfy  $\rho^+ - \rho^- \geq 2$ , then every integer  $\rho \in [\rho^- + 1, \rho^+ - 1]$  belongs to  $R(f_n)$  for  $n$  large enough. We deduce that  $R(f)$  is finite and that the set of rotation numbers of recurrent points of  $F$  is bounded. For the same reasons the set of rotation numbers of the recurrent points of each  $F_n$  is bounded. We want to find a uniform bound; it is sufficient to find it for  $n$  large enough.

Define  $M = \max_n |R(f_n)| \geq 0$  and consider  $q = M + 2$ . We can find an interval  $I \in \mathbf{Z}$  of cardinality  $M + 1$  such that if  $k \in I$ , the map  $F'$  of  $\mathbf{A}_q$  lifted by  $f' = T^{-k} \circ f$  is different from the identity. If  $F$  is different from the identity, any choice is good; if  $F$  is the identity, there exists  $p$  such that  $T^{-p} \circ f$  is the identity and we choose  $I$  disjoint from  $p + q\mathbf{Z}$ . For every  $n \geq 0$ , there exists  $k \in I$  such that  $R(f_n)$  doesn't meet  $I$ . That means that the homeomorphism  $F'_n$  of  $\mathbf{A}_q$  lifted by  $f'_n = T^{-k} \circ f_n$  is fixed point free. To prove the corollary it is sufficient to fix  $k \in I$  and to show that the rotation numbers of the recurrent points of  $F_n$  are uniformly bounded if  $F'_n$  is fixed point free. Consider a free transverse triangulation  $\mathcal{D}$  of  $\mathbf{A}_q \setminus \text{Fix}(F')$  and the set  $\mathcal{Q}(\mathcal{D})$  defined for the lift  $f'$  or  $F'$ , fix a number  $\rho \in \mathcal{Q}(\mathcal{D})$  and take a finite sub-family  $\mathcal{D}_1$  such that  $\rho \in \mathcal{Q}(\mathcal{D}_1)$ . If  $n$  is large enough, then  $\mathcal{D}_1$  is a family of disks whose closure is free for  $F'_n$ . If  $z$  is a recurrent point of  $F_n$  of rotation number  $\rho'$ , then by extending the family  $\mathcal{D}_1$  and perturbing it we can construct a free transverse triangulation  $\mathcal{D}$  of  $\mathbf{A}_q$  for  $F'_n$  such that  $\rho \in \mathcal{Q}(\mathcal{D})$  and such that every preimage of  $z$  in  $\mathbf{A}_q$  belongs to a disk of  $\mathcal{D}$ . We deduce that we can find in  $\mathcal{Q}(\mathcal{D})$  rational numbers arbitrarily close to  $\rho'/q$ . This implies that  $\rho'/q \in [\rho - 1, \rho + 1]$  because  $F'_n$  is fixed point free.  $\square$

### 3. PROOFS OF THE THEOREMS

*Proof of Theorem 1.* We consider a probability measure  $\mu$  on  $\mathbf{A}$ , a homeomorphism  $F \in H_\mu(\mathbf{A})$  and a lift  $f \in h_\mu(\mathbf{A})$ . We suppose that the set  $R(f)$  is bounded. Every fixed point of  $F$  has a rotation number and the set of nonfixed points of  $F$  can be covered by free disks. So it is sufficient to prove that if  $U$  is a free disk of  $F$  such that  $\mu(U) > 0$ , then  $\mu$ -a.e. recurrent point of  $F$  in  $U$  has a rotation number.

We denote by  $\Phi$  the first return map defined a.e. on  $U$  and by  $\nu$  the time of first return. The measurable function  $\nu$  is integrable on  $U$  and we have

$$\int_U \nu d\mu = \mu \left( \bigcup_{k \geq 0} F^k(U) \right).$$

Indeed, we have the following measurable partitions (modulo sets of measure zero):

$$U = \bigsqcup_{i \geq 1} U_i \quad \text{and} \quad \bigcup_{k \geq 0} F^k(U) = \bigsqcup_{i \geq 1} \bigsqcup_{0 \leq j \leq i-1} F^j(U_i) ,$$

where  $U_i = \nu^{-1}(\{i\})$ ; therefore

$$\mu \left( \bigcup_{k \geq 0} F^k(U) \right) = \sum_{i \geq 1} \sum_{0 \leq j \leq i-1} \mu(U_i) = \sum_{i \geq 1} i \mu(U_i) = \int_U \nu \, d\mu.$$

Fix a connected component  $\tilde{U}$  of  $\pi^{-1}(U)$  and consider  $\tau : U \rightarrow \mathbf{Z}$  such that

$$\tau(z) = k \quad \text{iff} \quad f^{\nu(z)}(\tilde{z}) \in T^k(\tilde{U}) ,$$

where  $\tilde{z}$  is the preimage of  $z$  contained in  $\tilde{U}$ .

