PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 12, Pages 3531–3537
S 0002-9939(01)06013-0
Article electronically published on June 28, 2001

REGULARITY PROPERTIES OF DISTRIBUTIONS AND ULTRADISTRIBUTIONS

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(Communicated by Jonathan M. Borwein)

Abstract. We give necessary and sufficient conditions for a regularized net of a distribution in an open set Ω which imply that it is a smooth function or C^k function in Ω. We also give necessary and sufficient conditions for an ultradistribution to be an ultradifferentiable function of corresponding class.

Introduction

Algebras of generalized functions are usually defined as factor algebras of certain algebras of sequences (nets) of smooth functions (cf. [1], [2], [3], [8], [11]), and the classical spaces of functions, distributions and ultradistributions are embedded into the appropriate algebra through their regularizations; they are also equivalence classes of sequences (nets) of smooth functions. Solutions in algebras of generalized functions of PDE are also presented by such sequences (nets) and even differential operators (for example, with singular coefficients) are sometimes replaced by sequences (nets) of differential operators with smooth coefficients. So the natural question arises: Under what conditions is a given generalized function (a generalized solution of PDE) actually a classical function of appropriate class, distribution, or ultradistribution?

The partial answers to such questions are given in Propositions 1, 2 and more generally in Theorems 1, 2, respectively, in terms of asymptotic behaviour of sequences of functions provided that these sequences are evaluated on the ultraproduct $\Omega^{\mathbb{N}}$.

Denote by (θ_n) (respectively, (ϕ_n)) a δ-sequence, also called a sequence of mollifiers, of smooth functions (respectively, of appropriate ultradifferentiable functions). Precise definitions will be given in sections 2 and 5, respectively. We will prove:

Proposition 1. (i) *Let (T_n) be a regularized sequence of $T \in \mathcal{E}'(\Omega)$, i.e. $T_n = T * \theta_n, n \in \mathbb{N}$. If $(\exists m \in \mathbb{R})(\forall (x_n) \in \Omega^{\mathbb{N}})(\forall \alpha \in \mathbb{N}_0)(T_n^{(\alpha)}(x_n) = \mathcal{O}(n^m))$, then $T \in C_0^\infty(\Omega)$. ($\mathcal{O}$ is the Landau symbol.)*

(ii) *Let (T_n) be a regularized sequence of $T \in \mathcal{E}'^k(\Omega)$, $k \in \mathbb{N}_0$. If $(\forall (x_n) \in \Omega^{\mathbb{N}})(\forall \alpha \leq k)(T_n^{(\alpha)}(x_n) = \mathcal{O}(1))$, then $T \in C_0^k(\Omega)$.*

Let (M_p) be a sequence of positive numbers satisfying conditions (M.1)*, (M.2) and (M.3)'. These conditions and the definitions of corresponding ultradistribution spaces will be given in section 1.

Received by the editors February 21, 2000.
2000 *Mathematics Subject Classification.* Primary 46F05, 46F30, 03C20.

Proposition 2. *Let (T_n) be a regularized sequence of $T \in \mathcal{E}'^{(M_p)}(\Omega)$, respectively, $T \in \mathcal{E}'^{\{M_p\}}(\Omega)$, of the form $T_n = T * \phi_n, n \in \mathbb{N}$, where (ϕ_n) is a sequence of mollifiers of M_p-type such that:*

(i) $(\exists h > 0)(\exists m \in \mathbb{R})(\forall (x_n) \in \Omega^{\mathbb{N}})$ *respectively,* $(\forall h > 0)(\exists m \in \mathbb{R})(\forall (x_n) \in \Omega^{\mathbb{N}})$

$$\sup_{\alpha \in \mathbb{N}_0} \frac{h^\alpha |T_n^{(\alpha)}(x_n)|}{M_\alpha} = \mathcal{O}(e^{M^*(mn)}).$$

(ii) $(\exists b > 0)(\forall \phi \in \mathcal{E}^{\{M_p\}}(\Omega))$ *respectively,* $(\exists b > 0)(\forall \phi \in \mathcal{E}^{(M_p)}(\Omega))$

$$\langle T - T_n, \phi \rangle = \mathcal{O}(e^{-M^*(bn)}).$$

Then $T \in \mathcal{D}^{\{M_p\}}(\Omega)$, respectively, $T \in \mathcal{D}^{(M_p)}(\Omega)$.

