

A HOMOMORPHISM OF HARISH-CHANDRA AND DIRECT IMAGES OF \mathcal{D} -MODULES

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ABSTRACT. Harish-Chandra defined a homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{h})^W$ of algebras of invariant polynomial differential operators. The construction and existence of δ are somewhat mysterious. We show how δ naturally arises when one considers matters in the context of \mathcal{D} -modules.

0. INTRODUCTION

0.1. Let \mathfrak{g} be a complex semisimple Lie algebra with adjoint group $G = \text{Aut}(\mathfrak{g})^\circ$, Cartan subalgebra \mathfrak{h} , and Weyl group W . Harish-Chandra [4], [5] defined a homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{h})^W$ of algebras of invariant polynomial differential operators. We show that δ arises naturally in the following way: Consider the diagram

$$(*) \quad \begin{array}{ccc} & \mathfrak{g} & \\ & \downarrow \chi & \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W \end{array}$$

where π and χ are the quotient morphisms by W and the adjoint action of G , respectively. The sheaf of regular functions $\mathcal{O}_{\mathfrak{h}}$ on \mathfrak{h} is a $\mathcal{D}_{\mathfrak{h}}$ -module, where $\mathcal{D}_{\mathfrak{h}}$ denotes the sheaf of differential operators on \mathfrak{h} . Then we have the direct image $\pi_+ \mathcal{O}_{\mathfrak{h}}$ and its pull back $\mathcal{M} := \chi^* \pi_+ \mathcal{O}_{\mathfrak{h}}$, which is a $\mathcal{D}_{\mathfrak{g}}$ -module. The G -invariant global sections $\Gamma(\mathfrak{g}, \mathcal{M})^G$ contain a copy of $\mathcal{O}(\mathfrak{h})$ and are a $(\mathcal{D}(\mathfrak{g})^G = \Gamma(\mathfrak{g}, \mathcal{D}_{\mathfrak{g}})^G)$ -module.

Theorem. $\mathcal{D}(\mathfrak{g})^G$ acts on the image of $\mathcal{O}(\mathfrak{h})$ in $\Gamma(\mathfrak{g}, \mathcal{M})^G$ via δ .

Our methods are elementary and consist of a change of variables theorem for étale maps and a result about lifting differential operators on a quotient.

0.2. The $\mathcal{D}(\mathfrak{g})^G$ -module $\Gamma(\mathfrak{g}, \mathcal{M})^G$ is larger than $\mathcal{O}(\mathfrak{h})$. Using less elementary means, we show how one can construct a $\mathcal{D}(\mathfrak{g})^G$ -module which identifies canonically with $\mathcal{O}(\mathfrak{h})$: Fix a Borel subalgebra \mathfrak{b} of \mathfrak{g} containing \mathfrak{h} , and let $B \supset H$ be the corresponding subgroups of G . The adjoint action $G \times \mathfrak{g} \rightarrow \mathfrak{g}$ induces a morphism φ from $\tilde{\mathfrak{g}} := G \times^B \mathfrak{b}$ to \mathfrak{g} (the Grothendieck-Springer resolution). The projection

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$\mathfrak{b} \rightarrow \mathfrak{b}/\mathfrak{n} \simeq \mathfrak{h}$ induces a smooth map $\theta : \tilde{\mathfrak{g}} \rightarrow \mathfrak{h}$, where \mathfrak{n} is the maximal nilpotent subalgebra of \mathfrak{b} . There is a commutative diagram

$$\begin{array}{ccc}
 \tilde{\mathfrak{g}} & \xrightarrow{\varphi} & \mathfrak{g} \\
 \theta \downarrow & & \downarrow \chi \\
 \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W
 \end{array}$$

(**)

Now we obtain a $\mathcal{D}(\mathfrak{g})^G$ -module from $\mathcal{O}_{\mathfrak{h}}$ by going the other way around. Let \mathcal{N} denote $\varphi_+ \theta^* \mathcal{O}_{\mathfrak{h}} \simeq \varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}$.

Theorem. (1) *The space of invariants $\Gamma(\mathfrak{g}, \mathcal{N})^G$ identifies canonically with the polynomial functions $\mathcal{O}(\mathfrak{h})$ on \mathfrak{h} :*

$$\Gamma(\mathfrak{g}, \mathcal{N})^G = \mathcal{O}(\mathfrak{h}),$$

where $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h})$ via δ .

