PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 12, Pages 3741–3747
S 0002-9939(01)06002-6
Article electronically published on May 21, 2001

A WEAK ASPLUND SPACE WHOSE DUAL IS NOT WEAK* FRAGMENTABLE

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(Communicated by Jonathan M. Borwein)

ABSTRACT. Under the assumption that there exists in the unit interval $[0,1]$ an uncountable set A with the property that every continuous mapping from a Baire metric space B into A is constant on some non-empty open subset of B, we construct a Banach space X such that (X^*, weak^*) belongs to Stegall's class but (X^*, weak^*) is not fragmentable.

1. INTRODUCTION

We say that a Banach space X is *weak Asplund* if every continuous convex function defined on a non-empty open convex subset A of X is Gâteaux differentiable at the points of a residual subset of A. In the study of weak Asplund spaces Stegall [8] introduced the following class of topological spaces, which are defined in terms of minimal uscos. Recall that a set-valued mapping $\varphi : X \to 2^Y$ acting between topological spaces X and Y is called an *usco mapping* if for each $x \in X$, $\varphi(x)$ is a non-empty compact subset of Y and for each open set W in Y, $\{x \in X : \varphi(x) \subseteq W\}$ is open in X. An usco mapping $\varphi : X \to 2^Y$ is called *minimal* if its graph does not properly contain the graph of any other usco defined on X. We say that a topological space Y belongs to Stegall's *class($\mathcal{S}$)* if for every Baire space B and minimal usco $\varphi : B \to 2^Y$, φ is single-valued at the points of a residual subset of B. In [8] Stegall showed that a Banach space X is weak Asplund if (X^*, weak^*) lies in class($\mathcal{S}$). In fact, Stegall proved that if the dual unit ball B_{X^*} of X equipped with the weak* topology belongs to class($\mathcal{S}$), then X is weak Asplund. Another class of topological spaces that have played a significant role in the study of weak Asplund spaces is the class of fragmentable spaces. We say that a topological space Y is *fragmented* by a pseudo metric ρ if every non-empty subset of Y contains a non-empty relatively open set of arbitrarily small ρ-diameter. A space that is fragmented by some metric is called *fragmentable*. An easy argument shows that fragmentable spaces belong to Stegall's class($\mathcal{S}$) (see Theorem 5.1.11 in [2]). The converse question was considered in [4]. Indeed, in that paper the author shows

Received by the editors February 17, 2000 and, in revised form, April 22, 2000.
2000 *Mathematics Subject Classification.* Primary 54C60, 46B20, 54C10.
Key words and phrases. Stegall's class, fragmentability, weak Asplund space, double arrow space, Baire space, minimal usco.
The first author was partially supported by Grant MM-701/97 of the National Fund for Scientific Research of the Bulgarian Ministry of Education, Science and Technology.
The second author was supported by a Marsden fund grant, VUW 703, administered by the Royal Society of New Zealand.

that under some additional set-theoretic assumptions there are compact spaces in Stegall's class($\mathcal{S}$) that are not fragmentable. We show in this paper that under similar set-theoretic assumptions there are Banach spaces X such that (X^*, weak^*) lies in Stegall's class($\mathcal{S}$) but (X^*, weak^*) is not fragmentable.

2. Construction of a Banach space

Given a subset A of $(0,1)$ we shall consider the Banach space D_A of all real-valued functions on $(0,1]$ that have finite right-hand limits at the points of $[0,1)$, are left-continuous at the points of $(0,1]$ and are continuous at the points of $(0,1]\backslash A$, endowed with sup-norm. Then we shall characterise the duals of these spaces in terms of functions of bounded variation. Given bounded functions f and α defined on $(0,1]$ and $[0,1]$ respectively and a partition $P := \{t_k : 0 \leq k \leq n\}$ of $[0,1]$ where

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = 1,$$

the *Riemann-Stieltjes sum* of f with respect to α, determined by P, is the real number

$$S(P, f, \alpha) := \sum_{k=1}^{n} f(t_k) \cdot [\alpha(t_k) - \alpha(t_{k-1})].$$

We say that f is *Riemann-Stieltjes integrable with respect to* α if there exists a real number I such that for every $\varepsilon > 0$ there exists a partition P_ε of $[0,1]$ such that $|S(P, f, \alpha) - I| < \varepsilon$ for all partitions P that refine P_ε. In this case I is denoted by $I := \int_{[0,1]} f(t) \mathrm{d}\alpha(t)$ and is called the *Riemann-Stieltjes integral of f with respect to* α.