Note that in Proposition 1 we can use any regularized sequence, while in Proposition 2 we need a special regularized sequence. This is discussed in section 5.

In relation to distribution theory, our results can also be considered in the framework of asymptotic analysis; cf. [4] and references therein.

1. Regularized sequences and regularity properties

We assume that the reader is familiar with Schwartz's distribution theory and its traditional notation. Here, we will recall some basic facts concerning ultradistribution spaces (cf. [6]).

Denote by (M_p) a sequence of positive numbers with $M_0 = 1$ satisfying the assumptions

(M.1)* $(M_p^*)^2 \leq M_{p-1}^* M_{p+1}^*, p \in \mathbb{N}$.

(M.2) $M_p \leq AH^p M_q M_{p-q}, p \in \mathbb{N}, q \leq p$, for some $A > 0$ and $H > 0$.

(M.3)' $\sum_{p=1}^{\infty} M_{p-1}/M_p \leq \infty$.

Recall, $M_0^* = 1, M_p^* = M_p/p!, m_p = M_p/M_{p-1}, m_p^* = M_p^*/M_{p-1}^*, p \in \mathbb{N}$.

(M.1)* implies the well-known condition (M.1) of [6].

We refer the reader to [6], [7] and [9] for the analysis of these conditions.

The associated function M and the growth function M^* related to $(M_p)_p$ are defined by

$$M(t) = \sup_{p \in \mathbf{N}_0} ln \frac{t^p}{M_p}, \;\; M^*(t) = \sup_{p \in \mathbf{N}_0} ln \frac{t^p}{M_p^*}, \; t > 0.$$

We use the convention $M(x) = M(|x|), \; M^*(x) = M^*(|x|), \; x \in \mathbb{R}$ (cf. [6]).

Let Ω be an open set in $\mathbb{R}$. Then $K \subset\subset \Omega$ means that K (or its closure) is a compact subset of Ω. Recall, for $\varphi \in C^\infty(\Omega)$,

$$\|\varphi\|_{K,h,M_p} = \sup_{x \in K, \alpha \in \mathbb{N}_0} \frac{h^\alpha \mid \varphi^{(\alpha)}(x) \mid}{M_\alpha}, \; h > 0, \; K \subset\subset \Omega.$$

Denote by $\mathcal{E}^{M_p,h}(K)$ the space of smooth functions on Ω for which the above seminorm is finite and by $\mathcal{D}^{M_p,h}(K)$ its subspace consisting of smooth functions supported by K.

Define

$$\mathcal{E}^{(M_p)}(\Omega) = \text{proj} \lim_{K \to \Omega} \text{proj} \lim_{h \to \infty} \mathcal{E}^{M_p,h}(K) = \text{proj} \lim_{K \to \Omega} \mathcal{E}^{(M_p)}(K),$$

$$\mathcal{E}^{\{M_p\}}(\Omega) = \text{proj} \lim_{K \to \Omega} \text{ind} \lim_{h \to 0} \mathcal{E}^{M_p,h}(K) = \text{proj} \lim_{K \to \Omega} \mathcal{E}^{\{M_p\}}(K),$$

$$\mathcal{D}^{(M_p)}(\Omega) = \text{ind} \lim_{K \to \Omega} \text{proj} \lim_{h \to \infty} \mathcal{D}^{M_p,h}(K) = \text{ind} \lim_{K \to \Omega} \mathcal{D}^{(M_p)}(K),$$

$$\mathcal{D}^{\{M_p\}}(\Omega) = \text{ind} \lim_{K \to \Omega} \text{ind} \lim_{h \to 0} \mathcal{D}^{M_p,h}(K) = \text{ind} \lim_{K \to \Omega} \text{ind} \mathcal{D}^{\{M_p\}}(K).$$

($K \to \Omega$ means that K runs over all compact sets exhausting Ω.)

Their strong duals are spaces of compactly supported and general Beurling and Roumieu ultradistributions, respectively (cf. [6]).

2.