(2) *As a $\mathcal{D}_{\mathfrak{g}}$ -module, \mathcal{N} is generated by $\Gamma(\mathfrak{g}, \mathcal{N})^G$. Thus the inclusion $\mathcal{O}(\mathfrak{h}) \rightarrow \Gamma(\mathfrak{g}, \mathcal{M})^G$ induces an inclusion of $(\mathcal{D}_{\mathfrak{g}}, G)$ -modules*

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{M}.$$

The proof uses results of Evens [3], Hotta and Kashiwara [7] and Hunziker and Wallach [8].

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1. THE HOMOMORPHISM δ

1.1. Let V be a finite dimensional complex vector space with coordinate ring $\mathcal{O}(V)$. Let $\mathcal{D}(V)$ denote the algebra of polynomial differential operators on V . We identify the symmetric algebra $S(V)$ with the constant coefficient differential operators. The multiplication map $\mu : \mathcal{O}(V) \otimes S(V) \rightarrow \mathcal{D}(V)$ defined by $f \otimes P \mapsto f \cdot P$ is an isomorphism of vector spaces. Suppose that V is a rational G -module. Then $\mathcal{O}(V)$, $S(V)$, and $\mathcal{D}(V)$ are locally finite rational G -modules and φ is a G -isomorphism.

1.2. The map $\delta : \mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{h})^W$ has the following properties:

- ($\delta 1$) δ is an algebra homomorphism.
- ($\delta 2$) On $\mathcal{O}(\mathfrak{g})^G$, δ is the isomorphism given by restriction $\mathcal{O}(\mathfrak{g})^G \xrightarrow{\simeq} \mathcal{O}(\mathfrak{h})^W$.
- ($\delta 3$) On $S(\mathfrak{g})^G$, δ is the isomorphism $S(\mathfrak{g})^G \xrightarrow{\simeq} S(\mathfrak{h})^W$ induced by the canonical projection $\mathfrak{g} \rightarrow \mathfrak{h}$.
- ($\delta 4$) The kernel of δ is the ideal $I = \{P \in \mathcal{D}(\mathfrak{g})^G \mid P(f) = 0 \text{ for all } f \in \mathcal{O}(\mathfrak{g})^G\}$.

Harish-Chandra's construction of δ is as follows: Fix a system $R_+ \subset \mathfrak{h}^*$ of positive roots. Let $\sigma = \prod_{\alpha \in R_+} \alpha$. The set of regular elements $\mathfrak{h}' \subset \mathfrak{h}$ equals the subset \mathfrak{h}_σ where σ does not vanish. The Weyl group W acts freely on \mathfrak{h}' and the quotient map $\pi : \mathfrak{h}' \rightarrow \mathfrak{h}'/W$ is a covering (proper étale map). This implies that there is an isomorphism $\mathcal{D}(\mathfrak{h}')^W \xrightarrow{\simeq} \mathcal{D}(\mathfrak{h}'/W)$. We then obtain a homomorphism $\delta' : \mathcal{D}(\mathfrak{g})^G \rightarrow \mathcal{D}(\mathfrak{h}')^W$ as follows: Let \mathfrak{g}' denote the regular semisimple elements of \mathfrak{g} , and let ρ denote the isomorphisms of $\mathcal{O}(\mathfrak{g})^G$ and $\mathcal{O}(\mathfrak{g}')^G$ with $\mathcal{O}(\mathfrak{h})^W$ and $\mathcal{O}(\mathfrak{h}')^W$ induced by restriction. Then

$$\delta'(P)(f) = \rho(P(\rho^{-1}(f))), \text{ for } P \in \mathcal{D}(\mathfrak{g})^G, f \in \mathcal{O}(\mathfrak{h}')^W.$$

Define $\delta = m_\sigma \circ \delta' \circ m_\sigma^{-1}$ where m_σ denotes multiplication by σ . Assuming that $\delta(\mathcal{D}(\mathfrak{g})^G)$ lies in $\mathcal{D}(\mathfrak{h})^W \subset \mathcal{D}(\mathfrak{h}')^W$ (the hard part), one easily sees that ($\delta 1$), ($\delta 2$)

and (δ_4) hold, and they imply (δ_3) . See [12] for a short proof of the existence of δ along these lines. A completely different construction of δ is given in Hunziker and Wallach [8].

