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WEAK INFINITE PRODUCTS OF BLASCHKE PRODUCTS

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ABSTRACT. We study weak infinite products for sequences of Blaschke products. Using properties of these functions, F_σ-subsets of zero sets of functions in $H^\infty + C$ are studied. An affirmative answer is given to a problem on prime ideals of $H^\infty + C$ posed by Gorkin and Mortini.

1. INTRODUCTION

Let D be the open unit disk in the complex plane. Let L^∞ and H^∞ be the usual Banach algebras on the unit circle ∂D. A closed subalgebra B of L^∞ containing H^∞ properly is called a Douglas algebra. We denote by $M(B)$ the maximal ideal space of B. We may consider that $M(B)$ is a compact subset of $M(H^\infty)$. Also we may consider that $D \subset M(H^\infty)$, and the corona theorem says that D is dense in $M(H^\infty)$ [1]. The smallest Douglas algebra is $H^\infty + C$, where C is the space of continuous functions on ∂D, and it is known that $M(H^\infty + C) = M(H^\infty) \setminus D$; see [3]. For a subset E of $M(H^\infty + C)$, we denote by $\overline{E}$ and $int\, E$ the closure and the interior of E, respectively. For a function f in $H^\infty + C$, we put

$$Z(f) = \{\zeta \in M(H^\infty + C); f(\zeta) = 0\} \text{ and } \{|f| < 1\} = \{\zeta \in M(H^\infty + C); |f(\zeta)| < 1\}.$$

Similary, we define the set $\{f \neq 0\}$ in the space $M(H^\infty + C)$.

For a sequence $\{z_j\}_j$ in D such that $\sum_{j=1}^\infty (1 - |z_j|) < \infty$, there is the associated Blaschke product b given by

$$b(z) = \prod_{j=1}^\infty \frac{-\overline{z}_j}{|z_j|} \frac{z - z_j}{1 - \overline{z}_j z}, \quad z \in D.$$

It is known that $|b| = 1$ on $M(L^\infty)$; see [10]. We denote by $\mathcal{P}(b)$ the set of sequences of positive integers $p = (p_1, p_2, \dots)$ such that $\sum_{j=1}^\infty p_j(1 - |z_j|) < \infty$ and $p_j \to \infty$ as $j \to \infty$. Then we have a family of Blaschke products

$$b^p(z) = \prod_{j=1}^\infty \Big(\frac{-\overline{z}_j}{|z_j|} \frac{z - z_j}{1 - \overline{z}_j z} \Big)^{p_j}, \quad p \in \mathcal{P}(b).$$

In [13], these Blaschke products were studied and called weak infinite powers of b. In this paper, we consider a slight generalization of it.

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Let $B = (b_1, b_2, \dots)$ be a sequence of infinite Blaschke products and $\{z_{k,j}\}_j$ be the zeros of b_k. Then there is a sequence of positive integers $N = (n_1, n_2, \dots)$ such that

$$\sum_{k=1}^{\infty} \sum_{j=n_k}^{\infty} (1 - |z_{k,j}|) < \infty.$$

We denote by $\mathcal{N}(B)$ the set of such sequences N. We put

$$B_N = \prod_{k=1}^{\infty} \prod_{j=n_k}^{\infty} \frac{-\overline{z}_{k,j}}{|z_{k,j}|} \frac{z - z_{k,j}}{1 - \overline{z}_{k,j} z}, \quad N \in \mathcal{N}(B),$$

and we call B_N a weak infinite product of Blaschke products $\{b_k\}_k$. In this paper, using them we study topological properties of $Z(f), f \in H^\infty + C$, and apply them to study prime ideals of $H^\infty + C$.

In Theorem 2.1, we prove that if E is a G_δ-subset of $M(H^\infty + C)$ such that $\overline{\bigcup_{k=1}^\infty \{|b_k| < 1\}} \subset E$, then there exists $N \in \mathcal{N}(B)$ such that $\{|B_N| < 1\} \subset E$. Applying Theorem 2.1, we get similar results obtained in [13]. In Theorem 3.1, we prove that if $f \in H^\infty + C, f \neq 0$, and E is an F_σ-subset of $M(H^\infty + C)$ satisfying $E \subset int\, Z(f)$, then $\overline{E} \subset int\, Z(f)$.

