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INVERSE SCATTERING FOR THE NONLINEAR SCHRÖDINGER EQUATION II. RECONSTRUCTION OF THE POTENTIAL AND THE NONLINEARITY IN THE MULTIDIMENSIONAL CASE

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Abstract. We solve the inverse scattering problem for the nonlinear Schrödinger equation on $\mathbf{R}^n, n \geq 3$:

$$i\frac{\partial}{\partial t}u(t,x) = -\Delta u(t,x) + V_0(x)u(t,x) + \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u(t,x).$$

We prove that the small-amplitude limit of the scattering operator uniquely determines $V_j, j = 0, 1, \cdots$. Our proof gives a method for the reconstruction of the potentials $V_j, j = 0, 1, \cdots$. The results of this paper extend our previous results for the problem on the line.

1. Introduction

Let us consider the following nonlinear Schrödinger equation with a potential:

$$(1.1) \qquad i\frac{\partial}{\partial t}u(t,x) = -\Delta u(t,x) + V_0(x)u(t,x) + F(x,u), u(0,x) = \phi(x),$$

where $t \in \mathbf{R}, x \in \mathbf{R}^n, n \geq 3$. The potential, V_0, is a real-valued function, $F(x,u)$ is a complex-valued function, and $\Delta := \sum_{j=1}^n D_j^2$. We use the standard notation, $D_j := \frac{\partial}{\partial x_j}$ and for $\alpha := (\alpha_1, \cdots, \alpha_n)$, $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, with $|\alpha| := \sum_{j=1}^n \alpha_j$. We first construct the scattering operator for the nonlinear Schrödinger equation (1.1). For this purpose we introduce some assumptions and definitions.

Assumption A. Let p satisfy $\rho < p < 1 + \frac{4}{n-2}$, where ρ is the positive root of $\frac{n}{2}\frac{\rho-1}{\rho+1} = \frac{1}{\rho}$. Let k be an integer such that $k > \frac{n}{p+1}$. Let $F = F_1 + iF_2$ with F_1, F_2 real-valued, and $u = r_1 + ir_2, r_1, r_2 \in \mathbf{R}$. We suppose that $F(0) = 0$ and that for all integers β with $1 \leq \beta \leq k+1$ and all α with $\beta + |\alpha| \leq k+1$, we have that

$$(1.2) \qquad \sum_{j=1}^{2} \left| \frac{\partial^\beta}{\partial r_1^{\beta_1} \partial r_2^{\beta_2}} D^\alpha F_j(x,u) \right| \leq C|u|^{\max[0,p-\beta]} \text{ for } |u| \leq \gamma,$$

for some $\gamma > 0$, and for all nonnegative integers, β_1, β_2, with $\beta = \beta_1 + \beta_2$. □

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We denote by H_0 the self-adjoint realization of $-\Delta$ in $L^2(\mathbf{R}^n)$ with domain the Sobolev space $W_{2,2}$. For the definition of the Sobolev spaces $W_{j,p}, j = 1, 2, \cdots, 1 \leq p \leq \infty$, see [1].

Assumption B. We assume that V_0 is real valued and that for some $\delta > (3n/2)+1$,

$$\sup_{x\in\mathbf{R}^n} (1+|x|)^{\delta} \left(\int_{|x-y|\leq 1} |D^{\alpha} V_0(y)|^{p_0} \, dy \right)^{1/p_0} < \infty, \tag{1.3}$$

for all $|\alpha| \leq k + k_0$, with k as in Assumption A. If $n = 3$, $p_0 = 2$ and $k_0 = 0$, and if $n \geq 4$, $p_0 > n/2$ and $k_0 := [(n-1)/2]$, where $[\sigma]$ denotes the integral part of σ. Moreover, assume that zero is neither an eigenvalue nor half-bound state (a resonance) of $H := H_0 + V_0$. □

Zero is said to be a half-bound state of H if the equation $H\phi = 0$ has a solution $\phi \notin L^2(\mathbf{R}^n)$, such that $(1+|x|)^{-1-\epsilon}\phi \in L^2(\mathbf{R}^n)$ for all $\epsilon > 0$.

