

A REMARK ON THE DEBS–SAINT-RAYMOND THEOREM

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ABSTRACT. A theorem of Debs and Saint-Raymond gives sufficient conditions for a σ -ideal of compact sets to have the covering property. We discuss the necessity of these conditions. Namely, we show that there exists a Π_1^1 σ -ideal that is locally non-Borel, has no Borel basis and has the covering property. This partially answers a question posed by Kechris.

We start with basic definitions. Let E be a compact metric space, and let $\mathcal{K}(E)$ be the space of all closed subsets of E with the Hausdorff metric

$$\begin{aligned}\delta(K, L) &= \sup\{\max(\text{dist}(x, K), \text{dist}(y, L)); x \in L, y \in K\}, \text{ if } K, L \neq \emptyset, \\ &= \text{diam}(E) + 1, \text{ if } K = \emptyset, L \neq \emptyset, \text{ or } K \neq \emptyset, L = \emptyset, \\ &= 0, \text{ if } K = L = \emptyset.\end{aligned}$$

If $A \subset E$, then $\mathcal{K}(A)$ stands for the set of all compact subsets of A . A set $I \subset \mathcal{K}(E)$ is called a σ -ideal

- (i) if I is *hereditary*, i.e. if $K, L \in \mathcal{K}(E)$, $K \in I$, $L \subset K$, then $L \in I$,
- (ii) if $K, K_1, K_2, \dots \in \mathcal{K}(E)$, $K_n \in I$ for all $n \in \mathbb{N}$, and $K = \bigcup_{n=1}^{+\infty} K_n$, then $K \in I$.

We say that a σ -ideal I is *calibrated* if, for every $F \in \mathcal{K}(E)$, $F_n \in I$, $n \in \mathbb{N}$, with $\mathcal{K}(F \setminus \bigcup_{n=1}^{+\infty} F_n) \subset I$, we have $F \in I$. A σ -ideal I is called *locally non-Borel* if for every compact set $F \in \mathcal{K}(E) \setminus I$ the set $I \cap \mathcal{K}(F)$ is not Borel. We say that a σ -ideal $I \subset \mathcal{K}(E)$ has the *covering property* if every analytic subset of E which cannot be covered by countably many elements of I contains a closed set which is not in I . A *basis* for a σ -ideal $I \subset \mathcal{K}(E)$ is a subset $B \subset I$ such that

$$I = B_\sigma \stackrel{\text{def}}{=} \left\{ K \in \mathcal{K}(E); K = \bigcup_{n=1}^{\infty} K_n, K_n \in B \right\}$$

and B is hereditary. Let $I \subset \mathcal{K}(E)$ be a σ -ideal. Then we denote

$$I^{\text{loc}} = \{K \in \mathcal{K}(E); \text{ there exists an open set}$$

$$G \subset E \text{ with } K \cap G \neq \emptyset \text{ and } K \cap \overline{G} \in I\}.$$

The theory of σ -ideals of compact sets has been motivated by problems in harmonic analysis, namely by problems concerning U -sets and U_0 -sets. Results on the

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structure of σ -ideals of compact sets due to Kechris, Louveau and Woodin ([KLW]) were used by Debs and Saint-Raymond ([DSR]) to give a positive answer to the old question of whether every Borel U -set is meager. The following theorem plays the key role in their proof.

Theorem A ([DSR]). *Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a Π_1^1 σ -ideal satisfying*

- (i) *I is locally non-Borel,*
- (ii) *I is calibrated,*
- (iii) *I has a Borel basis.*

Then I has the covering property.

Kechris posed in [K1] the question of whether all assumptions in Theorem A are necessary. If I has the covering property, then clearly I is calibrated. The necessity of condition (i) is discussed in [K1] but this question remains open. We prove the following theorem which shows that condition (iii) is not necessary.

Theorem. *There exists a compact metric space E and a Π_1^1 σ -ideal $I \subset \mathcal{K}(E)$ such that*

- (i) *I is locally non-Borel,*
- (ii) *I has the covering property,*
- (iii) *I has no Borel basis.*

We need the following results.

Theorem B ([KLW, p. 271]). *Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a Π_1^1 σ -ideal. Then I has a Borel basis if and only if I^{loc} is Borel.*

Theorem C ([L, Chapter 3, pp. 48-54], see also [KLW, p. 271]). *Let E be a compact metric space, and let $I \subset \mathcal{K}(E)$ be a σ -ideal. Then I is Π_1^1 if and only if I has a Π_1^1 basis.*

Theorem D ([H]). *Let E be an uncountable compact metric space. Denote*

$$\mathcal{K}_\omega(E) = \{K \in \mathcal{K}(E); K \text{ is at most countable}\}.$$

Then $\mathcal{K}_\omega(E)$ is Π_1^1 and non-Borel.

