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A SHORT PROOF OF AN INDEX THEOREM

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Abstract. We give a KK-theoretical proof of an index theorem for Dirac-Schrödinger operators on a noncompact manifold.

1. Introduction

Index theorems, generally speaking, express an analytical index in terms of topological information. An analytical index is usually some generalization of the classical Fredholm index of a Fredholm operator, and the relevant topological information is usually given by the cohomological image of a K-theory/K-homology pairing, thus involving a Chern character map into a suitable cohomology theory. The best known example of an index theorem is the Atiyah-Singer index theorem [6]. A more complicated example is supplied by the Baum-Connes conjecture, where the analytical side is a K-group, and the other side is the topological K-homology (the elliptic operator group) of a suitable classifying space [8].

Anghel's index theorem is an index formula for Dirac-Schrödinger operators. These operators are of the form $D + iA$, where D is a (generalized) Dirac operator and iA is a skew-adjoint order zero operator, both acting on L^2 sections of some bundle. The theorem involves warped cones, which are manifolds that have a collar at infinity. More precisely, they are isomorphic outside a compact set to $\mathbb{R} \times N$ with Riemannian metric $dr^2 + f(r)^2\tilde{g}$, where $\tilde{g}$ is the Riemannian metric of the compact manifold N and f is a nondecreasing function $f : \mathbb{R} \to \mathbb{R}^+$.

We will give a short proof of the following theorem, the Euclidean space version of which was first proven by Callias [14], and then proven in greater generality by Anghel [1, 2].

Theorem 1.1. *Let $D + iA$ be a Dirac-Schrödinger operator over a warped cone M with compact even-dimensional base N. If A^2 becomes arbitrarily large outside a compact subset of M, and $[D, A]$ is bounded, then $D + iA$ is Fredholm, with index given by*

$$\int_N \widehat{A}(TN) \wedge \operatorname{ch} V^+ d(\mathrm{vol}_N),$$

where $\widehat{A}$ denotes Atiyah's A-genus and V^+ is the positive eigenbundle of A over a copy of N contained in a neighbourhood of infinity such that A is invertible in that neighbourhood.

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Anghel's original proof of this theorem used differential geometry and a "cutting and pasting" argument to reduce the problem to one that can be solved by a separation of variables technique.

2. Outline of the proof

We give a short proof of Anghel's theorem using KK-theory. The strategy of the proof is as follows:

First of all, we will show that the hypothesis of the theorem gives certain naturally defined K-theory or K-homology cycles over N and M:

i) The Dirac operators over N and over M give cycles $[D_N] \in KK^0(C(N), \mathbb{C})$ and $[D_M] \in KK^1(C_0(M), \mathbb{C})$.
ii) The geometrical properties of M give a cycle $[E] \in KK^1(C(N), C(M))$.
iii) The endomorphism A defines cycles $[A] \in KK^1(\mathbb{C}, C_0(M))$ and $[V^+] \in KK^0(\mathbb{C}, C(N))$.
iv) The index of $D + iA$ defines a cycle $[D + iA] \in KK(\mathbb{C}, \mathbb{C})$.

Then we show, by computing several Kasparov products, that

$$\operatorname{Ind}(D + iA) = [D + iA] = [A] \otimes_{C_0(M)} [D_M] \tag{1}$$

$$= ([V^+] \otimes_{C(N)} [E]) \otimes_{C_0(M)} [D_M] \tag{2}$$

$$= [V^+] \otimes_{C(N)} [D_N]. \tag{3}$$

The Kasparov product of K-theory and K-homology cycles over a *compact* manifold coincides with Atiyah's [5] pairing of $K_*(X)$ and $\operatorname{Ell}(X)$. We finish the proof by observing that the Atiyah-Singer index theorem, which expresses this pairing in terms of cohomology, gives exactly the result we are looking for:

Theorem 2.1 (Atiyah-Singer [6, 20])**.** *If N is a compact spin^c manifold, V is a vector bundle over N, and D_N is a (generalized) Dirac operator over N, then*

$$[V] \otimes_{C(N)} [D_N] = \int_N \widehat{A}(TN) \wedge \operatorname{ch} V^+ d(\operatorname{vol}_N).$$