Using the fact that the disk  $U$ , taken alone, defines a transitive family of free disks of  $F$  and using the properties of the sets  $Q(\alpha)$  stated in Section 2, we deduce that the function  $\tau/\nu$  is bounded. More precisely we recall that there exists a fixed point of  $f \circ T^{-k}$  for every integer  $k$  surrounded by two points in the image of  $\tau/\nu$ .

Since the function  $\nu$  is integrable, it is the same for  $\tau$  and we know by the Poincaré Recurrence Theorem and the Birkhoff Ergodic Theorem that almost every point of  $U$  satisfies the following properties:

- i)  $z$  is a recurrent point of  $\Phi$ ;
- ii)  $\lim_{n \rightarrow \pm\infty} \frac{\tau(z) + \dots + \tau(\Phi^{n-1}(z))}{n} = \tau^*(z)$  exists;
- ii)  $\lim_{n \rightarrow \pm\infty} \frac{\nu(z) + \dots + \nu(\Phi^{n-1}(z))}{n} = \nu^*(z)$  exists.

The recurrent points of  $F$  are exactly the recurrent points of  $\Phi$  because  $U$  is open. We deduce that almost every recurrent point of  $F$  in  $U$  has a rotation number which is

$$\lim_{n \rightarrow \pm\infty} \frac{\tau(z) + \dots + \tau(\Phi^{n-1}(z))}{\nu(z) + \dots + \nu(\Phi^{n-1}(z))} = \frac{\tau^*(z)}{\nu^*(z)}.$$

If the support of  $\mu$  is  $\mathbf{A}$  we know using the corollaries in Section 2 that the rotation numbers are uniformly bounded and that the measure has a rotation number. □

*Remark.* If the support of  $\mu$  is not  $\mathbf{A}$  the second part is obviously false. Suppose that  $f(x, k) = (x + k + 1/2, k)$  if  $k \in \mathbf{Z}$  and that  $F$  is fixed point free. One can construct easily a discrete invariant measure without rotation number.

*Proof of Theorem 2.* We consider a probability measure  $\mu$  on  $\mathbf{A}$ , a sequence  $(F_n)_{n \geq 0}$  in  $H_\mu(\mathbf{A})$  which converges to  $F$  and a sequence of lifts  $(f_n)_{n \geq 0}$  which converges to a lift  $f$  of  $F$ . We suppose that the integers  $|R(f_n)|$  are uniformly bounded. Corollary 2 tells us that  $R(f)$  is bounded and that the rotation numbers of recurrent points of the  $F_n$  are bounded independently of  $n$ . By Theorem 1 we can define the rotation number  $\rho_n$  of  $\mu$  for each  $f_n$ ,  $n \geq 0$ , and the rotation number  $\rho$  of  $\mu$  for  $f$ . We know that the sequence  $(\rho_n)_{n \geq 0}$  is bounded and want to prove that it converges to  $\rho$ . Consider an integer  $k \geq 1$  such that the rotation numbers of recurrent points of each  $F_n$  or  $F$  are contained in  $[-k, k]$ . Taking  $f'_n = T^{k+1} \circ f_n$  instead of  $f_n$  and  $\mathbf{A}_{2k+2}$  instead of  $\mathbf{A}$  we can suppose that every  $F_n$  (and  $F$ ) is fixed point free and that every  $\rho_n$  (and  $\rho$ ) belongs to  $]0, 1[$ . Using the results of Section 2 we deduce that for any free transverse triangulation  $\mathcal{D}$  of  $\mathbf{A}$  for  $F_n$  or  $F$ , we have  $Q(\mathcal{D}) \subset ]0, 1[$ .