In the next two sections, we show how the twist by σ arises naturally.

2. AN ALGEBRAIC CHANGE OF VARIABLES FORMULA

2.1. Let X be a (smooth) complex algebraic variety. We denote by \mathcal{O}_X the sheaf of regular functions and by \mathcal{D}_X the sheaf of algebraic differential operators. As usual, $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$ and $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$. If $\pi : X \rightarrow Y$ is a morphism between varieties we denote by π_*, π^* the usual functors between \mathcal{O} -modules. The inverse image of a \mathcal{D}_Y -module (looked upon as an \mathcal{O}_Y -module) always carries a natural structure of a \mathcal{D}_X -module (see [1, Chapter VI, §4]). The notion of direct image is, however, much more complicated for \mathcal{D} -modules. In general, to define a direct image functor π_+ one has to work in the derived category, i.e., one has to work with complexes of \mathcal{D} -modules (see [1, Chapter VI, §5]). The morphisms π that we are considering in this section are affine, in which case the functor π_+ exists on the module level. If π is étale, then π_+ is isomorphic to the functor π_* , and we give this isomorphism explicitly below.

2.2. Let $X = Y = \mathbb{C}^n$. We choose linear coordinates x_1, \dots, x_n on X and y_1, \dots, y_n on Y . Then $\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]$ and $\mathcal{O}(Y) = \mathbb{C}[y_1, \dots, y_n]$. Let $\pi : X \rightarrow Y$ be a polynomial map with coordinate functions $u_1 = \pi^*y_1, \dots, u_n = \pi^*y_n$. Set $a_{ij} = \frac{\partial u_i}{\partial x_j}$, $i, j = 1, \dots, n$, and let $\sigma = \det(a_{ij})$ denote the Jacobian. Then π is étale on the open subset $X_\sigma = \{x \in X \mid \sigma(x) \neq 0\}$. Let (b_{ij}) denote the matrix of cofactors of (a_{ij}) . Then (b_{ij}) is the matrix with entries in $\mathcal{O}(X)$ such that

$$\sum_{k=1}^n a_{ik} b_{kj} = \sigma \delta_{ij}.$$

We define vector fields ∂_{u_j} on X_σ by $\partial_{u_j} = \sigma^{-1} \sum_{i=1}^n b_{ij} \partial_{x_i}$, where $\partial_{x_i} = \partial/\partial x_i$, $i = 1, \dots, n$. Then we have

$$\partial_{u_j}(u_k) = \frac{1}{\sigma} \sum_{i=1}^n b_{ij} \frac{\partial u_k}{\partial x_i} = \frac{1}{\sigma} \sum_{i=1}^n b_{ij} a_{ki} = \delta_{kj}.$$

2.3. Let $\mathcal{D}(X \rightarrow Y) := \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y)$. It is a $(\mathcal{D}(X), \mathcal{D}(Y))$ -bimodule, where the right $\mathcal{D}(Y)$ -module structure is the obvious one given by right multiplication. The left action of $\mathcal{O}(X)$ is given by left multiplication, and the partial derivatives ∂_{x_i} act by

$$\partial_{x_i} \cdot (f \otimes P) = \frac{\partial f}{\partial x_i} \otimes P + \sum_{k=1}^n f \frac{\partial u_k}{\partial x_i} \otimes \partial_{y_k} P, \quad i = 1, \dots, n.$$

Suppose that we localize at X_σ , i.e., we consider the subset of X where π is étale. We have an injective homomorphism $\eta : \mathcal{D}(Y) \rightarrow \mathcal{D}(X_\sigma)$ where $\eta(y_i) = \pi^*(y_i) = u_i$ and $\eta(\partial_{y_i}) = \partial_{u_i}$, $i = 1, \dots, n$. We consider $\mathcal{D}(X_\sigma)$ as a $(\mathcal{D}(X_\sigma), \mathcal{D}(Y))$ -bimodule (via η for the $\mathcal{D}(Y)$ -module structure). Then

Proposition. *The map η induces an isomorphism of $(\mathcal{D}(X_\sigma), \mathcal{D}(Y))$ -bimodules:*

$$\mathcal{O}(X_\sigma) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y) \simeq \mathcal{D}(X_\sigma), \quad f \otimes P \mapsto f\eta(P), \quad f \in \mathcal{O}(X_\sigma), \quad P \in \mathcal{D}(Y).$$