3. Cycles given by Dirac operators

Let us begin with a discussion of Clifford bundles, principally because the fact that the index of the target KK-group of a Dirac operator depends on the dimension of the base manifold can be understood in terms of an isomorphism of Clifford algebras. Let $\Gamma_0(S)$ denote the C^*-algebra of C_0-sections of a bundle S. Recall that a bundle S with Riemannian metric and connection is a *complex $\mathbb{Z}_2$-graded Dirac bundle* [20] over M if the sections of S are left modules over the naturally $\mathbb{Z}_2$-graded bundle $\mathbb{C}l(M)$ which are compatible with the metric and connection on $\mathbb{C}l(M)$ in the sense that

i) $\langle ts_1, s_2\rangle + \langle s_1, ts_2\rangle = 0$ for all $t \in \Gamma_0(T^*M)$ and $s_i \in \Gamma_0 S$;
ii) $\nabla_t(\omega s) = (\nabla_t \omega)s + \omega \nabla_t s$ for all $t \in \Gamma_0(T^*M)$, $\omega \in \Gamma_0(\mathbb{C}l(M))$, $s \in \Gamma_0(S)$.

Such a bundle is said to be $\mathbb{Z}_2$-graded if there is a grading on S that is preserved by covariant differentiation and is compatible with the grading on $\mathbb{C}l(M)$. The basic example of a complex $\mathbb{Z}_2$-graded Dirac bundle is $\mathbb{C}l(M) \otimes E$, where E is some Riemannian vector bundle over M with connection ∇^E. If M is odd-dimensional, the volume form ω in $\mathbb{C}l(M)$ can be regarded as the generator of the complex Clifford algebra C_1, and $\mathbb{C}l(M) = \mathbb{C}l(M)^0 \otimes_{\mathbb{C}} C_1$. Therefore, a Dirac bundle over an odd-dimensional manifold can be regarded as a $C_1 \otimes \Gamma_0(M)$-module. The Dirac operator D on a $\mathbb{Z}_2$-graded complex Dirac bundle S is always of odd degree, so we can define a $KK(\Gamma_0(\mathbb{C}l(M)), \mathbb{C})$ cycle by $[D] := (S, \phi, D)$, where ϕ is the standard Clifford action. If the dimension of M is odd, this cycle is actually a $KK^1(\Gamma_0(\mathbb{C}l(M)^0), \mathbb{C})$ cycle, whereas if the dimension of M is even, we have a KK^0 cycle. Because of the topological Thom isomorphism [10], we obtain the same KK-groups if we forget the Clifford structure on the C^*-algebras, and regard (S, ϕ, D) as a $KK^0(\Gamma_0(M), \mathbb{C})$ cycle if M is even-dimensional, and as a $KK^0(\Gamma_0(M) \otimes C_1, \mathbb{C}) \cong KK^1(\Gamma_0(M), \mathbb{C})$ cycle if M is odd-dimensional. It is more convenient to work with cycles over $\Gamma_0(M)$ if we wish to apply the Atiyah-Singer index theorem.

Note that if D is a Dirac operator on S, we can tensor S with any bundle E, obtaining another Dirac operator. This new Dirac operator is usually called the Dirac operator on S with coefficients in E. For brevity we denote the L^2-spaces that the operators act on by $L^2(S)$ instead of $L^2(M, S)$, etc. Finally, we regard the graded complex Clifford algebra C_1 as two copies of $\mathbb{C}$. Elements of the form (e, e) are then of degree 0 for the grading, and elements of the form $(e, -e)$ are of degree 1.