Applying Theorem 3.1, we solve a problem posed by Gorkin and Mortini [7, Q4]. The study of the structure of ideals in H^∞ and $H^\infty + C$ is very attractive and many problems remain open (see [5, 6]). For a point $x \in M(H^\infty + C)$, we have two typical ideals. One is the maximal ideal $\{f \in H^\infty + C; f(x) = 0\}$ and the other is the ideal $J(x)$ which consists of functions in $H^\infty + C$ which vanish in a neighborhood of x. To investigate the structure of $H^\infty + C$, it is important to study the behavior of functions in $H^\infty + C$ in neighborhoods of x. In [7], the authors studied these ideals and posed the problem whether $J(x)$ is a prime ideal of $H^\infty + C$ or not. In Theorem 4.1, we prove that $J(x)$ is a prime ideal of $H^\infty + C$ for every $x \in M(H^\infty + C)$. This theorem will give some light on the study of prime ideals in H^∞ and $H^\infty + C$.

2. Weak infinite products

Let $B = (b_1, b_2, \dots)$ be a sequence of infinite Blaschke products. For $N \in \mathcal{N}(B), N = (n_1, n_2, \dots)$, and a positive integer k, put

$$b_{k,n_k}(z) = \prod_{j=n_k}^{\infty} \frac{-\overline{z}_{k,j}}{|z_{k,j}|} \frac{z - z_{k,j}}{1 - \overline{z}_{k,j} z}, \quad z \in D.$$

Then we have $B_N = \prod_{k=1}^\infty b_{k,n_k}$. Put

$$\tilde{B} = (b_1, b_1, b_2, b_1, b_2, b_3, b_1, b_2, b_3, b_4, b_1, \dots).$$

The sequence $\tilde{B}$ is called the associated sequence of B, and in the sequence $\tilde{B}$ each b_k appears infinitely many times. By these definitions, it is not difficult to see the following.

Lemma 2.1. *For each $N \in \mathcal{N}(\tilde{B})$, there exist $M \in \mathcal{N}(B), M = (m_1, m_2, \dots)$, and $p_k \in \mathcal{P}(b_{k,m_k})$ such that $\tilde{B}_N = \prod_{k=1}^\infty b_{k,m_k}^{p_k}$.*

For a Blaschke product b, we denote by $S(b)$ the closed subset of ∂D on which b cannot be extended analytically. For a subset $E \subset L^\infty$, $H^\infty[E]$ is denoted by the Douglas algebra generated by H^∞ and functions in E. In [13, Theorem 3.1], the author proved that $L^\infty = H^\infty[\overline{b^p}; p \in \mathcal{P}(b)]$ if and only if $S(b) = \partial D$, where the bar denotes the complex conjugate.

Proposition 2.1. *Let $B = (b_1, b_2, \dots)$ be a sequence of infinite Blaschke products and $\tilde{B}$ the associated sequence of B. If $\bigcup_{k=1}^\infty S(b_k)$ is dense in ∂D, then $L^\infty = H^\infty[\overline{\tilde{B}_N}; N \in \mathcal{N}(\tilde{B})]$.*

Proof. Fix $N_0 \in \mathcal{N}(\tilde{B})$ and put $b = \tilde{B}_{N_0}$. By Lemma 2.1, there exist $M \in \mathcal{N}(B), M = (m_1, m_2, \dots)$, and $p_k \in \mathcal{P}(b_{k,m_k})$ such that $b = \prod_{k=1}^\infty b_{k,m_k}^{p_k}$. Let $p \in \mathcal{P}(b)$. Then it is easy to see the existence of $N_1 \in \mathcal{N}(\tilde{B})$ such that $\tilde{B}_{N_1} = b^p$. Hence

$$\{b^p; p \in \mathcal{P}(b)\} \subset \{\tilde{B}_N; N \in \mathcal{N}(\tilde{B})\}. \tag{2.1}$$

Since $\bigcup_{k=1}^\infty S(b_k) = \bigcup_{k=1}^\infty S(b_{k,m_k}) \subset S(b)$, by our assumption we have $S(b) = \partial D$. By the result mentioned above, we have $L^\infty = H^\infty[\overline{b^p}; p \in \mathcal{P}(b)]$. Hence by (2.1), $L^\infty = H^\infty[\overline{\tilde{B}_N}; N \in \mathcal{N}(\tilde{B})\}$.