Under Assumption B H is self-adjoint with domain $W_{2,2}$ and it has no singular-continuous spectrum and no positive eigenvalues [5]. Moreover, the wave operators

$$W_{\pm} := s - \lim_{t\to\pm\infty} e^{itH} \, e^{-itH_0} \tag{1.4}$$

exist and Range$W_{\pm} = \mathcal{H}_c$, the subspace of continuity of H. The scattering operator for the linear Schrödinger equation (equation (1.1) with $F = 0$) is given by

$$S_L := W_+^* \, W_-. \tag{1.5}$$

Actually, these results are true under more general conditions. The crucial issue is that Yajima has proven that under Assumption B the wave operators and the adjoints, $W_{\pm}^*$, are bounded operators on $W_{l,p}, l = 0, 1, \cdots, k, 1 \leq p \leq \infty$. For this result see Theorem 1.2 of [25] and also [24]. This result and the intertwining relations for the wave operators, $e^{-itH}P_c = W_{\pm}e^{-itH_0}W_{\pm}^*$, imply that the following $L^p - L^{\acute{p}}$ estimate follows from the corresponding result for H_0 (see [24]):

$$\left\| e^{-itH} P_c \right\|_{\mathcal{B}(L^p, L^{\acute{p}})} \leq \frac{C}{t^{n(\frac{1}{p}-\frac{1}{2})}}, \, t > 0, \tag{1.6}$$

for $1 \leq p \leq 2$ and where $\frac{1}{p} + \frac{1}{\acute{p}} = 1$. By P_c we denote the orthogonal projector onto $\mathcal{H}_c$. For any pair of Banach spaces X, Y, we denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded operators from X into Y. The $L^p - L^{\acute{p}}$ estimate in $\mathbf{R}^n, n \geq 3$, was first proven, under slightly different conditions, in [8].

The results of Yajima [24] allow us to extend to the case of $n \geq 3$ the method for the construction of the scattering operator for (1.1) and for the solution of the inverse scattering problem that we gave in [23] in the case of $n = 1$. The $L^p - L^{\acute{p}}$ estimate and the continuity of the wave operators on $W_{k,p}$ for the problem on the line was proven in [20] and [21] (see also [6]).

Let us denote [13],

$$N_{\delta}(V_0) := \sup_{x\in\mathbf{R}^n} \left[\int_{|x-y|<\delta} |V(y)|^{p_0} dy \right]^{1/p_0}. \tag{1.7}$$

Assumption C. We assume that $N_{\delta}(D^{\alpha}V_0) < \infty, \delta > 0$, and that

$$\lim_{\delta\to 0} N_{\delta}(D^{\alpha}V_0) = 0, \tag{1.8}$$

where $|\alpha| \leq k - 1$, with k and p_0 as in Assumption B.

We designate

$$M := \left\{ u \in C(\mathbf{R}, W_{k,p+1}) : \sup_{t\in\mathbf{R}} (1+|t|)^d \|u\|_{W_{k,p+1}} < \infty \right\}, \quad \text{with norm} : \|u\|_M := \sup_{t\in\mathbf{R}} (1+|t|)^d \|u\|_{W_{k,p+1}}, \tag{1.9}$$

where $d := \frac{n}{2}\frac{p-1}{p+1}$. For functions $u(t,x)$ defined in $\mathbf{R}^{n+1}$ we denote $u(t)$ for $u(t,\cdot)$. In the following theorem we construct the small-amplitude scattering operator.

Theorem 1.1. *Suppose that Assumptions* A, B *and* C *are satisfied and that* H *has no eigenvalues. Then, there is a* $\delta > 0$ *such that for all* $\phi_- \in W_{k+1,2} \cap W_{k,1+\frac{1}{p}}$ *with* $\|\phi_-\|_{W_{k+1,2}} + \|\phi_-\|_{W_{k,1+\frac{1}{p}}} \leq \delta$ *there is a unique solution,* u*, to* (1.1) *such that* $u \in C(\mathbf{R}, W_{k,2}) \cap M$ *and*

$$\lim_{t\to-\infty} \|u(t) - e^{-itH}\phi_-\|_{W_{k,2}} = 0. \tag{1.10}$$

Moreover, there is a unique $\phi_+ \in W_{k,2}$ *such that*

$$\lim_{t\to\infty} \|u(t) - e^{-itH}\phi_+\|_{W_{k,2}} = 0. \tag{1.11}$$

Furthermore, $e^{-itH}\phi_\pm \in M$ *and*

$$\left\| u - e^{-itH}\phi_\pm \right\|_M \leq C \left\| e^{-itH}\phi_\pm \right\|_M^p, \tag{1.12}$$

$$\|\phi_+ - \phi_-\|_{W_{k,2}} \leq C \left[\|\phi_-\|_{W_{k+1,2}} + \|\phi_-\|_{W_{k,1+\frac{1}{p}}} \right]^p. \tag{1.13}$$

The scattering operator $S_{V_0} : \phi_- \hookrightarrow \phi_+$ *is injective on* $W_{k,1+\frac{1}{p}} \cap W_{k+1,2}$.