Lemma 3.1. *Suppose S is a Dirac bundle over M. Let H and C be the graded Hilbert spaces $L^2(S \otimes E) \oplus L^2(S \otimes E)$ and $L^2(S) \oplus L^2(S)$, respectively. Let D be the Dirac operator on S, and let D_E be the Dirac operator on S with coefficients in E. Suppose A is a smooth self-adjoint vector bundle endomorphism, such that $|D_E \pm iA|^2 - D_E^2$ is bounded below.*

Then, if the following three cycles are well-defined, the third one is the Kasparov product of the first two:

$$[D] := \left(C, \phi, \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}\right) \in KK(C_0(M) \otimes C_1, \mathbb{C}),$$

$$[A] := \left(\Gamma_0(E) \otimes C_1, 1, \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\right) \in KK(\mathbb{C}, C_0(M) \otimes C_1),$$

and

$$[D + iA] := \left(H, 1, \begin{pmatrix} 0 & D_E - iA \\ D_E + iA & 0 \end{pmatrix}\right) \in KK(\mathbb{C}, \mathbb{C}).$$

Remark. The main case of interest is one where the commutator of D_E and $1 \otimes_{\mathbb{C}} A$ is bounded. This immediately implies the semiboundedness of $|D_E \pm iA|^2 - D_E^2$.

Proof. The action of $\phi : C_0(M) \otimes C_1 \to \mathcal{L}(C)$ is given by

$$\phi : b \oplus b \mapsto \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \qquad \phi : b \oplus -b \mapsto \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix},$$

and a calculation with gradings shows that there is a Hilbert module isomorphism that identifies H with the inner tensor product of $B := \Gamma_0(E) \otimes C_1$ and $C = L^2(S) \oplus L^2(S)$ over ϕ.

In order to apply the criterion for an unbounded cycle to be the Kasparov product of two given cycles [23], we have to verify a semiboundedness condition and a connection condition. Define

$$\widetilde{L} := \begin{pmatrix} 0 & D_E - iA \\ D_E + iA & 0 \end{pmatrix} : H \to H \quad \text{and} \quad \widetilde{D} := \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} : C \to C.$$

We denote the image of $[D+iA]$ under the module isomorphism $\omega : H \to B\hat{\otimes}_\phi C$ by $(B\hat{\otimes}_\phi C, 1, G)$. We now verify the connection condition. We have to show that $T_b\widetilde{D} - (-1)^{\partial b} G T_b : C \to B\hat{\otimes}_\phi C$ is bounded for all homogeneous b in some dense subset of B, where $T_b : c \mapsto b \otimes c$. We take b to be smooth and compactly supported in order to satisfy the appropriate range and domain conditions.

If we take the case of $b = g \oplus -g$, we get

$$(\omega^{-1}T_{g\oplus -g}\widetilde{D} + \widetilde{L}\omega^{-1}T_{g\oplus -g})\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} -ig\otimes Dl_1 \\ ig\otimes Dl_2 \end{pmatrix} + \begin{pmatrix} D_E(ig\otimes l_1) \\ -D_E(ig\otimes l_2) \end{pmatrix} + \begin{pmatrix} Ag\otimes l_1 \\ Ag\otimes l_2 \end{pmatrix}.$$

Since g is compactly supported, Ag is bounded, and the term on the right can be neglected. The compactly supported first-order differential operators $l \mapsto g\otimes Dl$ and $l \mapsto D_E(g \otimes l)$ have the same symbols, and hence differ by a bounded operator. The case of b with even degree is similar.

We come to the semiboundedness condition. We are to show that

$$[G, (A\oplus -A)\hat{\otimes}_\phi \operatorname{Id}]$$

is semibounded below. As an operator on H, this commutator is

$$[\widetilde{L}, \widetilde{A}] = 2\begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix} + i\begin{pmatrix} DA - AD & 0 \\ 0 & AD - DA \end{pmatrix}, \tag{4}$$