Let $T \in \mathcal{D}'(\Omega), suppT = K \subset\subset \Omega,\ \theta \in \mathcal{D},\ \int \theta(x)dx = 1$ and $\theta_n(x) = n\theta(nx),\ n \in \mathbb{N}$. Let $T_n(x) = T * \theta_n(x),\ n \in \mathbb{N}$. For any $\psi \in \mathcal{E}$ we have

$$\langle T_n - T, \psi \rangle = \langle T, \psi * \check{\theta}_n - \psi \rangle = \mathcal{O}(n^{-1}).$$

This is clear because, by Taylor's formula, one has (for some $C > 0$, some compact set $K_1, K_1 \supset K$ and $k \in \mathbb{N}$)

$$\begin{aligned} |\langle T * \theta_n - T, \psi \rangle| &\le C|\psi * \check{\theta}_n - \psi|_{K_1,k} \\ &\le C|\int (\psi(x-y) - \psi(x))\check{\theta}_n(ny)ndy|_{K_1,k} \le Cn^{-1}. \end{aligned}$$

Thus, any regularized sequence for a distribution T satisfies condition b) of (i) and (ii) in Theorem 1 given below.

3.

In this section we prove more general assertions than the ones in Proposition 1.

Theorem 1. (i) *Let (f_n) be a sequence of C^∞ functions on Ω supported by a compact set $K \subset\subset \Omega$ and $T \in \mathcal{D}'(\Omega)$, $suppT \subset K$. Assume:*
  a) $(\exists m \in \mathbb{R})(\forall (x_n) \in \Omega^{\mathbb{N}})(\forall \alpha \in \mathbb{N})(f_n^{(\alpha)}(x_n) = \mathcal{O}(n^m))$.
  b) $(\exists b > 0)(\forall \phi \in \mathcal{E}(\Omega))(\langle T - f_n, \phi \rangle = \mathcal{O}(n^{-b}))$.
  Then $T \in C^\infty(\Omega)$.

(ii) *Let (f_n) be a sequence of C^k functions on Ω ($k \in \mathbb{N}$ is fixed) supported by $K \subset\subset \Omega$ and $T \in \mathcal{D}'(\Omega)$, $suppT \subset K$. Assume:*
  a) $(\forall (x_n) \in \Omega^{\mathbb{N}})(\forall \alpha \le k)(f_n^{(\alpha)}(x_n) = \mathcal{O}(1))$.
  b) $(\exists b > 0)(\forall \phi \in \mathcal{E}^k(\Omega))(\langle T - f_n, \phi \rangle = \mathcal{O}(n^{-b}))$.
  Then $T \in C^k(\Omega)$.

Remark 1. (i) If $T \in C_0^\infty(\Omega)$, respectively, $T \in C_0^k(\Omega)$, then one can prove that conditions in (i), respectively, (ii), hold for T and its regularized sequence (T_n).

(ii) Proposition 1 follows from Theorem 1.

Proof. (i) Hypothesis a) is equivalent to

$$\sup_{x \in \Omega} |f_n^{(\alpha)}(x)| = \mathcal{O}(n^m),\ \text{for every } \alpha \in \mathbb{N}.$$

In fact, if it is not so, then there would exist some $\alpha \in \mathbb{N}$ such that

$$(\forall C > 0)(\exists n_C \in \mathbb{N})(\exists x_C \in \Omega)(|f_{n_C}^{(\alpha)}(x_C)| > Cn_C^m).$$

By choosing $C = N \in \mathbb{N}$, we have a sequence (x_N) such that

$$|f_{n_N}^{(\alpha)}(x_N)| > Nn_N^m,\ n \in \mathbb{N}.$$

Putting $y_n = x_N, n_N \leq n < n_{N+1}$ (assuming that $(n_N)_N$ is increasing) we see that there does not exist $C > 0$ such that

$$|f_n^{(\alpha)}(y_n)| \leq Cn^m,\ n \in \mathbb{N}.$$

By taking $N > C$, this inequality does not hold for $y_n,\ n > N$, which contradicts the hypothesis.