For any subset A of $(0,1)$ we shall denote by $BV_A[0,1]$ the space of all real-valued functions of bounded variation on $[0,1]$ that are right-continuous at the points of $(0,1)\backslash A$ and map 0 to 0. We will consider this space endowed with the total variation norm, i.e. for each $\alpha \in BV_A[0,1]$

$$\|\alpha\| := \text{Var}(\alpha) = \sup\{\sum_{k=1}^{n} |\alpha(t_k) - \alpha(t_{k-1})| : \{t_k : 0 \leq k \leq n\} \text{ is a partition of } [0,1]\}.$$

The proof of the following lemma is straightforward.

Lemma 1 (Uniform approximation lemma)**.** *Let A be any dense subset of* $(0,1)$, *$f \in D_A$ and $\varepsilon > 0$. Then there exists a partition $P_\varepsilon := \{t_k : 0 \leq k \leq n\}$ of $[0,1]$ with $t_k \in A$ for all $1 \leq k < n$ such that $\|f - f_{P_\varepsilon}\|_\infty < \varepsilon$, where $f_{P_\varepsilon} : (0,1] \to \mathbb{R}$ is defined by* $f_{P_\varepsilon}(t) := \sum_{k=1}^{n} f(t_k) \cdot \chi_{(t_{k-1}, t_k]}(t)$.

One can now use the previous lemma to prove the following theorem.

Theorem 1. *Suppose that $\alpha : [0,1] \to \mathbb{R}$ has bounded variation and $f \in D_{(0,1)}$. Then f is Riemann-Stieltjes integrable with respect to α.*

Proof. First note that to show f is Riemann-Stieltjes integrable with respect to α we need only show that for every $\varepsilon > 0$ there exists a partition P_ε of $[0,1]$ such that $|S(P_\varepsilon, f, \alpha) - S(P', f, \alpha)| < \varepsilon$ for all partitions P' that refine P_ε. Further, an elementary calculation shows that for any $g, g' \in D_{(0,1)}$ and partition P we have that $|S(P,g,\alpha) - S(P,g',\alpha)| \leq \|g - g'\| \cdot \mathrm{Var}(\alpha)$. Therefore, if we fix $\varepsilon > 0$ and choose a partition P of $[0,1]$ such that $\|f - f_P\| < \varepsilon/(\mathrm{Var}(\alpha)+1)$, then

$$\begin{aligned} |S(P,f,\alpha) - S(P',f,\alpha)| &\leq |S(P,f,\alpha) - S(P,f_P,\alpha)| \\ &\quad + |S(P,f_P,\alpha) - S(P',f_P,\alpha)| \\ &\quad + |S(P',f_P,\alpha) - S(P',f,\alpha)| \\ &< 0 + 0 + \varepsilon = \varepsilon \end{aligned}$$

for all partitions P' that refine P. □

By a *slight* adaption of the proof of Riesz's representation theorem for the dual of $(C[0,1], \|\cdot\|_\infty)$ we can obtain the following representation theorem. Note: it is easiest to make the adaption to the proof of Riesz's representation theorem that relies upon the Hahn-Banach extension theorem. In fact the standard proof only *uses* extensions to the space $D_{(0,1)}$ and not to all of $B[0,1]$ - the space of bounded functions on $[0,1]$; see [1]. Further details may also be found in the paper [6].

Theorem 2. *Let A be any subset of $(0,1)$. Then the dual of D_A is isometrically isomorphic to $BV_A[0,1]$. In particular the mapping $T : BV_A[0,1] \to D_A^*$ defined by $T(\alpha)(x) := \int_{[0,1]} x(t)\mathrm{d}\alpha(t)$ for each $x \in D_A$ is an isometry from $BV_A[0,1]$ onto D_A^*.*

For a non-empty subset A of $[0,1]$ we shall denote by τ_A the topology (on $BV_A[0,1]$) of pointwise convergence on $A \cup \{1\}$. If A is dense in $[0,1]$, then τ_A is a Hausdorff topology. Moreover, the closed unit ball in $BV_A[0,1]$ (with respect to the total variation norm) is τ_A-compact.