which is semibounded below by hypothesis. □

In order to apply this lemma, one must verify that the 3 cycles mentioned are in fact cycles, if A goes to infinity at infinity as in the statement of Anghel's theorem. For the cycle denoted $[A]$, the only thing to check is that $(1+A^2)^{-1}$ is compact in the Hilbert module sense, which (for noncompact M) is true if and only if $A^2 - \lambda$ is positive, for any real λ, outside a compact subset of M. For the cycles given by Dirac operators, more work is required. The principal question is if the operator $D_E - iA$ from the above proof has an L^2-index at all. This corresponds to proving that the resolvent of $L := \left(\begin{smallmatrix} 0 & D_E - iA \\ D_E + iA & 0 \end{smallmatrix}\right) : H \to H$ is compact, or, equivalently, to proving that L has discrete spectrum. There is already a lot of literature on the subject of Fredholmness for first-order operators. In the case of operators of this particular type, Anghel has studied this question [2, 3, 4]. We give a different proof, based on a general technique adapted from Bochner's method as used by Gromov and Lawson [18]. First let us obtain a substitute for Lichnerowicz's identity by trivially rewriting our hypothesis in terms of quadratic forms.

Lemma 3.2. *If D and A have bounded commutator and A^2 goes to infinity, there is a semibounded endomorphism R that becomes arbitrarily large outside a compact set, such that*

$$\langle Ls, Ls\rangle = \langle Ds, Ds\rangle + \langle Rs, s\rangle$$

for all s in the domain of L and D.

Proof. This is a consequence of the already used fact that L and D are closed self-adjoint operators (for a proof of this, see Chernoff [16]). □

Next we prove a lemma about approximation of L on a subspace by bounded operators.

Lemma 3.3. *If H_c is a vector space such that $\|Ls\| \le c\|s\|$ for all $s \in H_c$, then H_c is finite-dimensional.*

Proof. Define $\chi(A)$ to be the characteristic function of the measurable set $A \subset M$, and let $\|s\|_A := \|\chi(A)s\|$. Since the norm is an integral norm, $\|s\| = \|s\|_A + \|s\|_{M\setminus A}$.

Let $-b$ be a lower bound for the endomorphism R in the lemma, and choose $c_0 > -b$ so large that $\frac{c_0 - c^2}{c_0 + b} > 1/2$. There is a compact set K such that $R \geq c_0 \operatorname{Id}$ outside K, and of course $R \geq -b \operatorname{Id}$ on K. If $s \in H_c$, then

$$\|Ds\|^2 + \langle Rs, s\rangle \leq c^2\|s\|^2$$

so that

$$\|Ds\|^2 + c_0\|s\|^2_{M\setminus K} \leq c^2\|s\|^2 + b\|s\|^2_K.$$

We conclude that

$$(c_0 - c^2)\|s\|^2 \leq (c_0 + b)\|s\|^2_K.$$

Now let $Q := (i + L/2c)^{-1}$ and $\mathcal{S} := \chi(K)(1 - QL/2c)$, so that

$$\|\mathcal{S}s\| + 1/2\|\chi(K)Q\|\,\|s\| \geq \|s\|_K \geq \frac{c_0 - c^2}{c_0 + b}\|s\|,$$

and because of the way that c_0 was chosen, there is a $c' > 0$ such that $\|\mathcal{S}s\| \geq c'\|s\|$ for all s in H_c, implying that $\mathcal{S} : H_c \to H$ is injective and has closed range. By the Rellich lemma, the operator $\mathcal{S}^* = -iQ^*\chi(K)$ is compact, so $\mathcal{S}$ is of finite rank, and H_c is finite-dimensional. □

Finally, we apply the well-known Glazman variational lemma [27, p. 233].

Proposition 3.4 (Glazman). *Let A be a self-adjoint Hilbert space operator that is semibounded from below. Let $N_h(\lambda)$ denote the number of eigenvalues in $(-\infty, \lambda]$, with multiplicity, and counting points of the continuous spectrum as points with infinite multiplicity. Then*

$$N_h(\lambda) = \sup_{H \in \mathcal{H}} \dim H,$$

where the supremum is taken over all subspaces H which are such that $\langle Ah, h\rangle \leq \lambda\langle h, h\rangle$ for all $h \in H$.

Setting $A = L^2$ and using Lemma 3.3, we find that L^2 has only a point spectrum, consisting of isolated points with no finite point of accumulation.