Theorem 2.1. *Let $B = (b_1, b_2, \dots)$ be a sequence of infinite Blaschke products. Let E be a G_δ-subset of $M(H^\infty + C)$ such that $\overline{\bigcup_{k=1}^\infty \{|b_k| < 1\}} \subset E$. Then there exists $N \in \mathcal{N}(B)$ such that $\{|B_N| < 1\} \subset E$.*

Proof. The idea of the proof is the same as the one of [13, Theorem 2.4]. For the reader's convenience, we give the proof.

Since E is a G_δ-set, there is a sequence of closed subsets $\{K_n\}_n$ of $M(H^\infty + C)$ such that

$$M(H^\infty + C) \setminus E = \bigcup_{n=1}^\infty K_n. \tag{2.2}$$

By our assumption, there is an open subset V_n of $M(H^\infty)$ such that

$$K_n \subset V_n \quad \text{and} \quad \overline{\bigcup_{k=1}^\infty \{|b_k| < 1\}} \cap \overline{V}_n = \emptyset. \tag{2.3}$$

By the corona theorem,

$$\overline{V}_n \setminus D = \overline{V_n \cap D} \setminus D. \tag{2.4}$$

By (2.3),

$$|b_k| = 1 \quad \text{on } \overline{V}_n \setminus D \text{ for every } k \text{ and } n. \tag{2.5}$$

Let $\{\varepsilon_k\}_k$ be a sequence of positive numbers with $0 < \varepsilon_k < 1$ such that

$$\prod_{k=1}^\infty \varepsilon_k > 0. \tag{2.6}$$

By (2.4) and (2.5), for each positive integer k there is a sufficiently large positive integer n_k such that

$$|b_{k,n_k}| > \varepsilon_k \quad \text{on } V_n \cap D \text{ for every } n, 1 \le n \le k. \tag{2.7}$$

Moreover, we may assume that $\sum_{k=1}^{\infty}\sum_{j=n_k}^{\infty}(1-|z_{k,j}|)<\infty$. Put $N=(n_1,n_2,\dots)$. Then $N\in\mathcal{N}(B)$ and we have a Blaschke product $B_N=\prod_{k=1}^{\infty}b_{k,n_k}$. Fix n arbitrary. By (2.7), for every $i\ge n$ we have

$$|B_N|\ge\Big(\prod_{k=1}^{i-1}|b_{k,n_k}|\Big)\prod_{k=i}^{\infty}\varepsilon_k \quad \text{on } V_n\cap D.$$

Then by (2.4) and (2.5), $|B_N|\ge\prod_{k=i}^{\infty}\varepsilon_k$ on $\overline{V}_n\setminus D$ for every $i\ge n$. Hence by (2.3) and (2.6), $|B_N|=1$ on K_n. Thus by (2.2), $|B_N|=1$ on $M(H^\infty+C)\setminus E$. Hence $\{|B_N|<1\}\subset E$.

Corollary 2.1. *Let $B=(b_1,b_2,\dots)$ be a sequence of infinite Blaschke products. Then*

$$\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}=\bigcap\big\{\overline{\{|B_N|<1\}};N\in\mathcal{N}(B)\big\}.$$

Proof. It is easy to see that $\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}\subset\overline{\{|B_N|<1\}}$ for every $N\in\mathcal{N}(B)$. Let $x\in M(H^\infty+C)\setminus\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}$. Take an open subset V of $M(H^\infty+C)$ such that $x\in V$ and $\overline{V}\cap\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}=\emptyset$. Then by Theorem 2.1, there exists $N\in\mathcal{N}(B)$ such that $|B_N|=1$ on $\overline{V}$. Hence $x\notin\overline{\{|B_N|<1\}}$.

Corollary 2.2. *Let $B=(b_1,b_2,\dots)$ be a sequence of infinite Blaschke products and $\tilde{B}$ the associated sequence of B. Let E be a G_δ-subset of $M(H^\infty+C)$ such that $\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}\subset E$. Then there is $N\in\mathcal{N}(\tilde{B})$ such that*

$$\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}\subset Z(\tilde{B}_N)\subset\{|\tilde{B}_N|<1\}\subset E.$$

Proof. Put $\tilde{B}=(c_1,c_2,\dots)$. Then by our assumption, $\overline{\bigcup_{k=1}^{\infty}\{|c_k|<1\}}\subset E$. By Theorem 2.1, there exists $N\in\mathcal{N}(\tilde{B})$ such that $\{|\tilde{B}_N|<1\}\subset E$. By Lemma 2.1, there exist $M\in\mathcal{N}(B), M=(m_1,m_2,\dots)$, and $p_k\in\mathcal{P}(b_{k,m_k})$ such that $\tilde{B}_N=\prod_{k=1}^{\infty}b_{k,m_k}^{p_k}$. By [13, Lemma 2.1], $\overline{\{|b_k|<1\}}=\overline{\{|b_{k,m_k}|<1\}}\subset Z(b_{k,m_k}^{p_b})$. Hence we get

$$\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}\subset Z(\tilde{B}_N)\subset\{|\tilde{B}_N|<1\}\subset E.$$

By Corollary 2.2, we have the following.

