

**UNIQUENESS AND ASYMPTOTIC BEHAVIOUR  
 FOR SOLUTIONS OF SEMILINEAR PROBLEMS  
 WITH BOUNDARY BLOW-UP**

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ABSTRACT. In this paper we prove uniqueness of positive solutions to logistic singular problems  $-\Delta u = \lambda(x)u - a(x)u^p$ ,  $u|_{\partial\Omega} = +\infty$ ,  $p > 1$ ,  $a > 0$  in  $\Omega$ , where the main feature is the fact that  $a|_{\partial\Omega} = 0$ . More importantly, we provide exact asymptotic estimates describing, in the form of a two-term expansion, the blow-up rate for the solutions near  $\partial\Omega$ . This expansion involves both the distance function  $d(x) = \text{dist}(x, \partial\Omega)$  and the mean curvature  $H$  of  $\partial\Omega$ .

1. INTRODUCTION AND RESULTS

This research originated with the recent paper [10] which contains an exhaustive study of positive solutions  $u$  to the logistic problem:

$$(L) \quad \begin{cases} -\Delta u = \lambda u - a(x)u^p, & x \in \Omega', \\ u = 0, & x \in \partial\Omega', \end{cases}$$

$p > 1$ ,  $\lambda > 0$  a parameter,  $\Omega' \subset \mathbb{R}^n$  a bounded domain and where  $a \in C(\overline{\Omega}')$  satisfies  $a > 0$  in a proper subdomain  $\Omega \subset \overline{\Omega} \subset \Omega'$  while  $a = 0$  in  $\overline{\Omega}' \setminus \Omega$ . It turns out that positive solutions  $u$  can only exist when  $\lambda_1(\Omega') < \lambda < \lambda_1(\Omega)$  ( $\lambda_1$  being the first Dirichlet eigenvalue in the corresponding domain) being unique in that range. Moreover, the solution  $u_\lambda$  to (L) satisfies  $u_\lambda \rightarrow \infty$  uniformly on  $\Omega' \setminus \Omega$  when  $\lambda \uparrow \lambda_1(\Omega)$  while  $u_\lambda \rightarrow \underline{u}$  in  $C^{2,\alpha}(\Omega)$  as  $\lambda \uparrow \lambda_1(\Omega)$ , for certain  $0 < \alpha < 1$ , where  $\underline{u}$  is the minimal solution to the singular boundary value problem,

$$(P) \quad \begin{cases} -\Delta u = \lambda(x)u - a(x)u^p, & x \in \Omega, \\ u = +\infty, & x \in \partial\Omega, \end{cases}$$

being  $\lambda(x) = \lambda_1(\Omega)$  in this precise case. The boundary condition is understood as  $u(x) \rightarrow +\infty$  when  $d(x) := \text{dist}(x, \partial\Omega) \rightarrow 0+$ . Observe that  $a > 0$  in  $\Omega$  while  $a = 0$  on  $\partial\Omega$ .

Our main objectives here are to show the uniqueness of positive solutions to (P) together with producing an exact two-term asymptotic expansion of the solutions  $u$

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to (P) near the boundary, involving the distance  $d$  together with the mean curvature  $H$  of  $\partial\Omega$ . We also analyze a class of perturbations of (P).

Singular boundary value problems such as (P) go back to the pioneering work [5] on automorphic functions and the equation  $-\Delta u = -e^u$  in the plane, and were later studied under the general form  $-\Delta u = -f(u)$  in  $n$ -dimensional domains in [12], [18]. The more subtle questions of blow-up rates near  $\partial\Omega$  and uniqueness of solutions are the goals of more recent literature. The problem  $-\Delta u = a(x)e^u$  plus  $u|_{\partial\Omega} = +\infty$  is shown to exhibit a unique solution in a smooth domain  $\Omega$  together with an estimate of the form  $u = \log d^{-2} + v(d)$  in [14] (where  $a(x) \geq a_0 > 0$  and  $v = O(1)$  as  $d \rightarrow 0$ ) and in [1] (where  $a \equiv 1, v = o(1)$  as  $d \rightarrow 0$ ).

The equation  $-\Delta u = -a(x)u^p$  is first considered in [16] ( $a \equiv 1, p = \frac{n+2}{n-2}$ ), [13], [2], [3], [20] ( $a \geq a_0 > 0, p \geq 3$  in [13]) and [8], [9] where the  $p$ -Laplacian extension  $-\Delta_p u = -a(x)u^q, q > p - 1, a \geq a_0 > 0$ , is studied. In all of them, uniqueness is achieved together with a blow-up estimate under the form  $u \sim Ad^{-\alpha}$  as  $d \rightarrow 0+$ , with  $\alpha = \frac{2}{p-1}$  and  $A = (2(p+1)/\{a(p-1)^2\})^{1/(p-1)}$ .