Let $a = b/2$. By assumption b), we have

$$\langle n^a(T - f_n), \phi\rangle = \mathcal{O}(n^{-a}),\ \phi \in \mathcal{E}(\Omega),\ n \in \mathbb{N}.$$

Thus, $(n^a(T - f_n))$ is a sequence in $\mathcal{E}'(\Omega)$ converging to 0 as $n \to \infty$. This implies that the convergence of this sequence takes place in some $C^k(\bar{\omega})$, where ω is an open bounded set such that $K \subset\subset \omega \subset\subset \Omega$ and that $(n^a(T - f_n))$ is a bounded sequence in the dual space $(C^k(\bar{\omega}))'$. By the Banach-Steinhaus theorem, we have

$$(\exists C > 0)(\forall \psi \in \mathcal{E}(\Omega))(\forall n \in \mathbb{N})(|n^a\langle T - f_n, \psi\rangle| \leq C||\psi||_{\bar{\omega},k})$$

$(|\psi|_{\bar{\omega},k} = \sup_{x\in\bar{\omega}, p\leq k} |\psi^{(p)}(x)|)$.

If $\psi_y = e^{-iy\cdot}$, we have $||\psi_y||_{\bar{\omega},k} \leq C(1 + |y|)^k$. For Fourier transforms $\hat{T}$ and $\hat{f}_n$, this implies

$$|n^a(\hat{T} - \hat{f}_n)(\xi)| \leq C(1 + |\xi|)^k,\ \xi \in \mathbf{R},\ n \in \mathbb{N}.$$

(C denotes positive constants which can be different.)

Note that $\hat{f}_n \in \mathcal{S}(\mathbb{R}),\ n \in \mathbb{N}$. We have the following estimates coming from the hypothesis on (f_n). For every $r > 0$, there exists $C_r > 0$ such that

$$|\hat{f}_n(\xi)| \leq C_r n^m (1 + |\xi|)^{-r},\ \xi \in \mathbb{R},\ n \in \mathbb{N}.$$

Thus, for $\xi \in \mathbb{R},\ n \in \mathbb{N}$,

$$|\hat{T}(\xi)| \leq n^{-a}C(1 + |\xi|)^k + n^m C_r(1 + |\xi|)^{-r}.$$

Now, for given $\xi \in \mathbb{R}$ we choose n such that

$$n^{-1} = [(1 + |\xi|)^{\frac{-p-k}{a}}].$$

By putting this in the previous inequality, we obtain

$$|\hat{T}(\xi)| \leq C((1 + |\xi|)^{-p} + (1 + |\xi|)^{\frac{p+k}{a} - r}),\ \xi \in \mathbb{R}.$$

Choose r such that $\frac{p+k}{a} - r < -p$. Thus, we conclude that $\hat{T}$ is of rapid decrease.

The above evaluation can be repeated for any derivative of $\hat{T}$. This implies that $\hat{T} \in \mathcal{S}(\mathbb{R})$ and that T is a smooth function.

(ii) The proof is similar to the proof of assertion (i). We will point out the differences.

The hypothesis a) is equivalent to $\sup_{x\in\Omega} |f_n^{(\alpha)}(x)| = \mathcal{O}(1)$, for every $\alpha \leq k$. With $a = b/2$, assumption b) implies

$$\langle n^a(T - f_n), \phi\rangle = \mathcal{O}(n^{-a}),\ \phi \in \mathcal{E}^k(\Omega),\ n \in \mathbb{N},$$

and that $(n^a(T - f_n))$ is a sequence in $\mathcal{E}'^k(\Omega)$ which converges to 0. This implies (again by the use of the Banach-Steinhaus theorem and the Paley-Wiener theorem; cf. [5])

$$|n^a(\hat{T} - \hat{f}_n)(\xi)| \leq C(1 + |\xi|)^k,\ \xi \in \mathbb{R},\ n \in \mathbb{N}.$$

Assumptions on (f_n) imply

$$|\hat{f}_n(\xi)| \leq C(1+|\xi|)^k,\ \xi \in \mathbb{R},\ n \in \mathbb{N}.$$

With $\xi \in \mathbb{R},\ n \in \mathbb{N}$ and suitable $C > 0$ we have,

$$|\hat{T}(\xi)| \leq n^{-a} C(1+|\xi|)^k + C(1+|\xi|)^{-k}.$$

By choosing $n = n(\xi)$ such that $n^{-1} = [(1+|\xi|)^{-2k}]$, we obtain

$$|\hat{T}(\xi)| \leq C(1+|\xi|)^{-k},\ \xi \in \mathbb{R}.$$

By the Paley-Wiener theorem, we have $T \in C^k(\Omega)$. □

Remark 2. If T is not a compactly supported distribution, we use a partition of unity and obtain a criterion for the local regularity.

