PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 12, Pages 3479–3484
S 0002-9939(01)06125-1
Article electronically published on April 25, 2001

FINITE GROUPS AND THE FIXED POINTS OF COPRIME AUTOMORPHISMS

PAVEL SHUMYATSKY

(Communicated by Stephen D. Smith)

ABSTRACT. Let p be a prime, and let G be a finite p'-group acted on by an elementary abelian p-group A. The following results are proved:

1. If $|A| \geq p^3$ and $C_G(a)$ is nilpotent of class at most c for any $a \in A^\#$, then the group G is nilpotent of $\{c, p\}$-bounded class.
2. If $|A| \geq p^4$ and $C_G(a)'$ is nilpotent of class at most c for any $a \in A^\#$, then the derived group G' is nilpotent of $\{c, p\}$-bounded class.

1. Introduction

Let G be a group admitting an action of a group A. We denote by $C_G(A)$ the set $C_G(A) = \{x \in G | x^a = x \text{ for any } a \in A\}$, the centralizer of A in G (the fixed-point group). Throughout this paper we assume that A is a noncyclic elementary abelian p-group, and G is a finite p'-group. Let $A^\#$ denote the set of non-identity elements of A. It follows from the classification of finite simple groups that if $C_G(a)$ is solvable for any $a \in A^\#$, then so is the group G (see [3]). The case $|A| \geq p^3$ does not require the classification: the result follows from Glauberman's theorem on solvable signalizer functors [1]. In certain specific situations much more can be said about the structure of G.

Ward showed that if A has rank at least 3, and if $C_G(a)$ is nilpotent for any $a \in A^\#$, then the group G is nilpotent [7]. Another of Ward's results is that if A has rank at least 4, and if $C_G(a)'$ is nilpotent for any $a \in A^\#$, then the derived group G' is nilpotent [8]. Later the author found that if, under these assumptions, $C_G(a)$ is nilpotent of class at most c (respectively $C_G(a)'$ is nilpotent of class at most c) for any $a \in A^\#$, and if G has derived length d, then the nilpotency class of G (respectively of G') is $\{c, d, p\}$-bounded [6]. In the present paper we show that actually much stronger results are valid: the bounds on the class of G and G' can be chosen independent of d.

Theorem 1.1. *Let A be an elementary abelian group of order p^3 acting on a finite p'-group G. Assume that $C_G(a)$ is nilpotent of class at most c for any $a \in A^\#$. Then G is nilpotent and the class of G is bounded by a function depending only on p and c.*

Received by the editors April 26, 2000.
1991 *Mathematics Subject Classification.* Primary 20D45.
Key words and phrases. Automorphisms, centralizers, associated Lie rings.
The author was supported by CNPq-Brazil.

Theorem 1.2. *Let A be an elementary abelian group of order p^4 acting on a finite p'-group G. Assume $C_L(a)'$ is nilpotent of class at most c for any $a \in A^{\#}$. Then G' is nilpotent and the class of G' is bounded by a function depending only on p and c.*

We conjecture that these results can be generalized in the following way.

Conjecture 1.3. *Let A be an elementary abelian group of order p^k with $k \geq 3$ acting on a finite p'-group G.*

1. *If $\gamma_{k-2}(C_G(a))$ is nilpotent of class at most c for any $a \in A^{\#}$, then $\gamma_{k-2}(G)$ is nilpotent and has $\{c, k, p\}$-bounded class.*
2. *If, for some integer d such that $2^d + 2 \leq k$, the dth derived group of $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is nilpotent and has $\{c, k, p\}$-bounded class.*

Our main evidence in favor of the above conjecture is Lie-theoretic: Theorem 2.7 obtained in Section 2 establishes the corresponding results for Lie algebras.

2. Action on Lie algebras

Throughout the paper the term Lie algebra means Lie algebra over an associative ring with unity. Let L be a Lie algebra. If $X, Y, X_1, \ldots, X_s$ are subsets of L we use $[X, Y]$ to denote the subspace of L spanned by the set $\{[x, y] | x \in X, y \in Y\}$. If $t \geq 2$ we write $[X, {}_tY]$ for $[[X, {}_{t-1}Y], Y]$ and $[X_1, \ldots, X_t]$ for $[[X_1, \ldots, X_{t-1}], X_t]$. For any positive integer w, define commutator-spaces of weight w in $X_1, \ldots, X_s$:

A subspace of L is a commutator-space of weight 1 in $X_1, \ldots, X_s$ if and only if it is the linear span of X_i for some $i \leq s$. A subspace M of L is a commutator-space of weight $w \geq 2$ in $X_1, \ldots, X_s$ if and only if $M = [M_1, M_2]$, where M_1 and M_2 are commutator-spaces of weights w_1 and w_2 respectively, such that $w_1 + w_2 = w$.