Uniqueness and blow-up rates of  $-\Delta u = -f(u)$  are treated in [15] and [4] for selective classes of nonlinearities extending  $f = e^u$  and  $f = u^p$ , where the explosive solution at  $x = 0$  of  $-u'' = -f(u)$  is involved. For instance in [4] and the case of  $-\Delta u = -u^p$  it is shown that  $u = Ad^{-\alpha}(1 + O(d))$  as  $d \rightarrow 0+$  allows the possible presence of a second explosive term in the expansion of  $u$  when  $\alpha > 1$  ( $p < 3$ ). An explicit expression for this second term has been recently found in [7] as  $u = Ad^{-\alpha}(1 + Bd + o(d))$ ,  $B$  given as a function of the mean curvature  $H$ , provided that  $1 < p < 3$  (in the radial case this term had been already computed in [4] for all  $p > 1$ ).

In this paper we will produce sharper results in two directions. First, uniqueness of solutions to (P) and some of its perturbations is achieved by removing the condition  $a > 0$  up to the boundary (uniqueness was obtained in [17] under the assumption  $a(x) \geq a_0 > 0$  in  $\bar{\Omega}$ , however considerably weakening the smoothness of  $\Omega$ ). It should be emphasized that  $a = 0$  on  $\partial\Omega$  is a *natural* restriction for  $a$  inherited from the logistic problem (L)! Second, we are extending the scope of the expansion  $u = Ad^{-\alpha}(1 + Bd + o(d))$  both to cover the full range  $p > 1$  and the more general nonlinearities in (P) and its perturbations.

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $C^k$  domain,  $k \geq 4$ , and  $\lambda, a \in C^\alpha(\bar{\Omega})$  such that  $a > 0$  in  $\Omega$ . Then the singular boundary value problem (P) admits a minimal and a maximal classical solution. If moreover  $a = 0$  on  $\partial\Omega$  and*

$$(1) \quad a(x) = C_0 d^\gamma + o(d^\gamma)$$

as  $d \rightarrow 0+$  with  $\gamma > 0$  and  $C_0 > 0$ , then every solution  $u \in C^2(\Omega)$  to (P) satisfies

$$(2) \quad \lim_{d \rightarrow 0+} \frac{u}{Ad^{-\alpha}} = 1,$$

where  $\alpha = (\gamma + 2)/(p - 1)$  and  $A = (\alpha(\alpha + 1)/C_0)^{\frac{1}{p-1}}$ . As a consequence, (P) admits a unique positive solution. In addition, if

$$(3) \quad a(x) = C_0 d^\gamma (1 + C_1 d + o(d))$$

as  $d \rightarrow 0+$ , a further estimate of the blow-up rate is available, namely

$$(4) \quad u(x) = Ad^{-\alpha}(1 + B(s)d + o(d))$$

as  $d \rightarrow 0+$ , where

$$(5) \quad B(s) = \frac{(n-1)H(s) - (\alpha+1)C_1}{\gamma+p+3},$$

with  $H(s)$  standing for the mean curvature of  $\partial\Omega$  in  $s$ , and for every  $x$  near  $\partial\Omega$ ,  $d(x) := \text{dist}(x, \partial\Omega)$  and  $s = s(x)$  means the projection of  $x$  over  $\partial\Omega$ .

*Remark 1.* a) The conclusions remain valid if the constants  $C_0$  and  $C_1$  are allowed to be functions of  $s$ ,  $C_0 = C_0(s)$ ,  $C_1 = C_1(s) \in C^2(\overline{\Omega})$  as long as  $C_0(s) > 0$  on  $\partial\Omega$ . Of course, the special case  $\gamma = 0$  in (1) and (3) falls within the scope of Theorem 1.

b) A special case of estimate (2) with  $a$  vanishing at  $\partial\Omega$  was obtained in [19].

c) Suppose more generally that  $\partial\Omega$  splits into  $\Gamma_1, \Gamma_2, \dots, \Gamma_m$  connected pieces and consider the problem  $-\Delta u = \lambda(x)u - a(x)u^p$  in  $\Omega$ ,  $u|_{\Gamma_i} = \sigma_i$ ,  $1 \leq i \leq m$ , with  $\sigma_i \geq 0$ , some of them  $+\infty$ . Then the conclusions of Theorem 1 hold, since estimates (2) and (4) are understood near the connected components  $\Gamma_i$  such that  $\sigma_i = +\infty$ .