Corollary 1. *For a non-empty subset A of $(0,1)$, $(BV_A[0,1], \tau_A)$ is homeomorphic to D_A^* endowed with the weak topology generated by the functions $\chi_{(0,a]}$ with $a \in A \cup \{1\}$. If A is dense in $(0,1]$, then τ_A is Hausdorff and the closed unit ball $B_{BV_A[0,1]}$ in $BV_A[0,1]$ with the τ_A-topology is homeomorphic to $(B_{D_A^*}, weak^*)$. In fact the mapping T defined in the previous theorem, restricted to the ball $B_{BV_A[0,1]}$, realizes such a homeomorphism.*

Proof. The proof of the first assertion is based upon the simple fact that for each $\alpha \in BV_A[0,1]$ and $t \in A \cup \{1\}$, $T(\alpha)(\chi_{(0,t]}) = \alpha(t)$. The fact that T restricted to $B_{BV_A[0,1]}$ realizes a homeomorphism onto $(B_{D_A^*}, \text{weak}^*)$ follows from the fact that on $B_{D_A^*}$ the relative weak* topology and the relative topology generated by the functions $\chi_{(0,t]}$, $t \in A \cup \{1\}$ coincide (see Lemma 1). □

3. $(BV_A[0,1], \tau_A)$ BELONGS TO CLASS($\mathcal{S}$)

We begin this section with the following preliminary theorem.

Theorem 3. *Let Y be a compact topological space and ρ a metric on it. Then Y belongs to class($\mathcal{S}$) if (and only if) for $\varepsilon > 0$, each Baire metric space B and each minimal usco $\varphi : B \to 2^Y$ there exists a point $x \in B$ such that ρ-diam $\varphi(x) \leq \varepsilon$.*

Proof. By the "factorization theorem" in [5] we need only show that for every complete metric space M and minimal usco $\varphi : M \to 2^Y$ there exists a residual set R of M such that φ is single-valued at the points of R. If we now apply the proof of Theorem 3.2.6 in [2] to our current situation we obtain the desired result. □

Lemma 2. *Let $\varphi : X \to 2^Y$ be a minimal usco acting between topological spaces X and Y and let $f : Y \to \mathbb{R}$ be a continuous function. Then there is a residual set R in X such that the composition mapping $f \circ \varphi : X \to 2^{\mathbb{R}}$ defined by $(f \circ \varphi)(x) := \{f(y) : y \in \varphi(x)\}$ is single-valued at the points of R.*

Proof. By Lemma 3.1.2(iv) in [2], $f \circ \varphi$ is a minimal usco on X and so the result follows from Theorem 5.1.11 in [2]. ☐

In the remainder of this section A will always denote a dense subset of $(0, 1)$ that satisfies the property: $(*)$ Every continuous function from a Baire metric space B into A is constant on some non-empty open subset of B.

Of course every countable dense subset of $(0, 1)$ has this property; however we shall be particularly interested in the case when A is uncountable, if indeed such a set exists.

Theorem 4. *Let A be a dense subset of $(0, 1)$ that satisfies property $(*)$. Then $(BV_A[0, 1], \tau_A)$ belongs to class($\mathcal{S}$).*

Proof. First, let us note that by Theorem 3.1.5, part(iv) in [2], we need only show that the closed unit ball $B_{BV_A[0,1]}$ of $BV_A[0, 1]$ belongs to class($\mathcal{S}$). In fact, we need only show that the (τ_A-compact) set $M_A[0, 1]$ of all non-decreasing functions in $B_{BV_A[0,1]}$, endowed with the τ_A-topology lies in Stegall's class($\mathcal{S}$). Since if $M_A[0, 1] \in$ class($\mathcal{S}$), then by Theorem 3.1.5, part(iii) in [2], $M_A[0, 1] \times M_A[0, 1] \in$ class($\mathcal{S}$). However, by the Jordan decomposition theorem $B_{BV_A[0,1]} \subseteq \Delta(M_A[0, 1] \times M_A[0, 1])$, where $\Delta : M_A[0, 1] \times M_A[0, 1] \to BV_A[0, 1]$ is defined by $\Delta(f, g) := f - g$. Hence the result follows from Theorem 3.1.5, part(i) in [2], since Δ is a perfect mapping. For any α, β in $M_A[0, 1]$ we define