A well-known theorem of Kreknin [5] says that if a Lie ring L admits a fixed-point-free automorphism of finite order n, then L is solvable and the derived length of L is bounded by a function of n. We will require the following extension of this result [4].

Theorem 2.1. *Let a Lie ring L admit an automorphism a of finite order n such that $[L, {}_tC_L(a)] = 0$. Assume that $nL = L$. Then L is solvable with derived length at most $(t+1)^{n-1} + log_2 t$.*

Lemma 2.2. *Let $t \geq 1$. Let L be a Lie algebra, and K a nilpotent subalgebra of class c. Assume K is generated by subspaces $X_1, \ldots, X_m$ such that for any commutator-space Y in $X_1, \ldots, X_m$ we have $[L, {}_tY] = 0$. Then there exists a $\{c, m, t\}$-bounded number u such that $[L, {}_uK] = 0$.*

Proof. This is by induction on c. Since K' is generated by commutator-spaces of weight ≥ 2 in $X_1, \ldots, X_m$ and since the number of such spaces is $\{c, m\}$-bounded, the inductive hypothesis will be that there exists a $\{c, m, t\}$-bounded number u_1 such that $[L, {}_{u_1}K'] = 0$. Now put $r = m(t-1) + 1$ and consider the space $M = [L, Y_1, \ldots, Y_r]$ for some choice of $Y_1, \ldots, Y_r \in \{X_1, \ldots, X_m\}$. Obviously, for any permutation π of the symbols $1, 2, \ldots, r$ we have $M \leq [L, Y_{\pi(1)}, \ldots, Y_{\pi(r)}] + [L, K']$. The number r is big enough to ensure that some X_i occurs in the list $Y_1, \ldots, Y_r$ at least t times. Thus, we obtain $M \leq [L, {}_tX_i, * \ldots, *] + [L, K']$, where the asterisks denote some spaces Y_j which, in view of the fact that $[L, {}_tX_i] = 0$, are of no consequence. Hence, $M \leq [L, K']$.

Now take $u = u_1 r$. Using the fact that $K = K' + \sum X_j$ and $M \leq [L, K']$ for any choice of $Y_1, \ldots, Y_r \in \{X_1, \ldots, X_m\}$, it is easy to see that $[L, {}_u K] \leq [L, {}_{u_1} K'] = 0$. □

Hypothesis 2.3. *Let ω be a primitive pth root of unity, and let L be a Lie algebra over $\mathbb{Z}[\omega]$ such that $L = pL$. Let A be an elementary abelian group of order p^k acting by automorphisms on L. Let $\hat{A}$ be the character group of A. For any $\alpha \in \hat{A}$ we set $L_\alpha = \{x \in K | x^a = \alpha(a)x$ for each $a \in A\}$.*

It is well-known that A and $\hat{A}$ are isomorphic, $[L_\alpha, L_\beta] \leq L_{\alpha\beta}$ for all $\alpha, \beta \in \hat{A}$ and $L = \bigoplus_\alpha L_\alpha$. For any positive integer n and any $\alpha_1, \ldots, \alpha_{2^n} \in \hat{A}$ define inductively

$$\gamma(\alpha_1) = L_{\alpha_1} \text{ and } \gamma(\alpha_1, \ldots, \alpha_n) = [\gamma(\alpha_1, \ldots, \alpha_{n-1}), L_{\alpha_n}],$$
$$\delta(\alpha_1) = L_{\alpha_1} \text{ and } \delta(\alpha_1, \ldots, \alpha_{2^n}) = [\delta(\alpha_1, \ldots, \alpha_{2^{n-1}}), \delta(\alpha_{2^{n-1}+1}, \ldots, \alpha_{2^n})].$$

As usual, $\gamma_n(L)$ and $L^{(n)}$ denote the nth term of the lower central series and the nth term of the derived series of L, respectively.

Lemma 2.4. *Under Hypothesis 2.3 we have $\gamma_n(L) = \sum \gamma(\alpha_1, \ldots, \alpha_n)$ and $L^{(n)} = \sum \delta(\alpha_1, \ldots, \alpha_{2^n})$, where $\alpha_1, \ldots, \alpha_{2^n}$ range independently through $\hat{A}$.*

Proof. Set $Q = \sum \gamma(\alpha_1, \ldots, \alpha_n)$. For any $\beta \in \hat{A}$ we have

$$[\gamma(\alpha_1, \ldots, \alpha_n), L_\beta] \leq \gamma(\alpha_1\alpha_2, \alpha_3, \ldots, \alpha_n, \beta) \leq Q,$$

which shows that Q is normalized by L_β and therefore is an ideal of L. It is easy to see that L/Q is nilpotent of class at most $n-1$ and so $\gamma_n(L) \leq Q$. The opposite inclusion is obvious, whence $\gamma_n(L) = Q$.