Consider now the more general class of problems:

$$(P)_g \quad \begin{cases} -\Delta u = \lambda(x)u - a(x)(u^p + g(x, u)), & x \in \Omega, \\ u = +\infty, & x \in \partial\Omega, \end{cases}$$

where  $g$  is of lower order than  $u^p$  at infinity. Then we have the following results:

**Theorem 2.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $C^k$  domain,  $k \geq 4$ , and  $\lambda, a \in C^\alpha(\overline{\Omega})$  such that  $a > 0$  in  $\Omega$ . If  $g \in C(\overline{\Omega} \times \mathbb{R})$  is locally Lipschitz in  $\overline{\Omega} \times \mathbb{R}$  and it satisfies the growth conditions*

$$(H)_g \quad \lim_{u \rightarrow 0} \frac{g(x, u)}{u} = 0, \quad \lim_{u \rightarrow +\infty} \frac{g(x, u)}{u^p} = 0$$

*uniformly in  $x \in \overline{\Omega}$ , then problem  $(P)_g$  admits at least a classical solution  $u \in C^{2,\alpha}(\Omega)$ . If  $a(x)$  vanishes on  $\partial\Omega$  and verifies (1), then estimate (2) holds for every possible solution to  $(P)_g$ . If, in addition,  $g$  is continuously differentiable with respect to  $u$ ,  $g'_u \in C(\overline{\Omega} \times \mathbb{R})$ , while*

$$(H)_{g'} \quad \lim_{u \rightarrow 0} \frac{g(x, u)}{u} = 0, \quad \lim_{u \rightarrow +\infty} \frac{g'_u(x, u)}{u^{p-1}} = 0$$

*uniformly in  $\overline{\Omega}$ , then  $(P)_g$  admits a unique positive solution, provided that  $|a|_\infty$  is small enough.*

*Remark 2.* a) The Lipschitz condition on  $g$  can be relaxed to continuity if only strong or weak solutions instead of classical solutions are considered.

b) When  $\lambda(x) > 0$  in  $\Omega$ , precise estimates on the size of  $|a|_\infty$  in order to get uniqueness can be given solely in terms of  $g$  and the geometry of  $\Omega$  (see section 4).

c) If  $a$  verifies (3) and  $g$ , together with its derivatives with respect to  $u$  up to order three, satisfies suitable growth conditions, then every solution to  $(P)_g$  admits in addition the asymptotic expansion (4).

d) The conclusions of Theorem 2 hold for the kind of perturbed equations  $-\Delta u = \lambda(x)u - a(x)u^p + a^*(x)g(x, u)$ ,  $g$  as above,  $a^* \in C^\alpha(\overline{\Omega})$ , but now with  $|a^*|_\infty$  small. However, it is required that  $a^*(x) = O(a(x))$  at  $\partial\Omega$ . Otherwise the problem could fall in the superlinear regime where even the existence of solutions may be lost.

The next sections are dedicated to the proof of Theorems 1 and 2. Section 2 is concerned with existence and uniqueness of (P). Estimate (4) is obtained in section 3 while the analysis of the perturbed problem (P)<sub>g</sub> is carried out in section 4.

2. EXISTENCE AND UNIQUENESS

*Proof of the existence of solutions.* Firstly, it should be remarked that positive solutions to the finite problem  $-\Delta u = \lambda(x)u - a(x)u^p$  in  $\Omega$ ,  $u|_{\partial\Omega} = \sigma(x)$  are unique as a consequence of the general result in [6]. Now choose  $\delta > 0$  such that  $p > 1 + \delta$ , and consider the problem

$$(P)_n \quad \begin{cases} -\Delta u = \lambda(x)u - (a(x) + \frac{1}{n^\delta})u^p, & x \in \Omega, \\ u = n, & x \in \partial\Omega. \end{cases}$$

Since 0 is a subsolution and  $n$  is a supersolution for  $n$  large, (P)<sub>n</sub> admits a solution  $u_n \in C^{2,\alpha}(\overline{\Omega})$  with  $u_n \leq n$ , which as quoted before is unique. Using [6], it can be shown that  $\{u_n\}_n$  is increasing. Our purpose is to pass to the limit as  $n \rightarrow \infty$  in (P)<sub>n</sub>.