Lemma. *Let X and Y be Polish spaces, and let $A \subset X \times Y$ be in Σ_1^1 . If the projection of A to Y is uncountable, then there exists an uncountable closed set $K \subset X \times Y$ such that $K^y \stackrel{\text{def}}{=} \{x \in X; [x, y] \in K\}$ contains at most one point for every $y \in Y$.*

Proof. Denote the projection of $X \times Y$ to Y by π and let $T \subset \pi(A)$ be a nonempty perfect set. According to the Jankov-von Neumann uniformization theorem (see [K2, 18.1]) there exists a $\sigma(\Sigma_1^1)$ -measurable function $\varphi : T \rightarrow X$ such that $(\varphi(y), y) \in A$ for every $y \in T$. The function φ is Baire measurable and therefore φ is continuous on some G_δ set H that is dense in T (see [K2, 8.38]). Find an uncountable compact set $L \subset H$ and put $K = \{[\varphi(y), y]; y \in L\}$. \square

Proof of the Theorem. Put $X = Y = 2^{\mathbb{N}}$ and $E = X \times Y$. The spaces X, Y, E are equipped with the usual topologies. Take a non-Borel set $C \subset Y$ in Π_1^1 (for the

existence of such a set see [K2, 14.2]). Put

$$\begin{aligned} B_0 &= \{[x, y]; x \in X, y \in Y\} \cup \{X \times \{y\}; y \in C\}, \\ B &= \{K \in \mathcal{K}(E); \text{ there exists } L \in B_0 \text{ with } K \subset L\}, \\ I &= B_\sigma. \end{aligned}$$

The projection of E to Y is denoted by π . The set B can be written as follows:

$$\begin{aligned} B &= \{[x, y]; x \in X, y \in Y\} \\ &\cup (\mathcal{K}(X \times C) \cap \{K \in \mathcal{K}(E); \pi(K) \text{ is a singleton}\}) \cup \{\emptyset\}. \end{aligned}$$

The set $X \times C$ is Π_1^1 and therefore $\mathcal{K}(X \times C)$ is Π_1^1 (cf. [K1]). The sets $\{[x, y]; x \in X, y \in Y\}$ and $\{K \in \mathcal{K}(E); \pi(K) \text{ is a singleton}\}$ are closed. Thus we see that B is Π_1^1 in $\mathcal{K}(E)$ and therefore I is a Π_1^1 σ -ideal by Theorem C.

Define a mapping $\varphi : Y \rightarrow \mathcal{K}(E)$ by $\varphi(y) = X \times \{y\}$. The mapping φ is clearly continuous. If $y \in C$, then $\varphi(y) \in B_0 \subset I^{\text{loc}}$. If $y \in Y \setminus C$ and an open set $G \subset E$ intersects $\varphi(y)$, then $\varphi(y) \cap \bar{G}$ is an uncountable closed set not intersecting $X \times C$. Therefore such a set cannot be covered by countably many elements of B and thus does not belong to I . This implies $\varphi(y) \notin I^{\text{loc}}$. Thus we have proved $\varphi^{-1}(I^{\text{loc}}) = C$. This gives that I^{loc} is non-Borel and I has no Borel basis by Theorem B.

Take $K \in \mathcal{K}(E) \setminus I$. Denote $H = K \setminus (X \times C)$. We distinguish two possibilities.

1) The set H is uncountable. The set H is analytic and therefore there exists a non-empty perfect set $K^* \subset H$. We have $\mathcal{K}(K^*) \cap I = \mathcal{K}_\omega(K^*)$. This and Theorem D show that $\mathcal{K}(K^*) \cap I$ is not Borel and consequently $\mathcal{K}(K) \cap I$ is not Borel.

2) The set H is countable. Then the set $P = K \cap (X \times C)$ is analytic. The set P cannot be covered by countably many elements of I , otherwise $K \in I$. This implies that the projection of P to Y is uncountable and, according to the Lemma, P contains an uncountable closed set K^* such that $(K^*)^y$ contains at most one point for every $y \in Y$. We have $\mathcal{K}(K^*) \cap I = \mathcal{K}_\omega(K^*)$ and $\mathcal{K}(K) \cap I$ is not Borel again.

It remains to prove that I has the covering property. Let $A \subset E$ be a Σ_1^1 set which cannot be covered by countably many elements of I . If $A \setminus (X \times C)$ is uncountable, then there exists an uncountable closed set $K \subset A \setminus (X \times C)$ and we have $K \notin I$. If $A \setminus (X \times C)$ is countable, then $\tilde{A} = A \cap (X \times C)$ is Σ_1^1 . The set \tilde{A} cannot be covered by countably many elements of I and therefore the projection of \tilde{A} to Y is uncountable. Using the Lemma we can find an uncountable closed set $K \subset \tilde{A}$ such that K^y contains at most one point for every $y \in Y$. This implies that $K \notin I$. \square

Remark. A similar construction of a σ -ideal used for another purpose is due to Dougherty (see [K1, p. 133]).

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