To prove the other claim we set $R_n = \sum \delta(\alpha_1, \ldots, \alpha_{2^n})$ and, arguing by induction on n, assume that $R_{n-1} = L^{(n-1)}$. We now need to show that $R_n = R'_{n-1}$.

For any $\beta_1, \ldots, \beta_{2^{n-1}} \in \hat{A}$ we see that

$$[\delta(\alpha_1, \ldots, \alpha_{2^n}), \delta(\beta_1, \ldots, \beta_{2^{n-1}})] \leq R_n,$$

which shows that $\delta(\beta_1, \ldots, \beta_{2^{n-1}})$ normalizes R_n. Therefore R_n is an ideal in R_{n-1} and it follows that $R_n = R'_{n-1}$. □

Corollary 2.5. *Assume Hypothesis 2.3. Then, for any $\beta \in \hat{A}$, we have $L_\beta \cap \gamma_n(L) = \sum \gamma(\alpha_1, \ldots, \alpha_n)$, where the summation is taken over those $\alpha_1, \ldots, \alpha_n \in \hat{A}$ for which $\alpha_1 \ldots \alpha_n = \beta$. Similarly, $L_\beta \cap L^{(n)} = \sum \delta(\alpha_1, \ldots, \alpha_{2^n})$, where $\alpha_1 \ldots \alpha_{2^n} = \beta$.*

Lemma 2.6. *Assume Hypothesis 2.3 with $k \geq 2$. Suppose there exists an integer u such that $[L, {}_u C_L(a)] = 0$ for any $a \in A^\#$. Then L is nilpotent of $\{p, u\}$-bounded class.*

Proof. By Theorem 2.1, L is solvable and the derived length d of L is at most $(u+1)^{p-1} + log_2 u$. We will prove the lemma by induction on d. Applying the inductive hypothesis to L' assume that L' is nilpotent of $\{p, u\}$-bounded class e, say.

Let B be any subgroup of A of order p^2, and let $B_1, \dots, B_{p+1}$ be the cyclic subgroups of B. We set $C_i = C_L(B_i)$, $1 \le i \le p+1$. Then $L = \sum_i C_i$. Let $r = (u-1)(p+1)+1$. If $Z = Z(L')$ we obviously have $[Z, X, Y] = [Z, Y, X]$ for any subsets X, Y of L. Having this in mind we write

$$[Z, {}_r L] = [Z, {}_r \sum_i C_i] = \sum [Z, {}_{u_1} C_1, \dots, {}_{u_{p+1}} C_{p+1}],$$

where $u_1 + u_2 + \dots u_{p+1} = r$. The number r is big enough to ensure that $u_i \ge u$ for some i, so it follows that $[Z, {}_{u_1} C_1, \dots, {}_{u_{p+1}} C_{p+1}] = 0$ since $[L, {}_u C_i] = 0$. Thus, $[Z, {}_r L] = 0$ and therefore $Z \le Z_r(L)$, where $Z_r(L)$ is the rth term of the upper central series of L. Applying this argument repeatedly to L/Z, $L/Z_2(K')$ and so on, we conclude that $L' \le Z_{er}(L)$ and therefore L is of nilpotency class at most $er+1$. □

Theorem 2.7. *Assume Hypothesis 2.3 with $k \ge 3$.*

1. *If $\gamma_{k-2}(C_L(a))$ is nilpotent of class at most c for any $a \in A^{\#}$, then $\gamma_{k-2}(L)$ is nilpotent and has $\{c, k, p\}$-bounded class.*
2. *If, for some integer d such that $2^d + 2 \le k$, the dth derived group of $C_L(a)$ is nilpotent of class at most c for any $a \in A^{\#}$, then $L^{(d)}$ is nilpotent and has $\{c, k, p\}$-bounded class.*