4.

With arguments similar to those in section 3, by using the Banach-Steinhaus theorem and Paley-Wiener type theorems for ultradifferentiable functions and ultradistributions (cf. [6], [7]), we can prove Theorem 2, which includes Proposition 2.

Theorem 2. (i) *Let (f_n) be a sequence in $\mathcal{E}^{\{M_p\}}(\Omega)$, respectively, $\mathcal{E}^{(M_p)}(\Omega)$, supported by $K \subset\subset \Omega$ and $T \in \mathcal{E}'^{\{M_p\}}(\Omega)$, respectively, $T \in \mathcal{E}'^{(M_p)}(\Omega)$, supp$T \subset K$. Assume:*

a) $(\exists h > 0)(\exists m \in \mathbb{R})(\forall (x_n) \in \Omega^{\mathbb{N}})$ *respectively,* $(\forall h > 0)(\exists m \in \mathbb{R})(\forall (x_n) \in \Omega^{\mathbb{N}})$,

$$\sup_{\alpha \in \mathbb{N}_0} \frac{h^\alpha |f_n^{(\alpha)}(x_n)|}{M_\alpha} = \mathcal{O}(e^{M^*(mn)}).$$

b) $(\exists b > 0)(\forall \phi \in \mathcal{E}^{\{M_p\}}(\Omega))$ *respectively,* $(\exists b > 0)(\forall \phi \in \mathcal{E}^{(M_p)}(\Omega))$,

$$\langle T - f_n, \phi\rangle = \mathcal{O}(e^{-M^*(bn)}).$$

Then $T \in \mathcal{D}^{\{M_p\}}(\Omega)$, respectively, $T \in \mathcal{D}^{(M_p)}(\Omega)$.

Proof. Hypothesis a) is equivalent to

$$\sup_{x \in \Omega} \frac{h^\alpha}{M_\alpha} |f_n^{(\alpha)}(x)| = \mathcal{O}(e^{M^*(mn)}).$$

Since

$$|\xi^\alpha \int_{\mathbb{R}} e^{-i\xi x} f_n(x) dx| = |\int_{\mathbb{R}} e^{-ix\xi} f_n^{(\alpha)}(x) dx| \leq C \sup_{x \in K} |f_n^{(\alpha)}(x)|,\ \xi \in \mathbb{R},$$

the definition of the associated function M implies that there exists $C_h > 0$ such that

$$|\widehat{f_n}(\xi)| \leq C_h e^{M^*(mn)} e^{-M(h\xi)}, \quad \xi \in \mathbb{R},\ \varepsilon \in (0,1),\ n \in \mathbb{N}. \tag{1}$$

Let $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R})$. Put $S_n = e^{M^*(an)}(T - f_n,)$ where we choose $a < b$ such that condition b) implies $S_n \to 0$ in $\mathcal{E}'^{\{M_p\}}(\Omega)$ as $n \to \infty$. By [6], there exists a compact set $K_1 \supset K$ such that for every $k_1 > 0$ there exists $C_1 > 0$ such that

$$|\langle S_n, \varphi\rangle| \leq C_1 \|\varphi\|_{K_1, k_1, M_p}, \quad \varphi \in \mathcal{E}^{(M_p)}(\mathbb{R}). \tag{2}$$

Therefore, (2) holds for every $\varphi \in \mathcal{E}^{k_1,M_p}(K_1)$. Note,

$$\|e^{i\xi x}\|_{K_1,k_1,M_p} \leq e^{M(k_1\xi)}, \quad \xi \in \mathbb{R}. \tag{3}$$

Thus, by (2) and (3), and by letting $\phi(x) = e^{i\xi x},\ x \in \mathbb{R}$, it follows that there exists $C > 0$ such that

$$|\hat{S}_n(\xi)| = e^{M^*(an)}|\widehat{T}(\xi) - \widehat{G_\varepsilon}(\xi)| \leq Ce^{M(k_1\xi)}, \quad \xi \in \mathbb{R},\ n \in \mathbb{N}.$$