Note that in Theorem 1.1 we do not restrict F in such a way that energy is conserved. Moreover, for $n = 3, \rho = 2$ and $\lim_{n\to\infty} \rho = 1$. We prove Theorem 1.1 in Section 2 extending to this case the proof given in [23] in the case of $n = 1$. We construct the solution $u(t,x)$ by means of the contraction mapping theorem in a ball, M_R, of M with small enough radius, R. It follows from Sobolev's imbedding theorem [1] that $|u(t,x)| < \gamma$, $t \in \mathbf{R}, x \in \mathbf{R}^n$, for all $u(t,x) \in M_R$. This is the reason why we only have to assume that (1.2) holds for $|u| \leq \gamma$. For results on scattering for the nonlinear Schrödinger equation in the case where $V_0 = 0$ see [16], [17], [18], [10], [9], [11], [3], [7], [2] and the references quoted there. In [8] the direct scattering for (1.1) with $F = F(u)$ was studied for $n \geq 3$. The corresponding inverse problem was considered in [19]. For the case of the nonlinear Klein-Gordon equation on the line see [22].

Since we wish to reconstruct the potential, V_0, we consider the scattering operator that relates asymptotic states that are solutions to the linear Schrödinger equation with potential zero ((1.1) with $V_0 = F = 0$):

$$S := W_+^* \, S_{V_0} \, W_-. \tag{1.14}$$

The first step is to reconstruct S_L from S.

Theorem 1.2. *Suppose that the assumptions of Theorem* 1.1 *are satisfied. Then for every* $\phi_- \in W_{k+1,2} \cap W_{k,1+\frac{1}{p}}$,

$$\frac{d}{d\epsilon} S(\epsilon\phi)\Big|_{\epsilon=0} = S_L\phi, \tag{1.15}$$

where the derivative in the left-hand side of (1.15) *exists in the strong convergence in* $W_{k,2}$.

Corollary 1.3. *Under the conditions of Theorem* 1.1 *the scattering operator,* S, *determines uniquely the potential* V_0.

Proof. By Theorem 1.2 S uniquely determines S_L. From S_L we uniquely reconstruct the potential V_0 using the known results on the inverse scattering problem for the linear Schrödinger equation. See [4]. □

Let us now consider the case where $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$. As we prove below we can also reconstruct the $V_j, j = 1, 2, \cdots$.

Lemma 1.4. *Suppose that the conditions of Theorem* 1.1 *are satisfied, and moreover, that* $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$, *for* $|u| \leq \gamma$, *for some* $\gamma > 0$, *where* j_0 *is an integer such that* $j_0 \geq (p-3)/2$, *and where* $V_j \in W_{k,\infty}$ *with* $\|V_j\|_{W_{k,\infty}} \leq C^j, j = 1, 2, \cdots$, *for some constant* C. *Then, for any* $\phi \in W_{k+1,2} \cap W_{k,1+\frac{1}{p}}$ *there is an* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$:

(1.16)

$$i\left((S_{V_0} - I)(\epsilon\phi),\, \phi\right)_{L^2} = \sum_{j=1}^{\infty} \epsilon^{2(j_0+j)+1}\left[\int\int dt\, dx\, V_j(x)\left|e^{-itH}\phi\right|^{2(j_0+j+1)} + Q_j\right],$$

where $Q_1 = 0$ *and* $Q_j, j > 1$, *depends only on* ϕ *and on* V_l *with* $l < j$. *Moreover, for any* $\acute{x} \in \mathbf{R}$, *and any* $\lambda \geq 1$, *we denote* $\phi_\lambda(x) := \phi(\lambda(x - \acute{x}))$. *Then, if* $\phi \neq 0$,

$$V_j(\acute{x}) = \frac{\lim_{\lambda\to\infty} \lambda^{n+2} \int\int dt\, dx\, V_j(x)\left|e^{-itH}\phi_\lambda\right|^{2(j_0+j+1)}}{\int\int dt\, dx\, \left|e^{-itH_0}\phi\right|^{2(j_0+j+1)}}. \tag{1.17}$$

Corollary 1.5. *Under the conditions of Lemma* 1.4 *the scattering operator,* S, *determines uniquely the potentials* $V_j, j = 0, 1, \cdots$.