2.4. Recall that formal transpose $P \mapsto P^t$ is an anti-involution of $\mathcal{D}(X)$ and $\mathcal{D}(Y)$. It is the identity on $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, and $\partial_{x_i}^t = -\partial_{x_i}$ and $\partial_{y_i}^t = -\partial_{y_i}$, $i = 1, \dots, n$. We now need to consider $\mathcal{D}(Y \leftarrow X)$ which is the transpose of $\mathcal{D}(X \rightarrow Y)$ for both the $\mathcal{D}(X)$ - and $\mathcal{D}(Y)$ -actions. As an $\mathcal{O}(X)$ - and $\mathcal{O}(Y)$ -module it is the same as $\mathcal{D}(X \rightarrow Y)$, but the actions of the ∂_{x_i} and ∂_{y_i} are transposed. If M is a left $\mathcal{D}(X)$ -module, we define the pushforward $\pi_+M := \mathcal{D}(Y \leftarrow X) \otimes_{\mathcal{D}(X)} M$. Then π_+M is a left $\mathcal{D}(Y)$ -module. If we restrict to X_σ , we have our identification of $\mathcal{D}(X_\sigma \rightarrow Y)$ with $\mathcal{D}(X_\sigma)$ where $\mathcal{D}(Y)$ acts via η . Let M_σ denote $\mathcal{D}(X_\sigma) \otimes_{\mathcal{D}(X)} M$. Then

$$\pi_+M_\sigma = \mathcal{D}(Y \leftarrow X_\sigma) \otimes_{\mathcal{D}(X_\sigma)} M_\sigma \simeq \mathcal{D}(X_\sigma) \otimes_{\mathcal{D}(X_\sigma)} M_\sigma \simeq M'_\sigma,$$

where M'_σ is a copy of M_σ on which elements $P \in \mathcal{D}(Y)$ act via $\eta(P^t)^t$.

Lemma. *Let $1 \leq i \leq n$. Then $y_i \in \mathcal{D}(Y)$ acts on M'_σ as multiplication by u_i and ∂_{y_i} acts as the differential operator $m_{\sigma^{-1}}\partial_{u_i}m_\sigma$.*

Proof. The action of y_i is via $\eta(y_i) = u_i$. The action of ∂_{y_i} is by the transpose of $\eta(-\partial_{y_i}) \in \mathcal{D}(X_\sigma)$. The transposed action of $\mathcal{D}(X)$ on $\mathcal{O}(X)$ can be seen as the action on smooth differential n -forms via (the negative of) the Lie derivative. In other words,

$$(-\partial_{x_i}^t(f))\omega_0 = \text{Lie}(\partial_{x_i})(f\omega_0),$$

where $\omega_0 := dx_1 \wedge \dots \wedge dx_n$. Set $\omega := \sigma\omega_0$. Then $\omega = du_1 \wedge \dots \wedge du_n$, so that

$$\text{Lie}(\partial_{u_i})(f\sigma\omega_0) = \text{Lie}(\partial_{u_i})(f\omega) = \frac{\partial f}{\partial u_i}\omega = \sigma \frac{\partial f}{\partial u_i}\omega_0,$$

proving the Lemma. □

Corollary. *Consider M_σ as a $\mathcal{D}(Y)$ -module via the action above. Then there is a commutative diagram of $\mathcal{O}(Y)$ -modules*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & \pi_+M \\ \downarrow & & \downarrow \\ M_\sigma & \xrightarrow[\sim]{\beta} & \pi_+M_\sigma \end{array}$$

where α is a homomorphism of $\mathcal{O}(Y)$ -modules and β is an isomorphism of $\mathcal{D}(Y)$ -modules. Here α sends an element $m \in M$ to $1 \otimes 1 \otimes m \in \mathcal{D}(Y \leftarrow X) \otimes_{\mathcal{D}(X)} M$, and similarly for β .

3. THE INVERSE IMAGE OF A \mathcal{D} -MODULE ON A QUOTIENT

3.1. Let G be a complex reductive group and Z an affine G -variety. There is a quotient variety $Z//G$ and a surjection $\varpi : Z \rightarrow Z//G$ such that ϖ^* identifies with the inclusion $\mathcal{O}(Z)^G \rightarrow \mathcal{O}(Z)$. There is a canonical homomorphism $\varpi_* : \mathcal{D}(Z)^G \rightarrow \mathcal{D}(Z//G)$ given by restriction of operators $P \mapsto P|_{\mathcal{O}(Z)^G}$.