This and (1) imply

$$|\widehat{T}(\xi)| \leq C(e^{-M^*(an)+M(k_1\xi)} + e^{M^*(rn)-M(h\xi)}),\ \xi \in \mathbb{R},\ n \in \mathbb{N}. \tag{4}$$

Let

$$n = \left[\frac{M^{*-1}(M((h-\delta)\xi))}{r}\right],\ \xi \in \mathbb{R}. \tag{5}$$

Note that $M^{*-1}(M((h-\delta)\xi)) \to \infty$ as $|\xi| \to \infty$. We choose constants $\delta < h$ and k_1 in (5) as follows. By (M.2), we have that there exist $C > 0$ and $c > 0$ such that

$$e^{-M^*(\frac{a}{r}t)} \leq Ce^{-cM^*(t)},\ t > 0$$

(cf. [6]). This implies that we have to choose k_1 such that there exists $s_1 > 0$ such that

$$-cM((h-\rho)\xi) + M(k_1\xi) \leq -M(s_1\xi),\ \xi \in \mathbb{R}.$$

Again, this is possible by (M.2). With this δ, k_1 and s_1 and by putting n of the form (5), we have

$$e^{-M^*(an)+M(k_1\xi)} \leq e^{-M(s_1\xi)},$$

$$e^{M^*(rn)-M(h\xi)} \leq e^{M((h-\delta)\xi)-M(h\xi)},\ \xi \in \mathbb{R}.$$

This implies that for (4) there exist $s > 0$ and $C_s > 0$ such that

$$|\widehat{T}(\xi)| \leq C_s e^{-M(s\xi)}, \quad \xi \in \mathbb{R}.$$

By Lemma 3.3 and Theorem 9.1 in [6], this implies that $T \in \mathcal{E}^{\{M_p\}}(\mathbb{R}),\ \operatorname{supp} T \subset K$. □

5.

We finish with the following question:

Does there exist a sequence of mollifiers (ϕ_n) in an appropriate space of ultradifferentiable functions such that condition (ii) of Proposition 2 holds?

We have proved in [11] the following result:

Proposition 3. *Let ϕ be a mollifier of $\{p!^{1+\rho}\}$-type.*

Let $f \in \mathcal{E}'^{(M_p)}(\Omega)$. Then for every $b > 0$,

$$\langle f * \phi_n - f, \psi\rangle = \mathcal{O}(e^{-M^*(bn)}), \psi \in \mathcal{D}^{(\frac{M_p}{p!^{1+\rho}})}(\Omega).$$

If $f \in \mathcal{E}'^{\{M_p\}}(\Omega)$, then for every $b > 0$,

$$\langle f * \phi_n - f, \psi\rangle = \mathcal{O}(e^{-M^*(bn)}), \psi \in \mathcal{D}^{\{\frac{M_p}{p!^{1+\rho}}\}}(\Omega).$$

The sequence of mollifiers (ϕ_n) is defined as follows:

Let $\theta \in \mathcal{D}^{\{p!^{1+\rho}\}}(\mathbb{R})$ be equal to one in a neighbourhood of zero. Then, its Fourier transformation $\phi = \mathcal{F}(\theta) = \hat{\theta}$ satisfies

$$\int_{\mathbb{R}} \phi(t)dt = 1,\ \int_{\mathbb{R}} t^n \phi(t)dt = 0,\ n = 1, 2, ...$$

Moreover, ϕ satisfies

$$\sigma_{h,p!^{1+\rho}}(\phi) = \sup_{x\in\mathbb{R},k,p\in\mathbb{N}_0} \frac{(1+|x|)^k |\phi^{(p)}(x)|}{h^{k+p} k!^{1+\rho} p!^{1+\rho}} < \infty \text{ for some } \ h > 0 \ \text{ (cf. [10])}.$$

Then ϕ is called the mollifier of $\{p!^{1+\rho}\}$-type; $\phi = n\phi(n\cdot)$ is a sequence of mollifiers.

Acknowledgement

We would like to thank ESI in Vienna, for its support. Also, we thank the referee for comments concerning the motivation behind the paper.

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