Proof. By Corollary 1.3, S uniquely determines V_0. Then the wave operators, $W_\pm$, are uniquely determined, and by (1.14), S uniquely determines S_{V_0}. Finally by (1.16) and (1.17) S_{V_0} uniquely determines $V_j, j = 1, 2, \cdots$.

We reconstruct the potentials $V_j, j = 0, 1, \cdots$, in the following way. First we obtain S_L from S using (1.15). By the method in [4] for inverse scaterring for the linear Schrödinger equation we reconstruct V_0. We then reconstruct S_{V_0} from S using (1.14). Finally (1.16) and (1.17) give us, recursively, $V_j, j = 1, 2, \cdots$. Our formula (1.17) is an extension to our case of the reconstruction algorithm of [15]. In [15] Strauss proved that in the case $V_0 = 0$ and $F(x,u) = V(x)|u|^{p-1}u$, $x \in \mathbf{R}^n$, $p > 4$ if $n = 1$, $p > 3$ if $n = 2$, $p \geq 3$ if $n \geq 3$, and $V(x)$ a real-valued potential whose derivatives up to order l are bounded, with $l > 3n/4$; then, the scattering operator uniquely determines V.

2. Scattering

By Theorem 3 on page 135 of [14],

$$\|\mathcal{F}^{-1}(1+q^2)^{k/2}(\mathcal{F}f)(q)\|_{L^p} \tag{2.1}$$

is a norm that is equivalent to the norm of $W_{k,p}, 1 < p < \infty$. $\mathcal{F}$ denotes the Fourier transform. Then, by equation (1.2) of [24]

$$\|(I+H)^{l/2} f\|_{L^p} \tag{2.2}$$

defines a norm that is equivalent to the norm of $W_{l,p}, l = 0, 1, \cdots, k, 1 < p < \infty$. We will use this equivalence below without further comments. This implies that estimate (1.6) holds in the norm on $\mathcal{B}(W_{l,p}, W_{l,\acute{p}}), l = 0, 1, \cdots, k$.

The following inequality is proven in Theorem 9.2 on page 141 of [13]:

$$\|(D^\alpha V_0)\phi\|_{L^2} \leq C_1 N_\delta(D^\alpha V_0) \|\phi\|_{W_{2,2}} + C_2 N_1(D^\alpha V_0) \|\phi\|_{L^2}, \tag{2.3}$$

where C_1 is independent of δ. Let us denote $R(\rho) := (H+\rho)^{-1}$ and $R_0(\rho) := (H_0+\rho)^{-1}$. Equation (2.3) implies that if Assumption C holds, given $a < 1$, there is $\rho_0 > 0$ such that

$$\|V_0 R_0(\rho)\|_{\mathcal{B}(L^2)} \leq a < 1 \tag{2.4}$$

for all $\rho \geq \rho_0 > 0$. Moreover, ρ_0 depends on V_0 only through $N_\delta(V_0)$. It follows that

$$R(\rho) = R_0(\rho)\left(I + V_0 R_0(\rho)\right)^{-1} = R_0(\rho)\sum_{l=0}^{\infty}(-1)^l (V_0 R_0(\rho))^l \tag{2.5}$$

for all $\rho \geq \rho_0$. Taking derivatives in (2.5) term by term we prove that

$$\|R(\rho)\|_{\mathcal{B}(W_{j,2}, W_{j+2,2})} \leq C, j = 0, 1, 2, \cdots, k-1. \tag{2.6}$$

It follows that if k is odd,

$$\left\|(H_0+\rho)^{(k+1)/2}(R(\rho))^{(k+1)/2}\right\|_{\mathcal{B}(L^2)} \leq C. \tag{2.7}$$

Then, for some constants C_1, C_2,

$$C_1 \|\phi\|_{W_{k+1,2}} \leq \left\|(I+H)^{(k+1)/2}\phi\right\|_{L^2} \leq C_2 \|\phi\|_{W_{k+1,2}}. \tag{2.8}$$