3.2. Lemma. *Assume that Z and $Z//G$ are smooth. Let \mathcal{L} be a $\mathcal{D}_{Z//G}$ -module. Then $\varpi^*\mathcal{L}$ is a (\mathcal{D}_Z, G) -module and*

$$\Gamma(Z, \varpi^*\mathcal{L})^G = \Gamma(Z//G, \mathcal{L}) \text{ as } \mathcal{D}(Z)^G\text{-modules,}$$

where $\mathcal{D}(Z)^G$ acts on $\Gamma(Z//G, \mathcal{L})$ via ϖ_* .

Proof. Considered as an $\mathcal{O}(Z)$ -module,

$$\Gamma(Z, \varpi^* \mathcal{L}) = \mathcal{O}(Z) \otimes_{\mathcal{O}(Z)^G} \Gamma(Z//G, \mathcal{L}) .$$

The G -action is given by the action on the left factor. Clearly, the canonical map $\Gamma(Z//G, \mathcal{L}) \rightarrow \Gamma(Z, \varpi^* \mathcal{L})^G$, $s \mapsto 1 \otimes s$ is an isomorphism of $\mathcal{O}(Z)^G$ -modules. We will prove that it is also an isomorphism of $\mathcal{D}(Z)^G$ -modules.

Using Luna’s slice theorem (see [11, §4]), we can reduce to the case that $Z//G = \mathbb{C}^n$ is a vector space (the only case we actually need, anyway). Then $\mathcal{O}(Z)^G = \mathbb{C}[u_1, \dots, u_n]$, where the u_i are algebraically independent functions. Let y_1, \dots, y_n denote coordinate functions on \mathbb{C}^n such that $u_i = \varpi^* y_i$. By definition, a vector field θ on Z acts by Leibniz’s rule on $\Gamma(Z, \varpi^* \mathcal{L})$:

$$\theta(f \otimes s) = \theta(f) \otimes s + \sum_{i=1}^n f \theta(u_i) \otimes \partial_{y_i}(s) .$$

Let $P \in \mathcal{D}(Z)^G$ and $1 \otimes s \in \Gamma(Z//G, \mathcal{L})$. By the formula above

$$P(1 \otimes s) = \sum_{\beta=(\beta_1, \dots, \beta_n)} f_\beta \otimes \partial_y^\beta(s) ,$$

where the functions $f_\beta \in \mathcal{O}(Z)$ only depend upon P . Here we use the usual multi-index notation $\partial_y^\beta = \partial_{y_1}^{\beta_1} \dots \partial_{y_n}^{\beta_n}$. Since $P(1 \otimes s)$ and the $\varpi^*(\partial_y^\beta(s))$ are G -invariant, we may replace the f_β by their projections to the invariants (Reynolds operator). Then $f_\beta = \varpi^* \bar{f}_\beta$, $\bar{f}_\beta \in \mathcal{O}(Z//G)$, and $P(1 \otimes s) = \varpi^*(\bar{P}s)$ where $\bar{P} = \sum_\beta \bar{f}_\beta \partial_y^\beta$. From the case $\mathcal{L} = \mathcal{O}_{Z//G}$ it is clear that \bar{P} is the image of P under ϖ_* . \square

Proof of Theorem 0.1. Recall diagram (*) of the introduction. Set $M = \mathcal{O}(\mathfrak{h}')$ and let $\pi : \mathfrak{h}' \rightarrow \mathfrak{h}/W \simeq \mathfrak{g}//G$ be the natural étale morphism with Jacobian σ . Let $Q \in \mathcal{D}(\mathfrak{g}//G)$ and $f \in \mathcal{O}(\mathfrak{h}')$. The Corollary in 2.4 shows that Q sends $f' := 1 \otimes 1 \otimes f \in \pi_+ \mathcal{O}(\mathfrak{h}')$ to $1 \otimes 1 \otimes (m_{\sigma^{-1}} \circ Q \circ m_\sigma) f$. The construction of δ (see 1.2) and Lemma 3.2 show that $P \in \mathcal{D}(\mathfrak{g})^G$ sends $\chi^* f'$ to $\chi^*(1 \otimes 1 \otimes (m_{\sigma^{-1}} \circ \delta'(P) \circ m_\sigma) f) = \chi^*(1 \otimes 1 \otimes \delta(P) f)$. Thus $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h}') \simeq \Gamma(\mathfrak{g}, \chi^* \pi_+ \mathcal{O}(\mathfrak{h}'))^G$ via δ . Since $\mathcal{O}(\mathfrak{h}) \rightarrow \mathcal{O}(\mathfrak{h}')$ is injective, we obtain the Theorem in 0.1. \square