We now know that $(H, \underline{1}, L)$ is an unbounded Kasparov cycle, corresponding to the element of $\mathbb{Z}$ given by $\operatorname{Ind}(1 + L^+L^-)^{-1/2}L^+$.

4. A geometrically defined $KK^1(C(N), C_0(M))$ cycle

We have now established all the claims made in section 2 except those involving the cycle $[E] \in KK^1(C(N), C_0(M))$. We need to show that taking Kasparov products with $[E]$ takes a Dirac operator cycle to a Dirac operator cycle, and takes $[V^+]$ to $[A]$. The existence of the cycle $[E]$ and the fact that it maps a Dirac operator cycle to a Dirac operator cycle is a KK-theoretical reformulation of one of the basic results of Baum-Douglas-Taylor's topological K-homology [9, 19].

Lemma 4.1. *The cycle $[E] \in KK^1(C(N), C_0(M))$ is given by*

$$\left(C_0(W) \otimes C_1, \phi, \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}\right),$$

where $W \cong (-\infty, \infty) \times N$ is the collar of M, ϕ is induced by the obvious map from W to N, and $h : (-\infty, \infty) \to \mathbb{R}$ is any continuous function that goes to infinity at *both endpoints. As an element of* $\mathrm{Ext}(C(N), C_0(M))$*, this cycle becomes the exact sequence*

$$0 \to C_0(M) \to C_v(M) \to C(N) \to 0,$$

where $C_v(M)$ *is the subalgebra of* $C_b(M)$ *given by functions having a radial limit at infinity.*

Proof. Under the isomorphism of $KK^1(C(N), C_0(M))$ and $\mathrm{Ext}(C(N), C_0(M))$ given by "cutting down" the representation, the given KK^1 cycle corresponds to the extension with Busby map $b : C(N) \to C_b(M)/C_0(M)$ given by $f \mapsto \tilde{f} \pmod{C_0(M)}$, where $\tilde{f} \in C_b(M)$ is any function which has f as a (uniform) radial limit at infinity. Functions $\tilde{f}$ having this property necessarily form a C^*-algebra, denoted $C_v(M)$, and hence the cycle $[E]$ is represented by the short exact sequence

$$0 \to C_0(M) \to C_v(M) \to C(N) \to 0. \quad \square$$

As mentioned above, it is known that $[E] \otimes_{C_0(M)} [D_M] = [D_N]$. However, this fact can be obtained from our previous calculations. If we observe that we can take h to be a potential function, in the sense of Anghel's theorem, then the proof of Lemma 3.1 shows that the Kasparov product of $(C_0(W) \otimes C_1, \phi \otimes 1, (\begin{smallmatrix} h & 0 \\ 0 & -h \end{smallmatrix})) \in KK(C(N), C_0(M))$ and $(L^2(M, S) \otimes C_1, \phi_2, (\begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix}))$ is

$$\left(L^2(W, S) \oplus L^2(W, S), \phi \oplus \phi, \begin{pmatrix} 0 & D_W + ih \\ D_W - ih & 0 \end{pmatrix} \right) \in KK(C(N), \mathbb{C}).$$

Now we write $D_W + ih$ as a sum of longitudinal and transverse parts, $D_N + (e_r \partial_r + ih)$. The transverse part commutes exactly with ϕ, since ϕ is constant in the transverse direction. We can use a homotopy to get rid of the transverse part and to simultaneously replace $L^2(W, S)$ by the module of radially constant functions $i^* L^2(N, S)$.

Next we proceed to show that $[A] = [V^+] \otimes_{C(N)} [E]$, where $[A] \in KK^1(\mathbb{C}, C_0(M))$ is $(\Gamma_0(E) \otimes C_1, 1, (\begin{smallmatrix} A & 0 \\ 0 & -A \end{smallmatrix}))$ and $[V^+] \in K_0(N)$ is the cycle corresponding to the positive eigenbundle of A over some leaf N which is in the component of infinity of the subset of M where r is invertible.