In the case when k is even we have that

$$R(\rho)^{(k+1)/2} = R(\rho)^{k/2}(H+\rho)^{-1/2}. \tag{2.9}$$

Again using Theorem 9.2 on page 141 of [13] we obtain that

$$\left\||V_0|^{1/2}\phi\right\| \leq C_1[N_\delta(V_0)]^{1/2}\|\phi\|_{W_{1,2}} + C_2[N_1(V_0)]^{1/2}\|\phi\|_{L^2}. \tag{2.10}$$

Then, if ρ is large enough,

$$\left\|(H+\rho)^{1/2}\phi\right\|^2 = ((H_0+V_0+\rho)\phi, \phi) \geq C\|\phi\|^2_{W_{1,2}}, \tag{2.11}$$

and we have that

$$\left\|(H+\rho)^{-1/2}\right\|_{\mathcal{B}(L^2, W_{1,2})} \leq C. \tag{2.12}$$

Hence, by (2.6), (2.9) and (2.12), equation (2.8) also holds for k even.

The proofs of Theorem 1.1, Theorem 1.2, and Lemma 1.4 follow as in [23]. We give details below for the convenience of the reader.

Proof of Theorem 1.1. By Sobolev's imbedding theorem [1] L^∞ is continuously imbedded in $W_{k,1+p}$ and it follows by standard arguments (see [9] and (2.16) below) that $u \in C(\mathbf{R}, W_{k,2}) \cap M$ is a solution to (1.1) with $\lim_{t\to-\infty}\|u(t) - e^{-itH}\phi\|_{W_{k,2}} = 0$ for some $\phi \in W_{k,2}$, if and only if u is a solution to the following integral equation:

$$u(t) = e^{-itH}\phi + \frac{1}{i}\int_{-\infty}^{t} e^{-i(t-\tau)H}F(x, u(\tau))\,d\tau. \tag{2.13}$$

Let us designate

$$\mathcal{Q}u(t) := \frac{1}{i}\int_{-\infty}^{t} e^{-i(t-\tau)H} F(x,u(\tau))\,d\tau. \tag{2.14}$$

For $R>0$ let us denote $M_R := \{u \in M : \|u\|_M \le R\}$. By Assumption A, (1.6) and since $L_\infty \subset W_{k,p+1}$, there is an $R_0>0$ such that if $u \in M_{R_0}$,

$$\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\|_{W_{k,p+1}} \le C\,(1+|t|)^{-d}(\|u\|_M + \|v\|_M)^{p-1}\|u-v\|_M, \tag{2.15}$$

where we used that $d>1$ and that $p\,d>1$. Moreover, by (2.15) with $v(t)=0$,

$$\begin{aligned} \|\mathcal{Q}u(t)\|^2_{W_{k,2}} &\le C\Re\int_{-\infty}^{t} d\tau\,\left((I+H)^{k/2}F(x,u(\tau)), (I+H)^{k/2}\mathcal{Q}u(\tau)\right)_{L^2} \\ &\le C\int_{-\infty}^{t} d\tau\,\|F(x,u)(\tau)\|_{W_{k,1+1/p}} \times (1+|\tau|)^{-d}\,\|u\|^p_M \\ &\le C\int_{-\infty}^{t} d\tau\,\|u\|^p_{W_{k,p+1}}\,(1+|\tau|)^{-d}\|u\|^p_M \\ &\le C\int_{-\infty}^{t} d\tau\,(1+|\tau|)^{-d(p+1)}\,\|u\|^{2p}_M \\ &\le C(1+\max[0,-t])^{-(d+dp-1)}\,\|u\|^{2p}_M. \end{aligned} \tag{2.16}$$

Let us first prove the uniqueness. For u, v any pair of solutions to (1.1) that satisfy (1.10) we have that

$$u(t)-v(t) = \mathcal{Q}u(t) - \mathcal{Q}v(t). \tag{2.17}$$

Let us denote $u_T := \chi_{(-\infty,T)}(t)\,u(t)$, where $\chi_{(-\infty,T)}(t)$ is the characteristic function of $(-\infty,T), T\in \mathbf{R}$. v_T is similarly defined. It follows from (2.17) that

$$\|u_T(t)-v_T(t)\|_{\tilde{M}} < 1/2\|u_T(t)-v_T(t)\|_{\tilde{M}} \text{ for some } T \text{ negative enough,} \tag{2.18}$$

where $\tilde{M}$ is defined as M, but with a slightly smaller p. Then, $u(t)=v(t)$ for $t\le T$, and the standard uniqueness result implies that $u=v$. The uniqueness of ϕ_+ is obvious by the unitarity of e^{-itH} in L^2.