Proof. 1. Obviously, for any $\beta, \alpha_1, \dots, \alpha_{k-2} \in \hat{A}$ there exists $a \in A^{\#}$ such that $L_\beta, L_{\alpha_1}, \dots, L_{\alpha_{k-2}} \le C_L(a)$. Since $\gamma_{k-2}(C_L(a))$ is nilpotent of class at most c, it follows that $[L_\beta, {}_{c+2}\gamma(\alpha_1, \dots, \alpha_{k-2})] = 0$. Now, using that $L = \bigoplus_\beta L_\beta$, we derive that $[L, {}_{c+2}\gamma(\alpha_1, \dots, \alpha_{k-2})] = 0$. Corollary 2.5 shows that $C_L(a) \cap \gamma_{k-2}(L) = \sum \gamma(\alpha_1, \dots, \alpha_{k-2})$, where the summation is taken over all those $\alpha_1, \dots, \alpha_{k-2}$ for which $\alpha_1 \dots \alpha_{k-2}(a) = 1$. We now apply Lemma 2.2 with $K = C_L(a) \cap \gamma_{k-2}(L)$ and the spaces $\gamma(\alpha_1, \dots, \alpha_{k-2}) \le C_L(a)$ in place of X_i to deduce that there exists a $\{c, k, p\}$-bounded number u such that $[L, {}_u K] = 0$. But then it follows from Lemma 2.6 that $\gamma_{k-2}(L)$ is nilpotent of $\{p, u\}$-bounded class.

2. The proof of the second claim is not really much different from what we have done above. We establish first that $[L, {}_{c+2}\delta(\alpha_1, \dots, \alpha_{2^d})] = 0$ for all $\alpha_1, \dots, \alpha_{2^d} \in \hat{A}$. Next, we apply Lemma 2.2 to deduce that there exists a $\{c, k, p\}$-bounded number u such that $[L, {}_u C_{L^{(d)}}(a)] = 0$ for all $a \in A$. Finally, we observe that the required assertion follows from Lemma 2.6. □

3. Main results

The next lemma is well-known (see [2, 6.2.2, 6.2.4] for the proof).

Lemma 3.1. *Let A be a finite p-group acting on a finite p'-group G.*

1. *If N is an A-invariant normal subgroup of G, then $C_{G/N}(A) = C_G(A)N/N$.*
2. *If A is an elementary abelian group, and if $A_1, \dots, A_s$ are the maximal subgroups of A, then $G = \langle C_G(A_i) | 1 \le i \le s \rangle$.*

Lemma 3.2. *Let p be a prime, and G a finite p'-group acted on by an elementary abelian p-group A of rank at least 3. Let $A_1, \dots, A_s$ be the maximal subgroups of A. Then*

$$G' = \langle [C_G(A_i), C_G(A_j)] | 1 \le i, j \le s \rangle.$$

Proof. By Lemma 3.1 $G = \langle C_G(A_1), \ldots, C_G(A_s)\rangle$. Consider the subgroup $R = \langle [C_G(A_i), C_G(A_j)] | 1 \le i, j \le s\rangle$. Obviously R is A-invariant so $R = \langle C_R(A_1), \ldots, C_R(A_s)\rangle$. To show that R is normal it is sufficient to establish that $y^x \in R$ for any $y \in C_R(A_i)$ and $x \in C_G(A_j)$. We have $y^x = y^x y^{-1} y$ and obviously both $y^x y^{-1}$ and y belong to R. Hence $y^x \in R$ and R is normal. Using that $G = \langle C_G(A_1), \ldots, C_G(A_s)\rangle$, it is now easy to see that G/R is abelian, as required. □

We are now ready to prove the main results.

Theorem 1.1. *Let A be an elementary abelian group of order p^3 acting on a finite p'-group G. Assume that $C_G(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. Then G is nilpotent and the class of G is bounded by a function depending only on p and c.*

Proof. We know from Ward's result cited in the Introduction that G is nilpotent. Let $L(G)$ be the Lie ring corresponding to the lower central series of G. The construction associating the Lie ring with G is well-known. Let γ_i denote the ith term of the lower central series of G. Set $L_i = \gamma_i/\gamma_{i+1}$ and view L_i as an additive abelian group. Then $L(G) = \bigoplus_i L_i$. If $x \in \gamma_i$, $y \in \gamma_j$, then, for corresponding elements $x\gamma_{i+1}$, $y\gamma_{j+1}$ of $L(G)$, we set $[x\gamma_{i+1}, y\gamma_{j+1}] = [x, y]\gamma_{i+j+1}$. This operation can be uniquely extended by linearity on the additive abelian group $L(G)$ and, equipped with the product, $L(G)$ becomes a Lie ring. The Lie ring has the same nilpotency class as G. In our situation the group A acts naturally on each quotient γ_i/γ_{i+1} and this action extends uniquely to an action by automorphisms on the Lie ring $L(G)$. Lemma 3.1 shows that if $a \in A$, then $C_{L(G)}(a)$ is the direct sum of the quotients $C_{\gamma_i}(a)\gamma_{i+1}/\gamma_{i+1}$. It follows that $C_{L(G)}(a)$ is nilpotent of class at most c for any $a \in A^{\#}$. Finally, we note that $L(G)$ is finite and has the same order as G. Therefore $pL(G) = L(G)$. Set $L = L(G) \otimes \mathbb{Z}[\omega]$. We can view L as a Lie algebra over $\mathbb{Z}[\omega]$ and A as a group acting on L. By Theorem 2.7 L is nilpotent of $\{c, p\}$-bounded class and so is G. □