The quickest way to do this calculation is probably via the isomorphism with K-theory, interpreting $\cdot \otimes_{C(N)} E$ as the exponential map δ in the 6-term exact sequence:

$$\begin{array}{ccccc} K_1(C(N)) & \longleftarrow & K_1(C_v) & \longleftarrow & K_1(C_0(M)) \\ \big\downarrow{\scriptstyle \partial} & & & & \big\uparrow{\scriptstyle \delta} \\ K_0(C_0(M)) & \longrightarrow & K_0(C_v) & \longrightarrow & K_0(C(N)) \end{array}$$

Supposing that p is a projection onto the given n-dimensional positive eigenbundle of A, we recall that the K-theory exponential map takes $[p] - [I_n]$ to $[\exp(2\pi i P)] \in i^* K_1(C_0(M))$, where P is a self-adjoint lifting of $p \in M_\infty(C(N))$ to $M_\infty(C_v(M))$. An obvious quotient exact sequence gives another exponential map, $\delta' : K_0(Q) \to K_1(C_0(M))$, where Q denotes the stable outer multiplier algebra $Q := C_{b,s}(M, \mathcal{L})/C_0(M, \mathcal{K})$.

Combining these maps with the Paschke-Valette-Skandalis duality map [29, 28, 19, 26] from $K_0(Q)$ to $\mathrm{Ext}(\mathbb{C}, C_0(M))$, we have the diagram

$$\begin{array}{ccccc} K_1(C_0(M)) & = & K_1(C_0(M)) & \longleftarrow & KK^1(\mathbb{C}, C_0(M)) \\ \Big\uparrow{\scriptstyle \delta} & & \Big\uparrow{\scriptstyle \delta'} & & \Big\downarrow{\scriptstyle \approx} \\ [p] \in K_0(C(N)) & \xrightarrow{\ i\ } & K_0(Q) & \xrightarrow{\ PV\ } & \mathrm{Ext}(\mathbb{C}, C_0(M)) \end{array}$$

where all the maps except δ and i are isomorphisms. Comparing the definition of the two exponential maps δ and δ', we see that the map i takes the projection p to the class of P in the K_0 group of Q, if P is regarded as a matrix with coefficients in $C_b(M)$. Since the Busby map of an extension in $\mathrm{Ext}(\mathbb{C}, C_0(M))$ is given by a projection in $C_b(M, \mathcal{L})/C_0(M, \mathcal{K})$, the duality map just maps P to itself, under the above identification. The Busby map of the $KK^1(\mathbb{C}, C_0(M))$ cycle $[A] := (\Gamma_0(E) \otimes C_1, 1, (\begin{smallmatrix} A & 0 \\ 0 & -A \end{smallmatrix}))$ is the image of $l(A)$ in $C_b(M, \mathcal{L})/C_0(M, \mathcal{K})$, where $l : \mathbb{R} \to \mathbb{R}$ is any bounded function with limit 1 at infinity to the right and limit 0 at infinity to the left. But if we restrict $b(A)$ to a leaf N_r, then the condition on the endomorphism A implies that $b(A)$ approaches a projection in the class of p as r approaches infinity.

This completes the proof of the theorem.

Finally, let us point out that the standard differential geometry construction of a tubular neighbourhood gives a large supply of manifolds with a collar at infinity. Hence one has theorems such as the following known result:

Corollary 4.2 ([4, 21])**.** *Let $D + iA$ be a Dirac-Schrodinger operator which is bounded strictly away from zero outside a compact set $K \subset M$. Then the index of $D + iA$ is given by*

$$\int_N \widehat{A}(TN) \wedge \operatorname{ch} V^+ d(\mathrm{vol}_N),$$

where V^+ is the positive eigenbundle of A on N, the boundary of any compact set containing K.

In terms of KK-theory, this theorem reflects the fact that Fredholm cycles are stable under coning operations, as shown in the work of J. Cheeger [15]. We only outline a proof.

Proof. Choose a tubular neighbourhood of N, obtaining a warped product. Blow up the metric at N, so that the commutator $[D, A]$ goes to zero at N, and rescale the potential (outside a compact set) by $1/\|[D, A]\|$. This puts us in the situation of the previous theorem. □

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