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A SHORT PROOF OF ERGODICITY OF BABILLOT-LEDRAPPIER MEASURES

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ABSTRACT. Let M be a compact manifold, and let ϕ_t be a transitive homologically full Anosov flow on M. Let $\widetilde{M}$ be a $\mathbb{Z}^d$-cover for M, and let $\widetilde{\phi_t}$ be the lift of ϕ_t to $\widetilde{M}$. Babillot and Ledrappier exhibited a family of measures on $\widetilde{M}$, which are invariant and ergodic with respect to the strong stable foliation of $\widetilde{\phi_t}$. We provide a new short proof of ergodicity.

Let M be the unit tangent bundle of a compact Riemann surface, and let ϕ_t be the geodesic flow on M. Let $\widetilde{M}$ be a $\mathbb{Z}^d$-cover of M, and let $\widetilde{\phi_t}$ be the lift of ϕ_t. Babillot and Ledrappier [BL96] constructed a family of measures $\{\mu_v\}_{v\in\mathbb{R}^d}$ on $\widetilde{M}$, which are preserved by the horocycle flow on $\widetilde{M}$, and proved that they are ergodic. Kaimanovich [K98] gave an alternative proof of ergodicity in case of Liouville measure ($=\mu_0$).

The construction of Babillot and Ledrappier generalizes to a $\mathbb{Z}^d$-extension of an arbitrary Anosov flow, where it gives a family of measures $\{\mu_v\}_{v\in\mathbb{R}^d}$ invariant w.r.t. the strong stable foliation for $\widetilde{\phi_t}$. The lift of the measure of maximal entropy coincides with μ_0. In this setting Pollicott [P98] proved ergodicity of μ_0 by a different method, and Coudene [C99] obtained a generalization to all $\{\mu_v\}_{v\in\mathbb{R}^d}$. We give a new proof of ergodicity of μ_v for all $v \in \mathbb{R}^d$ using a theorem of Guivarc'h [G89].

The flow $\phi_t : M \to M$ is said to be *homologically full* if there exists a closed orbit of ϕ_t in each homology class; see [S93].

Theorem (Babillot and Ledrappier [BL96], Pollicott [P98], Coudene [C99]). *Let $(\widetilde{M}, \widetilde{\phi_t})$ be a $\mathbb{Z}^d$-extension of a transitive Anosov flow, and let $\{\mu_v\}_{v\in\mathbb{R}^d}$ be the family of Babillot-Ledrappier measures on $\widetilde{M}$. If ϕ_t is homologically full, then for any $v \in \mathbb{R}^d$ the strong stable foliation of $\widetilde{\phi_t}$ is ergodic w.r.t. μ_v.*

Proof. Using a method of [BM77], the problem can be reformulated in terms of symbolic dynamics; see [P98]. Let (Σ, T) be a topologically mixing one-sided shift of finite type (see e.g. [B75] for the definitions). Let $r : \Sigma \to \mathbb{R}^+$ and $\psi : \Sigma \to \mathbb{Z}^d$ be Hölder continuous functions, and assume that ψ depends only on the first two coordinates, i.e. $\psi(x) = \psi(x_1, x_2)$. Consider the set

$$\Lambda = \{(x, t, i) : 0 \le t \le r(x)\} \subset \Sigma \times \mathbb{R}^+ \times \mathbb{Z}^d$$

with identifications $(x, r(x), j) = (Tx, 0, j + \psi(x))$.

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Define an equivalence relation $R_\Lambda \subset \Lambda \times \Lambda$ by setting $(x,t,i) \sim (y,s,j)$ if there exist $m,n \geq 0$ such that

$$T^m(x) = T^n(y), \tag{1}$$

$$r_m(x) - t = r_n(y) - s, \tag{2}$$

$$i + \psi_m(x) = j + \psi_n(y), \tag{3}$$

where r_n and ψ_n denote the corresponding ergodic sums: $r_n(x) = r(x) + r(Tx) + \ldots + r(T^{n-1}x)$ and $\psi_n(x) = \psi(x) + \psi(Tx) + \ldots + \psi(T^{n-1}x)$.