Note that we have not shown that $\mathcal{D}(\mathfrak{g})^G$ leaves $\mathcal{O}(\mathfrak{h}) \subset \mathcal{O}(\mathfrak{h}')$ stable. This still requires a separate proof (see [12, §3], for example).

3.3. For the rest of this paper we assume that $\mathcal{D}(\mathfrak{g})^G$ leaves $\mathcal{O}(\mathfrak{h})$ stable. We show how to replace $\mathcal{M} = \chi^* \pi_+ \mathcal{O}(\mathfrak{h})$ by another module whose G -invariants are $\mathcal{O}(\mathfrak{h})$ on the nose.

4. THE HOMOMORPHISM δ AND THE GROTHENDIECK-SPRINGER RESOLUTION

4.1. In [8], Wallach and Hunziker gave a different construction of δ . Starting with a triangular decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ they define a homomorphism $\gamma : \mathcal{D}(\mathfrak{g})^H \rightarrow \mathcal{D}(\mathfrak{h})$ such that $\mathcal{D}(\mathfrak{g})^G$ lands inside $\mathcal{D}(\mathfrak{h})^W$. From the construction of γ it is immediate that properties $(\delta 1)$, $(\delta 2)$ and $(\delta 3)$ hold. The hard part is to show that $(\delta 4)$ holds, i.e., that γ vanishes on the ideal $(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))^G$. In the proof of this fact in [8], a certain $(\mathcal{D}_{\mathfrak{g}}, G)$ -module was introduced, which we will denote by \mathcal{N}_0 . The construction of \mathcal{N}_0 was such that the space of G -invariant global sections, looked

upon as a $\mathcal{D}(\mathfrak{g})^G$ -module, identifies canonically with the polynomial functions on \mathfrak{h} :

$$\Gamma(\mathfrak{g}, \mathcal{N}_0)^G = \mathcal{O}(\mathfrak{h}).$$

Here the $\mathcal{D}(\mathfrak{g})^G$ -action on $\mathcal{O}(\mathfrak{h})$ is given via γ .

4.2. In [3], Evens related the module \mathcal{N}_0 to constructions of Hotta and Kashiwara [7]. Recall diagram (**) of 0.2 (also reproduced in 4.3 below).

Theorem (Evens [3, §1]). *The $(\mathcal{D}_{\mathfrak{g}}, G)$ -module \mathcal{N}_0 is naturally isomorphic to the direct image \mathcal{N} of the regular functions on $\tilde{\mathfrak{g}}$ by φ :*

$$\mathcal{N}_0 \simeq \mathcal{N} = \varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}.$$

Remark. A priori, $\varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}$ is an object in the derived category. However, by a result due to Hotta and Kashiwara [7, Corollary 4.23], $h^i(\varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}) = 0$ for $i \neq 0$ and hence we may regard $\varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}$ as a $\mathcal{D}_{\mathfrak{g}}$ -module.

4.3. We now want to relate the module $\mathcal{N} = \varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}$ to the compatible $(\mathcal{D}_{\mathfrak{g}}, G)$ -module $\mathcal{M} = \chi^* \pi_+ \mathcal{O}_{\mathfrak{h}}$ that we studied in §3. Recall the Grothendieck-Springer resolution diagram:

$$(**) \quad \begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\varphi} & \mathfrak{g} \\ \theta \downarrow & & \downarrow \chi \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W \end{array}$$

The morphism $\varphi \times \theta : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \times \mathfrak{h}$ is a birational map onto the fiber product $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} = \{(x, t) \mid u(x) = u(t) \text{ for all } u \in \mathcal{O}(\mathfrak{g})^G\}$. Let \mathfrak{g}' be the set of regular semisimple elements in \mathfrak{g} and let $\tilde{\mathfrak{g}}' = \theta^{-1}(\mathfrak{h}')$. Then $(\varphi \times \theta)|_{\tilde{\mathfrak{g}}'} : \tilde{\mathfrak{g}}' \xrightarrow{\sim} \mathfrak{g}' \times_{\mathfrak{h}'/W} \mathfrak{h}'$ is an isomorphism. This isomorphism is the key to the following lemma that is proved by base change.