$$\rho_1(\alpha, \beta) := |(\alpha - \beta)(1)|, \quad \rho_I(\alpha, \beta) := \int_0^1 |(\alpha - \beta)(t)| \mathrm{d}t,$$
$$\rho_J(\alpha, \beta) := \sum_{t \in A} |(\alpha - \beta)(t^+) - (\alpha - \beta)(t)|.$$

Note: $\{t \in A : |(\alpha - \beta)(t^+) - (\alpha - \beta)(t)| > 0\}$ is at most countable. Then we define $\rho(\alpha, \beta) := \rho_1(\alpha, \beta) + \rho_I(\alpha, \beta) + \rho_J(\alpha, \beta)$. With a little thought it should be clear that ρ defines a metric on the set $M_A[0, 1]$. We now proceed via Theorem 3. To this end, let $\varepsilon > 0$, B be a Baire metric space and $\varphi : B \to 2^{M_A[0,1]}$ be a minimal usco.

Step 1. It is not too difficult to check that ρ_I is a continuous pseudo-metric on $M_A[0, 1]$, i.e. for each $\alpha \in M_A[0, 1]$ and $r > 0$ the set $\{\beta \in M_A[0, 1] : \rho_I(\alpha, \beta) < r\}$ is τ_A-open in $M_A[0, 1]$. Hence it follows that ρ_I "fragments" $M_A[0, 1]$. It is also very easy to see that ρ_1 is a continuous pseudo-metric on $M_A[0, 1]$ and so ρ_1 also "fragments" $M_A[0, 1]$. In particular this means that there is a residual set $R \subseteq B$ such that both ρ_1-diam $\varphi(x) = 0$ and ρ_I-diam $\varphi(x) = 0$ at each point $x \in R$ (see the proof of Theorem 5.1.11 in [2]). Therefore by restricting φ to R and re-labeling we may assume, without loss of generality, that both ρ_1-diam $\varphi(x) = 0$ and ρ_I-diam $\varphi(x) = 0$ for all $x \in B$. One immediate consequence of this is that for each $x \in B$ we may unambiguously refer to the left-hand and right-hand limits of $\varphi(x)$, since if α, $\beta \in \varphi(x)$, then both the left-hand and right-hand limits of α and β coincide on $[0, 1]$.

Step 2. In this step we decompose the space $M_A[0,1]$ into countably many parts, $\{M_{m,n,(F,f)} : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F}\}$, but first we introduce some notation. For each $\alpha \in M_A[0,1]$ and $m \in \mathbb{N}$,

$$S(\alpha,m) := \{t \in A : \alpha(t^+) - \alpha(t^-) > 1/m\} \quad \text{and}$$
$$L^1(\alpha,m) := \sum_{t \in S(\alpha,m)} [\alpha(t^+) - \alpha(t^-)].$$

The notation $S(\alpha,\infty)$ and $L^1(\alpha,\infty)$ will have the expected meaning. For each $m \in \mathbb{N}$ we define, $M_m := \{\alpha \in M_A[0,1] : L^1(\alpha,m) > L^1(\alpha,\infty) - \varepsilon/2\}$ and for each partition $P := \{t_k : 0 \le k \le n\}$ of $[0,1]$ we let $I_k(P) := [t_{k-1}, t_k]$, $1 \le k \le n$. Then for each $n \in \mathbb{N}$ we let P_n denote the uniform $1/n$-partition of $[0,1]$ and we define

$$M_{m,n} := \{\alpha \in M_m : P_n \cap S(\alpha,m) = \emptyset \text{ and}$$
$$\operatorname{card}[S(\alpha,m) \cap I_k(P_n)] \le 1 \text{ for } k \in \{1,2,\cdots,n\}\}.$$