Corollary 2.3. *Let $B=(b_1,b_2,\dots)$ be a sequence of infinite Blaschke products and $\tilde{B}$ the associated sequence of B. Then*

$$\overline{\bigcup_{k=1}^{\infty}\{|b_k|<1\}}=\bigcap\{Z(\tilde{B}_N);N\in\mathcal{N}(\tilde{B})\}=\bigcap\big\{\{|\tilde{B}_N|<1\};N\in\mathcal{N}(\tilde{B})\big\}.$$

3. F_σ-sets

For a point $x\in M(H^\infty)$, there is a unique probability measure μ_x on $M(L^\infty)$ such that $f(x)=\int_{M(L^\infty)}f\,d\mu_x$ for every $f\in H^\infty$; see [10]. We denote by $\operatorname{supp}\mu_x$ the closed support set of μ_x. Let b be a Blaschke product. Then $|b(x)|<1$ if and only if b is not constant on $\operatorname{supp}\mu_x$. A sequence $\{z_n\}_n$ in D is called interpolating if for every sequence of bounded complex numbers $\{a_n\}_n$ there exists $h\in H^\infty$ such that $h(z_n) = a_n$ for every n. A Blaschke product b is called also interpolating if the sequence of zeros of b in D is interpolating. We denote by G the set of all points x in $M(H^\infty + C)$ such that $b(x) = 0$ for some interpolating Blaschke product b. The structure of G is well studied in [11]. For $x, y \in M(H^\infty)$, let $\rho(x, y) = \sup\{|f(y)|; f(x) = 0, \|f\| \le 1, f \in H^\infty\}$. Put $P(x) = \{\zeta \in M(H^\infty); \rho(x, \zeta) < 1\}$. Hoffman showed that $G = \bigcup\{P(x); P(x) \ne \{x\}, x \in M(H^\infty) \setminus D\}$, and if $P(x) \ne \{x\}$ there is a one-to-one continuous map L_x from D onto $P(x)$ such that $f \circ L_x \in H^\infty$ for every $f \in H^\infty$ and $L_x(0) = x$. Moreover if $f(x) = 0$, then we can define order of zero of f at x, $ord\,(f, x)$, as the order of zero of $f \circ L_x$ at 0 in D. When $P(x) = \{x\}$ and $f(x) = 0$, we put $ord\,(f, x) = \infty$. Hence $ord\,(f, x) = \infty$ if and only if $f = 0$ on $P(x)$. When $f \in J(x)$, we have $ord\,(f, x) = \infty$.

For a function $f \in L^\infty$, let

$$\tilde{f}(\zeta) = \int_{M(L^\infty)} f \, d\mu_\zeta \quad \text{for } \zeta \in M(H^\infty). \tag{3.1}$$

Then $\tilde{f}$ is a continuous function on $M(H^\infty)$; see [3, p. 375].

The following is the main theorem in this paper.

Theorem 3.1. *Let $f \in H^\infty + C, f \ne 0$ and $int\, Z(f) \ne \emptyset$. Let E be an F_σ-subset of $M(H^\infty + C)$ such that $E \subset int\, Z(f)$. Then $\overline{E} \subset int\, Z(f)$.*

To prove our theorem, we use the following lemmas.

Lemma 3.1. *Let E be an F_σ-subset of $M(H^\infty + C)$ such that $E \cap M(L^\infty) = \emptyset$. Then $\overline{E} \cap M(L^\infty) = \emptyset$.*

Proof. Since E is an F_σ-set, it is not difficult to see the existence of a sequence of Blaschke products $\{b_n\}_n$ such that $E \subset \bigcup_{n=1}^\infty \{|b_n| < 1\}$. Hence the assertion follows from [15].

Lemma 3.2. *Let $f \in H^\infty + C$ and b a product of finitely many interpolating Blaschke products. If $ord\,(f, x) = \infty$ for every $x \in Z(b)$, then $\{|b| < 1\} \subset Z(f)$.*

The proof follows from [8, 9].