For any $v \in \mathbb{R}^d$, consider the potential $\phi_v : \Sigma \to \mathbb{R}$, given by $\phi_v = \lambda_v r(x) - \langle \psi(x) | \, v \rangle$, where $\langle \cdot \, | \, \cdot \rangle$ stands for the scalar product and λ_v is the unique real number such that topological pressure $P(\phi_v) = 0$. Let ν_v be the eigenmeasure for the corresponding Perron-Frobenius-Ruelle operator (the reader is referred to e.g. [B75] for the definitions). Set

$$\mu_v = \nu_v \times e^{-\lambda t} dt \times e^{\langle z | v \rangle} dz. \tag{4}$$

Ergodicity of the strong stable foliation w.r.t. Babillot-Ledrappier measures amounts to ergodicity of R_Λ w.r.t. μ_v; see [BL96].

It is convenient to consider the space $Y = \Sigma \times \mathbb{R} \times \mathbb{Z}^d$ rather than Λ, where the equivalence relation R_Y and measures μ_v are defined by (1)-(3) and (4), respectively. It suffices to prove ergodicity of R_Y w.r.t. m_v, or, equivalently, w.r.t. $\nu_v \times dt \times dz$, where dz is Haar measure on $\mathbb{Z}^d$.

Recall that a Hölder continuous function $f : \Sigma \to G$, where G is a locally compact abelian polish group, is called *periodic* (see [G89]) if there exist a nonconstant measurable $g : \Sigma \to \mathbb{S}^1$, a constant $z \in \mathbb{S}^1$ and a character $\gamma \in \hat{G}$ such that $\gamma \circ f(x) = z g(Tx) \bar{g}(x)$. If f is not periodic, it is called *aperiodic*. We set $G = \mathbb{R} \times \mathbb{Z}^d$ and $f(x) = (-r(x), \psi(x))$.

Lemma. *If ϕ_t is homologically full, then f is aperiodic.*

(The proof is given below.)

Consider the skew-product transformation $T_f : \Sigma \times G \to \Sigma \times G$ defined by

$$T_f(x,y) = (Tx, y + f(x)).$$

By a theorem of Guivarc'h [G89] (see also [AD99]), T_f is exact w.r.t. $p \times m$, where p is any Gibbs measure and m is Haar measure on G. Now, let R' be the tail equivalence relation for T_f, i.e. $(x,y) \sim (x',y')$ iff for some n

$$\begin{aligned} T^n(x) &= T^n(x'), \\ y + f_n(x) &= y' + f_n(x'). \end{aligned}$$

Clearly, ergodicity of R' amounts to the exactness of T_f, so R' is ergodic. But R' is smaller than R_Y, so R_Y is ergodic.

Proof of the Lemma. If f is periodic, then there exist $\gamma \in \hat{G}$ and $z \in \mathbb{S}^1$ such that $\gamma \circ f_n(x) = z^n$ for any periodic $x \in \Sigma$ with period n. Fix $\alpha \in \mathbb{Z}^d$ and $\delta > 0$. By a result of Sharp [S93], the number of periodic x with $\psi_n(x) = \alpha$ and $r_n(x) \in (T - \delta, T]$, where n is the period of x, is exponential in T. Since the roof function r is strictly positive, the number of possible periods n is linear in T. So, one can find x, y with the same period n, such that

$$0 < |r_n(x) - r_n(y)| < 2\delta, \quad \psi_n(x) = \psi_n(y) = \alpha.$$

This shows that $\gamma(\cdot, \alpha)$ is constant in the first coordinate. Similarly, for any α, $\beta \in \mathbb{Z}^d$ one can find x, y of the same period n with $\psi_n(x) = \alpha$, $\psi_n(y) = \beta$, which shows that γ is constant. Thus, f is aperiodic.

Remark. The theorem of Guivarc'h [G89] was applied in the same way in [ANSS].

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