Indeed the set  $Q(\mathcal{D})$  meets  $]0, 1[$  because the rotation number of any recurrent point belongs to this interval and cannot meet  $] - \infty, 0]$  or  $[1, \infty[$ , otherwise they would be a fixed point of  $F$  (or  $F_n$ ).

Consider a full free transverse triangulation  $\mathcal{D} = (V_\alpha)_{\alpha \in A}$  of  $F$  such that every disk has a diameter  $< 1$ . Then a finite subfamily  $\mathcal{D}_1 = (V_\alpha)_{\alpha \in A_1}$  of  $\mathcal{D}$  such that the measure of  $W = \text{int}(\overline{\bigcup_{\alpha \in A_1} V_\alpha})$  is larger than  $1 - \varepsilon$ . By the transitivity of  $\mathcal{D}$  we know that there exists a larger finite subfamily  $\mathcal{D}_2 = (V_\alpha)_{\alpha \in A_2}$  and  $M_1 > 0$  such that for every  $\alpha$  and  $\alpha'$  in  $A_1$  there exists a  $\mathcal{D}_2$ -chain from  $V_\alpha$  to  $V_{\alpha'}$  of length  $\leq M_1$ .

Let  $\tilde{\mathcal{D}} = (V_\alpha)_{\alpha \in \tilde{A}}$ ,  $\tilde{\mathcal{D}}_1 = (V_\alpha)_{\alpha \in \tilde{A}_1}$  and  $\tilde{\mathcal{D}}_2 = (V_\alpha)_{\alpha \in \tilde{A}_2}$  be the lifted families. There exists  $M_2 > 0$  such that for every  $\alpha \in \tilde{A}_1$  and  $\alpha' \in \tilde{A}_1$  there is an integer  $k \in \mathbf{Z}$  and a  $\tilde{\mathcal{D}}_2$ -chain from  $V_\alpha$  to  $V_{\alpha'+k}$  of length  $\leq M_1$  and such that  $V_\alpha$  and  $V_{\alpha'+k}$  can be projected by  $p_1$  on an interval of length  $\leq M_2$ .

Consider the first return map  $\Phi$  defined  $\mu$ -a.e. on  $W$ , the time of first return  $\nu$ , and the function

$$\lambda : z \mapsto \sum_{i=0}^{\nu(z)-1} r(F^i(z)) = p_1 \circ f^{\nu(z)}(\tilde{z}) - p_1(\tilde{z}) \text{ , } \tilde{z} \in \pi^{-1}(\{z\}).$$

If  $\Phi(z)$  is defined, using Baire's theorem, we can find  $z'$  close to  $z$  such that

- every point  $F^i(z')$ ,  $0 \leq i \leq \nu(z)$ , belongs to  $V_{\alpha_i}$ ,  $\alpha_i \in A$ ;
- $\alpha_0$  and  $\alpha_{\nu(z)}$  are in  $A_1$ ;
- $z \in \overline{V}_{\alpha_0}$  and  $F(z) \in \overline{V}_{\alpha_{\nu(z)}}$ .

If  $\tilde{z} \in \pi^{-1}(\{z\})$ , we can lift the chain  $(V_{\alpha_i})_{0 \leq i \leq \nu(z)}$  to get a  $\tilde{\mathcal{D}}$ -chain from  $V_\alpha$  to  $V_{\alpha'}$  where

- $\alpha$  and  $\alpha'$  are in  $\tilde{A}_1$ ;
- $\tilde{z} \in \overline{V}_\alpha$  and  $f(\tilde{z}) \in \overline{V}_{\alpha'}$ .

We can add to this chain a  $\tilde{\mathcal{D}}_2$ -chain from  $V_{\alpha'}$  to a translated  $V_{\alpha+k}$  to get a  $\tilde{\mathcal{D}}$ -chain from  $V_\alpha$  to  $V_{\alpha+k}$  of length between  $\nu(z)$  and  $\nu(z) + M_1$ . The fact that each disk has a diameter  $< 1$  and that  $V_\alpha$  and  $V_{\alpha+k}$  are projected in an interval of length  $\leq M_2$  implies that  $|k - \lambda(z)| \leq M_2 + 2$ . The fact that  $Q(\mathcal{D}) \subset ]0, 1[$  implies that  $0 < k < \nu(z) + M_1$ . Therefore

$$\begin{aligned} -\nu(z)(M_1 + M_2 + 3) &< -M_2 - 2 < \lambda(z) \\ &< \nu(z) + M_1 + M_2 + 2 \leq \nu(z)(M_1 + M_2 + 3) \end{aligned}$$

and

$$\left| \frac{\lambda(z)}{\nu(z)} \right| \leq M_1 + M_2 + 3 = M_3.$$

As in the proof of Theorem 1,  $\nu$  and  $\lambda$  are integrable. From Theorem 1 we know that for  $\mu$ -a.e. point  $z \in W$ :