One can check that $\bigcup\{M_{m,n} : (m,n) \in \mathbb{N}^2\} = M_A[0,1]$. Now, with m and n fixed we further decompose $M_A[0,1]$ as follows: For each fixed non-empty subset $F \subseteq \{1,2,\cdots,n\}$ and function $f : F \to \mathbb{Q}^2$, i.e. $f(k) := (f_1(k), f_2(k)) \in \mathbb{Q}^2$, we consider the set

$$M_{m,n,(F,f)} := \{\alpha \in M_{m,n} : \operatorname{card}[I_k(P_n) \cap S(\alpha,m)] = 1 \text{ if, and only if, } k \in F,$$
$$\text{and } \max\{|\alpha(t^-) - f_1(k)|, |\alpha(t^+) - f_2(k)|\} < 1/(4m)$$
$$\text{for each } t \in I_k(P_n) \cap S(\alpha,m) \text{ and } k \in F\}.$$

If we let $\mathcal{F}$ denote the family of all such pairs (F,f), then $\mathcal{F}$ is at most countable. Hence $\{M_{m,n,(F,f)} : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F}\}$ is a countable decomposition of $M_A[0,1]$.

Step 3. For any subset $X \subseteq M_A[0,1]$ we define $\varphi^{-1}(X) := \{x \in B : \varphi(x) \cap X \neq \emptyset\}$. Now since $M_A[0,1] = \bigcup\{M_{m,n,(F,f)} : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F}\}$, it follows that $\bigcup\{\varphi^{-1}(M_{m,n,(F,f)}) : (m,n,(F,f)) \in \mathbb{N}^2 \times \mathcal{F}\} = B$. Therefore there must be some $(m',n',(F',f')) \in \mathbb{N}^2 \times \mathcal{F}$ such that $\varphi^{-1}(M_{m',n',(F',f')})$ is second (Baire) category in B. Moreover, since the set $M_{m',n',(F',f')}$ is defined solely in terms of the left-hand and right-hand limits of its members it follows, by the note at the end of Step 1, that

$$\varphi(\varphi^{-1}(M_{m',n',(F',f')})) \subseteq M_{m',n',(F',f')}.$$

Further, by Proposition 3.2.5 in [2] there exists a non-empty open set U in B such that $B' := U \cap \varphi^{-1}(M_{m',n',(F',f')})$ is dense in U and a Baire space with the relative topology. Now by applying Lemma 2 in [4] twice we see that the restriction of φ to B' is a minimal usco. In this way, we see that there is no loss of generality in assuming that $\varphi(B) \subseteq M_{m',n',(F',f')}$.

Step 4. For each $k \in F' \subseteq \{1,2,\cdots,n'\}$ we define the function $g_k : B \to A$ by $g_k(x) := S(\varphi(x),m') \cap I_k(P_{n'})$. Note: this definition is sensible since for each $x \in B$ and α, $\beta \in \varphi(x)$, $S(\alpha,m') = S(\beta,m')$. It now follows from the τ_A-upper semi-continuity of φ and the definition of $M_{m',n',(F',f')}$ that each g_k is continuous on B. Hence by property $(*)$ there exists a non-empty open subset U of B such that each g_k, $k \in F'$, is constant on U.

Step 5. For each $k \in F'$ define $t_k := g_k(x)$, $x \in U$. Then by Lemma 2 there exists a residual set R in U such that each of the uscos $\hat{t}_k \circ \varphi : U \to 2^{\mathbb{R}}$ defined by $(\hat{t}_k \circ \varphi)(x) := \{\alpha(t_k) : \alpha \in \varphi(x)\}$ are single-valued on R. We claim that ρ-diam $\varphi(x) \leq \varepsilon$ for each $x \in R$. To see this, first note that it is sufficient to show that ρ_J-diam $\varphi(x) \leq \varepsilon$ for each $x \in R$. Now fix $x_0 \in R$ and consider α, $\beta \in \varphi(x_0)$; then

$$\rho_J(\alpha,\beta) = \sum_{t \in A} |(\alpha-\beta)(t^+) - (\alpha-\beta)(t)| = \sum_{t \in S(\alpha,\infty)} |(\alpha-\beta)(t^+) - (\alpha-\beta)(t)|.$$