By Sobolev's imbedding theorem,

$$\begin{aligned} \|e^{-itH}\phi_-\|_{W_{k,p+1}} &\le C\|e^{-itH}\phi_-\|_{W_{k+1,2}} \le C\left\|(H+I)^{(k+1)/2}e^{-itH}\phi_-\right\|_{L^2} \\ &= C\left\|(H+I)^{(k+1)/2}\phi_-\right\|_{L^2} \le C\,\|\phi_-\|_{W_{k+1,2}}. \end{aligned} \tag{2.19}$$

By (1.6) and (2.19):

$$\left\|e^{-itH}\phi_-\right\|_M \le C\left[\|\phi_-\|_{W_{k+1,2}} + \|\phi_-\|_{W_{k,1+\frac{1}{p}}}\right]. \tag{2.20}$$

Let us take $R\le R_0$ so small that $C(2R)^{p-1}\le 1/2$, with C as in (2.15), and $\delta>0$ such that $C\delta \le R/4$, with C as in (2.20). Then, the map $u \hookrightarrow e^{-itH}\phi_- + \mathcal{Q}\,u$ is a contraction from M_R into M_R for all $\phi_- \in W_{k+1,2}\cap W_{k,1+\frac{1}{p}}$ with $\|\phi_-\|_{W_{k+1,2}} +$

$\|\phi_-\|_{W_{k,1+\frac{1}{p}}} \leq \delta$. The contraction mapping theorem implies that this map has a unique fixed point that is a solution to (2.13) in M_R. Moreover,

$$\|u\|_M \leq \left\|e^{-itH}\phi_-\right\|_M + \frac{1}{2}\|u\|_M, \tag{2.21}$$

and then

$$\|u\|_M \leq C\left\|e^{-itH}\phi_-\right\|_M. \tag{2.22}$$

Equation (1.12) for ϕ_- follows from (2.13), (2.15) with $v = 0$ and (2.22). By (2.13) and (2.16) $u \in C(\mathbf{R}, W_{k,2})$ and (1.10) holds.

We now define

$$\phi_+ = \phi_- + \frac{1}{i}\int_{-\infty}^{\infty} e^{i\tau H}F(x, u(\tau))\,d\tau. \tag{2.23}$$

Estimating as in (2.16) we prove that $\phi_+ \in W_{k,2}$ and that

$$\|\phi_+ - \phi_-\|_{W_{k,2}} \leq C\|u\|_M^p. \tag{2.24}$$

Equation (1.13) follows from (2.20), (2.22) and (2.24). By (2.13) and (2.23)

$$u(t) = e^{-itH}\phi_+ - \frac{1}{i}\int_t^{\infty} e^{-i(t-\tau)H}F(x, u(\tau))\,d\tau. \tag{2.25}$$

We prove (1.11) estimating as in (2.16). In a similar way we prove that

$$\left\|\int_t^{\infty} e^{-i(t-\tau)H}F(x, u(\tau))\,d\tau\right\|_M \leq C\|u\|_M^p, \tag{2.26}$$

and it follows that

$$\|u\|_M \leq C\left\|e^{-itH}\phi_+\right\|_M. \tag{2.27}$$

Equation (1.12) for ϕ_+ follows from (2.25), (2.26) and (2.27). We prove that S_{V_0} is injective as in the proof of uniqueness above.

Proof of Theorem 1.2. Since $S(0) = 0$ and $W_\pm$ are bounded on $W_{k,2}$, it is enough to prove that

$$s - \lim_{\epsilon \downarrow 0}\frac{1}{\epsilon}\left(S_V(\epsilon\phi) - \epsilon\phi\right) = 0. \tag{2.28}$$

By (2.20) and (2.22) with ϕ_- replaced by $\epsilon\phi$ we have

$$\|u\|_M \leq C\epsilon\left[\|\phi_-\|_{W_{k+1,2}} + \|\phi_-\|_{W_{k,1+\frac{1}{p}}}\right]. \tag{2.29}$$

To prove (2.28) we estimate the integral on the right-hand side of (2.23) as in (2.16), with the aid of (2.29).