- i)  $z$  is a recurrent point of  $\Phi$ ;
- ii)  $\lim_{n \rightarrow \pm\infty} \frac{\nu(z) + \dots + \nu(\Phi^{n-1}(z))}{n} = \nu^*(z)$  exists;
- iii)  $\lim_{n \rightarrow \pm\infty} \frac{\lambda(z) + \dots + \lambda(\Phi^{n-1}(z))}{n} = \lambda^*(z)$  exists;
- iv) the rotation number  $\rho(z)$  exists and we have  $\rho(z) = \frac{\lambda^*(z)}{\nu^*(z)}$ .

Using the fact that  $\rho$  is  $F$ -invariant, that  $\mu$  is  $\Phi$ -invariant and the Birkhoff Ergodic Theorem we obtain

$$\int_{\bigcup_{k \geq 0} F^k(W)} \rho d\mu = \int_W \nu \rho d\mu = \int_W \nu^* \rho d\mu = \int_W \lambda^* d\mu = \int_W \lambda d\mu.$$

The first equality is obtained by arguing as in the proof of Theorem 1.

For  $n$  large enough  $\mathcal{D}_2$  is a family of free disks of  $f_n$  and we have the same situation: we define  $\Phi_n, \nu_n$  and  $\lambda_n$  on  $W$  with the same inequalities, we define  $\rho_n$  on  $\mathbf{A}, \nu_n^*$  and  $\lambda_n^*$  on  $W$  with the same equality.

We write

$$\begin{aligned} \left| \int_{\bigcup_{k \geq 0} F^k(W)} \rho d\mu - \int_{\bigcup_{k \geq 0} F_n^k(W)} \rho_n d\mu \right| &= \int_W (\lambda - \lambda_n) d\mu \\ &= \int_W \left( \frac{\lambda}{\nu} - \frac{\lambda_n}{\nu_n} \right) \nu d\mu + \int_W \frac{\lambda_n}{\nu_n} (\nu - \nu_n) d\mu. \end{aligned}$$

First we obtain

$$\left| \int_W \frac{\lambda_n}{\nu_n} (\nu - \nu_n) d\mu \right| \leq \int_W M_3 |\nu' - \nu'_n| d\mu \leq M_3 \int_W (\nu' + \nu'_n) d\mu \leq 2M_3 \varepsilon,$$

where  $\nu' = \nu - 1$  and  $\nu'_n = \nu'_n - 1$ . Then we write

$$\left| \frac{\lambda}{\nu} - \frac{\lambda_n}{\nu_n} \right| \nu \leq 2M_3 \nu.$$

Using the fact that the boundary of  $W$  has measure 0 (because the triangulation is full)) we deduce that the function  $\frac{\lambda}{\nu} - \frac{\lambda_n}{\nu_n}$  tends to zero a.e. when  $n \rightarrow +\infty$ . So by the Lebesgue Dominated Convergence Theorem, for  $n$  large enough we have

$$\left| \int_W \left( \frac{\lambda}{\nu} - \frac{\lambda_n}{\nu_n} \right) \nu, d\mu \right| \leq \varepsilon.$$

Using the fact that all the rotation numbers belong to  $[0, 1]$  we deduce that

$$\left| \int_{\mathbf{A}} \rho d\mu - \int_{\bigcup_{k \geq 0} F^k(W)} \rho d\mu \right| \leq \varepsilon$$

and

$$\left| \int_{\mathbf{A}} \rho_n d\mu - \int_{\bigcup_{k \geq 0} F_n^k(W)} \rho_n d\mu \right| \leq \varepsilon.$$

We have obtained

$$|\rho_n - \rho| \leq (2M_3 + 3)\varepsilon$$

for  $n$  large enough, and the sequence  $(\rho_n)_{n \geq 0}$  converges to  $\rho$ . □

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