Theorem 1.2. *Let A be an elementary abelian group of order p^4 acting on a finite p'-group G. Assume $C_L(a)'$ is nilpotent of class at most c for any $a \in A^{\#}$. Then G' is nilpotent and the class of G' is bounded by a function depending only on p and c.*

Proof. Let $A_1, \ldots, A_s$ be the maximal subgroups of A. Then, by Lemma 3.2, $G' = \langle [C_G(A_i), C_G(A_j)] | 1 \le i, j \le s\rangle$. We know that G' is nilpotent. Let $L(G')$ be the Lie ring corresponding to the lower central series of G'. Set $L = L(G') \otimes \mathbb{Z}[\omega]$. We will view L as a Lie algebra over $\mathbb{Z}[\omega]$ and A as a group acting on L. By Theorem 2.7 L' is nilpotent of $\{c, p\}$-bounded class, say e. Let $X_1, \ldots, X_t$ be the images of various subgroups of the form $[C_G(A_i), C_G(A_j)]$ in G'/G''. So L is generated by the sets $X_1, \ldots, X_t$. For any $i, j, k \le s$ we observe that there exists some $a \in A^{\#}$ such that the centralizers $C_G(A_i)$, $C_G(A_j)$, $C_G(A_k)$ are all contained in $C_G(a)$. Therefore $[C_G(A_k), {}_{c+2}[C_G(A_i), C_G(A_j)]] = 1$. Now, if X_l is the image of $[C_G(A_i), C_G(A_j)]$ in G'/G'', it follows that $[C_L(A_k), {}_{c+2}X_l] = 0$, whence $[L, {}_{c+2}X_l] = 0$.

Set $r = (c+1)t + 1$. If $Z = Z(L')$, we obviously have $[Z, X, Y] = [Z, Y, X]$ for any subsets X, Y of L. Having this in mind, and taking into account that L is generated by the sets X_i, we write

$$[Z, {}_r L] = \sum [Z, {}_{u_1} X_1, \ldots, {}_{u_t} X_t],$$

where $u_1 + u_2 + \cdots + u_t = r$. The number r is big enough to ensure that $u_j \geq c+2$ for some j, so it follows that $[Z, {}_{u_1}X_1, \ldots, {}_{u_t}X_t] = 0$ since $[L, {}_{c+2}X_j] = 0$. Thus, $[Z, {}_rL] = 0$ and therefore $Z \leq Z_r(L)$, where $Z_r(L)$ is the rth term of the upper central series of L. Applying this argument repeatedly to L/Z, $L/Z_2(K')$ and so on, we conclude that $L' \leq Z_{er}(L)$ and therefore L is of nilpotency class at most $er + 1$. □

References

[1] G. Glauberman, On solvable signalizer functors in finite groups, Proc. London Math. Soc. **33** (1976), 1–27. MR **54:**5341

[2] D. Gorenstein, Finite groups, New York, Evanston, London: Harper and Row, 1968. MR **38:**229

[3] R. Guralnick and P. Shumyatsky, Derived subgroups of fixed points, preprint, 2000.

[4] E. I. Khukhro and P. Shumyatsky, On fixed points of automorphisms of Lie rings and locally finite groups, Algebra and Logic **34** (1995), 395–405. MR **97c:**20059

[5] V.A. Kreknin, Solvability of Lie algebras with a regular automorphism of finite period, Soviet Math. Dokl. **4** (1963), 683–685.

[6] P. Shumyatsky, On periodic soluble groups and the fixed point groups of operators, Comm. Algebra **20(10)** (1992), 2815–2820. MR **93i:**20038

[7] J.N. Ward, On finite groups admitting automorphisms with nilpotent fixed-point group, Bull. Austral. Math. Soc. **5** (1971), 281–282. MR **45:**5224

[8] J.N. Ward, On finite soluble groups and the fixed-point groups of automorphisms, Bull. Austral. Math. Soc. **5** (1971), 375–378. MR **45:**3572

Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil

E-mail address: `pavel@ipe.mat.unb.br`