Proof of Lemma 1.4. By the contraction mapping theorem,

$$u(t) = e^{-itH}\epsilon\phi + \sum_{n=1}^{\infty}\mathcal{Q}^n e^{-itH}\epsilon\phi. \tag{2.30}$$

Equation (1.16) follows from (2.23) and (2.30). By Sobolev's imbedding theorem [1], $W_{k+1,2} \subset L^q$, $2 \leq q \leq \infty$. Then, estimating as in (2.19) we prove that

$\left\|e^{-itH}\phi\right\|_{L^q} \leq C_q \left\|e^{-itH}\phi\right\|_{W_{k+1,2}} \leq C_q \left\|\phi\right\|_{W_{k+1,2}}, 2 \leq q \leq \infty$, and it follows from (1.6) that

$$\int\int dt\,dx\,\left|e^{-itH}\phi\right|^{2(j_0+j+1)} < \infty,\; j = 1, 2, \cdots . \tag{2.31}$$

For $\lambda \geq 1$ and $\acute{x} \in \mathbf{R}^n$ we denote by H_λ the following self-adjoint operator in L^2:

$$H_\lambda := H_0 + V_\lambda(x), \text{ where } V_\lambda(x) = \frac{1}{\lambda^2} V_0\left(\frac{x}{\lambda} + \acute{x}\right). \tag{2.32}$$

Since H has no eigenvalues, we have that H_λ has no eigenvalues, i.e., $H_\lambda > 0$. Moreover, as $N_\delta(D^\alpha V_\lambda) \leq N_\delta(D^\alpha V_0)$ for $\lambda \geq 1$, equation (2.8) holds with H_λ instead of H with the same C_1, C_2 for all $\lambda \geq 1$.

Let us denote $\tilde{t} := \lambda^2 t$ and $\tilde{x} := \lambda(x - \acute{x})$. We have that

$$\left(e^{-i\tilde{t}H_\lambda}\phi\right)(\tilde{x}) = \left(e^{-itH}\phi_\lambda\right)(x). \tag{2.33}$$

Equation (2.33) implies that

$$\begin{aligned} I_j &:= \lambda^{n+2} \int\int dt\,dx\,V_j(x)\left|e^{-itH}\phi_\lambda\right|^{2(j_0+j+1)} \\ &= \int\int d\tilde{t}\,d\tilde{x}\,V_j\left(\frac{\tilde{x}}{\lambda} + \acute{x}\right)\left|e^{-i\tilde{t}H_\lambda}\phi\right|^{2(j_0+j+1)}(\tilde{x}). \end{aligned} \tag{2.34}$$

By Theorem VIII.20 on page 286 of [12] and (2.32)

$$s - \lim_{\lambda\to\infty} e^{-i\tilde{t}H_\lambda}\phi = e^{-i\tilde{t}H_0}\phi, \tag{2.35}$$

where the limit exists in the strong topology on $W_{k+1,2}$. By Sobolev's imbedding theorem, the limit in (2.35) also exists in the strong topology on L^q, $2 \leq q \leq \infty$. Moreover,

$$\left\|e^{-i\tilde{t}H_\lambda}\phi\right\|_{L^\infty} \leq C\left\|\phi\right\|_{W_{k+1,2}}. \tag{2.36}$$

By (1.6) and(2.33),

(2.37)
$$\begin{aligned} \left\|e^{-i\tilde{t}H_\lambda}\phi\right\|_{L^{p+1}}^{p+1} &= \lambda^n \left\|e^{-itH}\phi_\lambda\right\|_{L^{p+1}}^{p+1} \leq C\frac{1}{t^{d(p+1)}}\lambda^n \left\|\phi_\lambda\right\|_{L^{1+1/p}}^{p+1} \\ &= C\frac{1}{\tilde{t}^{d(p+1)}}\left\|\phi\right\|_{L^{1+1/p}}^{p+1}, \end{aligned}$$

with $d := \frac{n}{2}\frac{p-1}{p+1}$. Equation (1.17) follows from (2.34), (2.35), (2.36), (2.37) and the dominated convergence theorem. Note that $2(j_0+j+1) \geq p+1$, that $d(p+1) > 1$, and that V_j is continuous.

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