However, if $t \in S(\alpha, m')$, then $|(\alpha-\beta)(t^+) - (\alpha-\beta)(t)| = 0$ since (by Step 1) $\alpha(t^+) = \beta(t^+)$ and (as just noted) $\alpha(t) = \beta(t)$. On the other hand, if we write $S_{tail} := S(\alpha,\infty) \backslash S(\alpha, m')$, then we have

$$\begin{aligned}\sum_{t \in S_{tail}} |(\alpha-\beta)(t^+) - (\alpha-\beta)(t)| &\leq \sum_{t \in S_{tail}} \alpha(t^+) - \alpha(t) + \sum_{t \in S_{tail}} \beta(t^+) - \beta(t) \\ &\leq \sum_{t \in S_{tail}} \alpha(t^+) - \alpha(t^-) + \sum_{t \in S_{tail}} \beta(t^+) - \beta(t^-) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

This shows that $\rho(\alpha,\beta) \leq \varepsilon$ and so ρ-diam $\varphi(x_0) \leq \varepsilon$, which completes the proof. □

Corollary 2. *Let A be any dense subset of $(0,1)$ that satisfies property $(*)$. Then $(D_A^*, weak^*)$ belongs to class$(\mathcal{S})$.*

We end this section of the paper by returning to the question of the existence of uncountable sets that have property $(*)$. The good news is that there is such a subset A of $(0,1)$ that satisfies property $(*)$ in Gödel's universe $(V = L)$ and hence the set $A' := A \cup [(0,1) \cap \mathbb{Q}]$ will serve our needs; see [7]. However, the set A necessarily relies upon additional axioms, as it is known that if we assume the existence of a precipitous ideal over ω_1, then for every uncountable separable metric space A there exists a Baire metric space B and a continuous function $f : B \to A$ such that $\mathrm{int}(f^{-1}(a)) = \emptyset$ for each $a \in A$ (see [3]).

4. When is $(BV_A[0,1], \tau_A)$ fragmentable?

We will show that for every set $A \subseteq (0,1)$, D_A is isometrically isomorphic to $C(K_A)$ for some compact Hausdorff space K_A. Indeed, if $\emptyset \neq A \subseteq (0,1)$, then we may define K_A in the following manner: $K_A := [(\{0\} \cup A) \times \{1\}] \cup [(0,1] \times \{0\}]$. We endow K_A with the order topology (on K_A) generated by the lexographic (i.e. dictionary) ordering, i.e. $(s_1, s_2) \leq (t_1, t_2)$ if, and only if, either $s_1 < t_1$ or $s_1 = t_1$ and $s_2 \leq t_2$. It is shown in [4] (see Proposition 2) that K_A is always Hausdorff and compact. It is also shown that K_A is fragmentable if, and only if, A is countable and this occurs if, and only if, K_A is metrizable.

Theorem 5. *Let A be a non-empty subset of $(0,1)$. Then $(D_A^*, weak^*)$ is fragmentable if, and only if, A is countable.*

Proof. We define an isometry T from D_A onto $C(K_A)$ in the following way: $T(f)((t,0)) := f(t)$ for all $t \in (0,1]$ and $T(f)((t,1)) := \lim_{t' \to t^+} f(t')$ for $t \in \{0\} \cup A$. One can check, as in ([2], p. 47), that T is in fact an isometry from D_A onto $C(K_A)$. Indeed, it is routine to verify that T is a linear isometry into $C(K_A)$, so it suffices to check that T is surjective. To this end, let $g \in C(K_A)$ and define $f : (0,1] \to \mathbb{R}$ by $f(t) := g((t,0))$ for all $t \in (0,1]$. Then $f \in D_A$ and $T(f) = g$. □

Corollary 3. *If A is an uncountable dense subset of $(0,1)$ that satisfies property $(*)$, then $(D_A^*, weak^*)$ belongs to class$(\mathcal{S})$ (and so D_A is weak Asplund) but $(D_A^*, weak^*)$ is not fragmentable.*

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