Lemma. *We have a natural isomorphism of compatible $(\mathcal{D}_{\mathfrak{g}'}, G)$ -modules:*

$$\mathcal{N}|_{\mathfrak{g}'} \simeq \mathcal{M}|_{\mathfrak{g}'}.$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} & & \tilde{\mathfrak{g}}' & \xrightarrow{\varphi} & \mathfrak{g}' \\ & \theta \swarrow & \downarrow \pi & \swarrow \chi & \downarrow j \\ \mathfrak{h}' & \xrightarrow{\pi} & \mathfrak{h}'/W & & \mathfrak{g}' \\ & \downarrow & \downarrow & & \downarrow j \\ & \downarrow j & \tilde{\mathfrak{g}} & \xrightarrow{\varphi} & \mathfrak{g} \\ & \theta \swarrow & \downarrow \pi & \swarrow \chi & \downarrow j \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W & & \mathfrak{g} \end{array}$$

The vertical arrows j are the inclusion maps. Every face except the bottom face of this cube is a Cartesian diagram. Note also that every map in this diagram is G -equivariant if we define trivial G -actions on \mathfrak{h} and \mathfrak{h}/W , respectively. We then have the following natural isomorphisms of functors (in the equivariant derived

category):

$$\begin{aligned}
 j^! \circ \chi^! \circ \pi_+ &= \chi^! \circ j^! \circ \pi_+ && \text{(composition law, right face)} \\
 &= \chi^! \circ \pi_+ \circ j^! && \text{(base change, front face)} \\
 &= \varphi_+ \circ \theta^! \circ j^! && \text{(base change, top face)} \\
 &= \varphi_+ \circ j^! \circ \theta^! && \text{(composition law, left face)} \\
 &= j^! \circ \varphi_+ \circ \theta^! && \text{(base change, back face)}.
 \end{aligned}$$

(We refer the reader to the appendix below for notation and basic background on operations on \mathcal{D} -modules.) Thus we obtain a natural isomorphism

$$(\chi^! \pi_+ \mathcal{O}_{\mathfrak{h}})|_{\mathfrak{g}'} \simeq (\varphi_+ \theta^! \mathcal{O}_{\mathfrak{h}})|_{\mathfrak{g}'}$$

Since $\theta^! \mathcal{O}_{\mathfrak{h}} = \theta^* \mathcal{O}_{\mathfrak{h}}[d_{\mathfrak{g}/\mathfrak{h}}] = \mathcal{O}_{\mathfrak{g}}[d_{\mathfrak{g}/\mathfrak{h}}]$, the lemma follows. □

An interpretation of Lemma 4.3. Recall that we have a natural injective homomorphism $\mathcal{O}(\mathfrak{h}) \rightarrow \Gamma(\mathfrak{g}, \mathcal{M})^G$ of $\mathcal{D}(\mathfrak{g})^G$ -modules, where $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h})$ via δ . On the other hand, we have a natural isomorphism $\mathcal{O}(\mathfrak{h}) \xrightarrow{\sim} \Gamma(\mathfrak{g}, \mathcal{N}_0)^G$, where $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h})$ via γ . The isomorphism $\mathcal{N}|_{\mathfrak{g}'} \simeq \mathcal{M}|_{\mathfrak{g}'}$ identifies $\Gamma(\mathfrak{g}', \mathcal{M})^G$ with $\Gamma(\mathfrak{g}', \mathcal{N})^G$. We therefore have a geometric explanation that $\gamma = \delta$. More precisely, in light of the proof of the Lemma in 4.3 we may say that $\gamma = \delta$ via base change.