Proof of Theorem 3.1. By our assumption,

$$E \subset int\, Z(f), \tag{3.2}$$

and let $\{K_n\}_n$ be a sequence of compact subsets of $M(H^\infty + C)$ such that

$$E = \bigcup_{n=1}^\infty K_n. \tag{3.3}$$

First, we prove the following case.

Case 1. $E \cap M(L^\infty) = \emptyset$.

Let n be a fixed positive integer. Take $\zeta \in K_n$ arbitrary. When $\zeta \in G$, by (3.2) and (3.3) there exists an interpolating Blaschke product q_ζ such that $q_\zeta(\zeta) = 0$ and $Z(q_\zeta) \subset int\, Z(f)$. When $\zeta \notin G$, by [4, Corollary 3.2] there exists $\xi \in G \cap int\, Z(f)$ such that

$$\operatorname{supp} \mu_\xi \subset \operatorname{supp} \mu_\zeta. \tag{3.4}$$

Take an interpolating Blaschke product q_ζ such that $q_\zeta(\xi) = 0$ and $Z(q_\zeta) \subset \mathit{int}\, Z(f)$. By (3.4), $|q_\zeta(\zeta)| < 1$. Hence $K_n \subset \bigcup_{\zeta \in K_n} \{|q_\zeta| < 1\}$. Since K_n is compact, there exist $\zeta_1, \zeta_2, \dots, \zeta_k$ in K_n such that

$$K_n \subset \Big\{ \prod_{i=1}^k |q_{\zeta_i}| < 1 \Big\} \quad \text{and} \quad Z\Big(\prod_{i=1}^k q_{\zeta_i} \Big) \subset \mathit{int}\, Z(f).$$

Put $b_n = \prod_{i=1}^k q_{\zeta_i}$. Then by the above,

$$K_n \subset \{|b_n| < 1\} \tag{3.5}$$

and

$$Z(b_n) \subset \mathit{int}\, Z(f). \tag{3.6}$$

We have $\mathit{ord}\,(f, \zeta) = \infty$ for every $\zeta \in \mathit{int}\, Z(f)$. Then by (3.6) and Lemma 3.2 we have $\{|b_n| < 1\} \subset Z(f)$. Since $Z(f)$ is a G_δ-set, we can apply Corollary 2.2. Put $B = (b_1, b_2, \dots)$, and let $\tilde{B}$ be the associated sequence of B. Then by Corollary 2.2, there exists $N \in \mathcal{N}(\tilde{B})$ such that

$$\overline{\bigcup_{n=1}^\infty \{|b_n| < 1\}} \subset Z(\tilde{B}_N) \subset \{|\tilde{B}_N| < 1\} \subset Z(f).$$

Hence by (3.3) and (3.5), $\overline{E} \subset Z(\tilde{B}_N) \subset \{|\tilde{B}_N| < 1\} \subset Z(f)$. Thus we get our assertion.

Next, we prove the following case.

Case 2. $E \cap M(L^\infty) \neq \emptyset$.

To prove our assertion, suppose that $\overline{E} \not\subset \mathit{int}\, Z(f)$. Take a point x_0 satisfying

$$x_0 \in \overline{E} \quad \text{and} \quad x_0 \notin \mathit{int}\, Z(f). \tag{3.7}$$

Then two cases occur: $x_0 \notin M(L^\infty)$ and $x_0 \in M(L^\infty)$.

First suppose that $x_0 \notin M(L^\infty)$. By Newman's theorem [14], there is a Blaschke product φ such that $\varphi(x_0) = 0$. By (3.7), we have

$$x_0 \in \overline{E \cap \{|\varphi| < 1\}}. \tag{3.8}$$

Since E and $\{|\varphi| < 1\}$ are F_σ-sets, $E \cap \{|\varphi| < 1\}$ is an F_σ-set. Since

$$(E \cap \{|\varphi| < 1\}) \cap M(L^\infty) = \emptyset,$$

we can apply Case 1. Hence $\overline{E \cap \{|\varphi| < 1\}} \subset \mathit{int}\, Z(f)$. By (3.8), $x_0 \in \mathit{int}\, Z(f)$ and this contradicts (3.7).