Proof of Theorem 0.2. We have part (1) of the theorem, i.e., that $\Gamma(\mathfrak{g}, \mathcal{N})^G \simeq \mathcal{O}(\mathfrak{h})$. It was shown by Hotta and Kashiwara that $\mathcal{N} \simeq \varphi_+ \mathcal{O}_{\mathfrak{h}}$ is a regular holonomic $\mathcal{D}_{\mathfrak{g}}$ -module equal to the minimal extension of $(\varphi_+ \mathcal{O}_{\mathfrak{g}})|_{\mathfrak{g}'}$. Thus the restriction map $\Gamma(\mathfrak{g}, \mathcal{N}) \rightarrow \Gamma(\mathfrak{g}', \mathcal{N})$ is injective. It also follows from their work that $\Gamma(\mathfrak{g}, \mathcal{N})$ is the quotient of $\mathcal{D}(\mathfrak{g})$ by a G -stable left ideal, in particular, \mathcal{N} is generated by $\Gamma(\mathfrak{g}, \mathcal{N})^G$. Thus the canonical inclusion $\mathcal{O}(\mathfrak{h}) \rightarrow \Gamma(\mathfrak{g}, \mathcal{M})^G$ induces an inclusion $\mathcal{N} \rightarrow \mathcal{M}$ of $(\mathcal{D}_{\mathfrak{g}}, G)$ -modules, and we have part (2). □

APPENDIX: OPERATIONS ON \mathcal{D} -MODULES

A.1. If X is a smooth variety we denote by $D^b(\mathcal{D}_X)$ the derived category of bounded complexes of \mathcal{D}_X -modules. If $\mathcal{M} \in D^b(\mathcal{D}_X)$ we write $h^i(\mathcal{M})$ for the i -th cohomology module of the complex. We will identify the category of \mathcal{D}_X -modules with the full category of complexes $\mathcal{M} \in D^b(\mathcal{D}_X)$ such that $h^i(\mathcal{M}) = 0$ for $i \neq 0$.

If $\pi : X \rightarrow Y$ is a morphism between smooth varieties, then there are functors

$$\pi_+ : D^b(\mathcal{D}_X) \rightarrow D^b(\mathcal{D}_Y), \quad \pi^! : D^b(\mathcal{D}_Y) \rightarrow D^b(\mathcal{D}_X).$$

If $\psi : Y \rightarrow Z$ is another morphism between smooth varieties we have the following composition laws:

$$(\psi \circ \pi)_+ = \psi_+ \circ \pi_+, \quad (\psi \circ \pi)^! = \pi^! \circ \psi^!.$$

A.2. Now suppose that X is a smooth G -variety, where G is some linear algebraic group. Bernstein defined an equivariant derived category $D_G^b(\mathcal{D}_X)$ together with a forgetful functor $D_G^b(\mathcal{D}_X) \rightarrow D^b(\mathcal{D}_X)$. If X is an affine G -variety and if \mathcal{M} is an object in $D_G^b(\mathcal{D}_X)$, then the \mathcal{D}_X -modules $h^i(\mathcal{M})$ inherit a structure of a compatible

(\mathcal{D}_X, G) -module. If $\pi : X \rightarrow Y$ is a G -equivariant morphism between smooth G -varieties, then there are functors $\pi_+ : D_G^b(\mathcal{D}_X) \rightarrow D_G^b(\mathcal{D}_Y)$ and $\pi^! : D_G^b(\mathcal{D}_Y) \rightarrow D_G^b(\mathcal{D}_X)$.

A.3. We recall the principle of base change for operations on \mathcal{D} -modules (see [6, Chapter 1]). Consider a Cartesian square of smooth varieties:

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\pi}} & Z \\ \tilde{\varphi} \downarrow & \square & \downarrow \varphi \\ X & \xrightarrow{\pi} & Y \end{array}$$

Cartesian means that

$$\tilde{Z} = X \times_Y Z = \{(x, z) \in X \times Z \mid \pi(x) = \varphi(z)\}.$$

Then we have a natural isomorphism of functors from $D^b(\mathcal{D}_X)$ to $D^b(\mathcal{D}_Z)$:

$$\varphi^! \circ \pi_+ = \tilde{\pi}_+ \circ \tilde{\varphi}^!.$$

For equivariant maps we have natural isomorphisms of functors between equivariant derived categories.

A.4. Let $\varphi : Z \rightarrow Y$ be a smooth morphism and let $d_{Z/Y} = \dim(Z) - \dim(Y)$. Then $h^i \circ \chi^! = 0$ for $i \neq d_{Z/Y}$ and for every \mathcal{D}_Y -module \mathcal{L} , $\chi^! \mathcal{L} = \chi^* \mathcal{L}[d_{Z/Y}]$. Here, as usual, if \mathcal{C} is a complex, then $\mathcal{C}[d]$ denotes the shift of the complex by d .

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