Now, suppose that $x_0 \in M(L^\infty)$. Put

$$S_1 = \overline{M(L^\infty) \cap \mathit{int}\, Z(f)} \quad \text{and} \quad S_2 = \overline{M(L^\infty) \cap \{f \neq 0\}}. \tag{3.9}$$

By (3.2), $M(L^\infty) \cap \mathit{int}\, Z(f) \neq \emptyset$. Since $M(L^\infty)$ is extremely disconnected [2, p. 18], S_1 and S_2 are non-empty open-closed subsets of $M(L^\infty)$, and

$$S_1 \cap S_2 = \emptyset. \tag{3.10}$$

Put

$$E_1 = E \cap \{\tilde{\chi}_{S_1} < 1\} \quad \text{and} \quad E_2 = E \cap \{\tilde{\chi}_{S_1} = 1\}. \tag{3.11}$$

Then E_1 is an F_σ-set. By (3.2), (3.9), and (3.11), we have $E_1 \cap M(L^\infty) = \emptyset$. Hence by Lemma 3.1, $\overline{E_1} \cap M(L^\infty) = \emptyset$. Thus $x_0 \notin \overline{E_1}$. By (3.11), $E = E_1 \cup E_2$, so that by (3.7) we have

$$x_0 \in \overline{E_2} \subset \{\tilde{\chi}_{S_1} = 1\}. \tag{3.12}$$

Put

$$V_1 = \{f \neq 0\} \cap \{\tilde{\chi}_{S_2} < 1\} \quad \text{and} \quad V_2 = \{f \neq 0\} \cap \{\tilde{\chi}_{S_2} = 1\}. \tag{3.13}$$

Then V_1 is an F_σ-set, and by (3.9) $V_1 \cap M(L^\infty) = \emptyset$. Hence by Lemma 3.1, $\overline{V}_1 \cap M(L^\infty) = \emptyset$. Thus $x_0 \notin \overline{V_1}$. By (3.13), $\{f \neq 0\} = V_1 \cup V_2$. By (3.7), $x_0 \in \overline{\{f \neq 0\}}$ so that

$$x_0 \in \overline{V_2} \subset \{\tilde{\chi}_{S_2} = 1\}. \tag{3.14}$$

By (3.1) and (3.10), we have $\{\tilde{\chi}_{S_1} = 1\} \cap \{\tilde{\chi}_{S_2} = 1\} = \emptyset$. But this contradicts (3.12) and (3.14). This completes the proof.

4. Prime ideals in $H^\infty + C$

The following theorem answers the problem posed by Gorkin and Mortini [7, Q4].

Theorem 4.1. *For every $x \in M(H^\infty + C)$, $J(x)$ is a prime ideal of $H^\infty + C$.*

Proof. Let $f, g \in H^\infty + C$ such that $fg \in J(x)$. Since our topology of $M(H^\infty + C)$ is the weak*-topology, fundamental neighborhoods of x are open F_σ-sets. Hence there is an open F_σ-subset U of $M(H^\infty + C)$ such that $x \in U$ and

$$fg = 0 \quad \text{on } U. \tag{4.1}$$

To prove our theorem, suppose not; that is,

$$f \notin J(x) \quad \text{and} \quad g \notin J(x). \tag{4.2}$$

If $f(x) \neq 0$, then by (4.1) we have $g \in J(x)$. So we may assume that

$$f(x) = 0 \quad \text{and} \quad g(x) = 0. \tag{4.3}$$

Let

$$U_f = U \cap \{f \neq 0\} \quad \text{and} \quad U_g = U \cap \{g \neq 0\}. \tag{4.4}$$

Then by (4.2) and (4.3),

$$x \in \overline{U_f} \cap \overline{U_g}. \tag{4.5}$$

Since U is an F_σ-set, U_f is an F_σ-subset of $M(H^\infty + C)$, and by (4.1) and (4.4), $U_f \subset \mathit{int}\, Z(g)$. Then by Theorem 3.1, $\overline{U_f} \subset \mathit{int}\, Z(g)$. Since $U_g \subset \{g \neq 0\}$, we have $\overline{U_f} \cap \overline{U_g} = \emptyset$. This contradicts (4.5).

Let $x \in M(H^\infty + C)$. Then by Newman's theorem [14], $J(x) \cap H^\infty = \{0\}$ if and only if $x \in M(L^\infty)$. By Theorem 4.1, we have the following corollary.

Corollary 4.1. *Let $x \in M(H^\infty + C) \setminus M(L^\infty)$. Then $J(x) \cap H^\infty$ is a prime ideal of H^∞.*

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