The validity of the following lemma (cf. [10]) is the key to guaranteeing the existence of the limit.

**Lemma 3.** *Let  $B \subset \mathbb{R}^n$  be an arbitrary ball and consider the problem*

$$(6) \quad \begin{cases} -\Delta u = \lambda u - Au^p, & x \in B, \\ u = \sigma, & x \in \partial B, \end{cases}$$

where  $A, \lambda$  are positive constants and  $\sigma > (\lambda/A)^{\frac{1}{p-1}}$ . Then the unique solution  $u_{\sigma,B}$  to (6) is a radial function,  $u_{\sigma,B} \in C^\infty(\overline{B})$ , while  $u_{\sigma,B}$  converges as  $\sigma \rightarrow +\infty$  to  $\underline{u}_B$  in  $C^k(B)$  for every  $k \in \mathbb{N}$ . Moreover,  $\underline{u}_B$  defines the minimal solution to (6) with  $\sigma = +\infty$ . Finally, the following estimate holds:

$$\inf_B \underline{u}_B \geq \frac{B_0^2}{2(p+1)AR^2},$$

where  $B_0 = B(\frac{p-1}{2(p+1)}, \frac{1}{2})$  and  $B$  is the Euler Beta function.

Let us finish the proof of the existence assertion. If  $u_n$  is the solution to (P)<sub>n</sub> we have  $-\Delta u_n \leq \sup_B \lambda(x)u_n - \inf_B a(x)u_n^p$  in  $B$ . Since  $u_n \leq n$  in  $\overline{B}$ , we obtain  $u_n < u_{n,B} < \underline{u}_B$  in  $B$ , where  $u_{n,B}$  is as in Lemma 3. This together with the monotonicity of  $\{u_n\}$  leads to  $u_n \rightarrow \underline{u}$ ,  $\underline{u} \in L^\infty_{loc}(\Omega)$ . By elliptic estimates and a standard bootstrapping argument we achieve  $u_n \rightarrow \underline{u}$  in  $C^{2,\alpha}(\Omega)$ .

To prove the minimal character of  $\underline{u}$ , let  $u$  be an arbitrary solution to (P). Since  $u \geq n$  near  $\partial\Omega$ , for every positive integer  $n$ , an inner approximation argument leads to  $u \geq u_n$ , and consequently  $u \geq \underline{u}$ , as was to be proved.

The existence of a maximal solution follows a similar reasoning. □

The following result is especially useful when dealing with the perturbed problem (P)<sub>g</sub>. The proof, which is omitted for brevity, is carried out by applying the sub- and supersolutions method to problem (P) in domains  $\{x \in \Omega : d(x) > 1/n\}$ , and doing  $n \rightarrow \infty$  through a diagonal process.

**Lemma 4.** *Let  $v, w \in C^2(\Omega)$  be functions such that  $v(x) \leq w(x)$  in  $\Omega$ ,  $-\Delta v \leq \lambda(x)v - a(x)v^p$ ,  $-\Delta w \geq \lambda(x)w - a(x)w^p$  in  $\Omega$  and  $\lim_{d(x) \rightarrow 0^+} v(x) = \lim_{d(x) \rightarrow 0^+} w(x) = +\infty$ . Then there exists at least a solution  $u \in C^2(\Omega)$  of problem (P) satisfying  $v(x) \leq u(x) \leq w(x)$  in  $\Omega$ .*

*Remark 3.* The features described in Lemma 4 also hold for the extended problem introduced in Remark 1 c).

*Proof of estimate (2).* We are only assuming that  $a$  verifies hypothesis (1). Fix  $\delta_0 > 0$  so that  $d(x) := \text{dist}(x, \partial\Omega)$  is a  $C^k$  function in  $0 < d(x) < \delta_0$ . Fixing an arbitrary  $\varepsilon > 0$ , we have  $(C_0 - \varepsilon)d^\gamma \leq a(x) \leq (C_0 + \varepsilon)d^\gamma$ , as long as  $0 \leq d < \delta_1$ . Choosing

$$A^+ = \left\{ \frac{\alpha(\alpha + 1)}{C_0 - 2\varepsilon} \right\}^{\frac{1}{p-1}},$$

it is easy to prove that  $u^+ = A^+d^{-\alpha}$ , defined for small  $d > 0$ , is a supersolution. Indeed, it is enough to prove  $-\Delta u^+ \geq \lambda(x)u^+ - (C_0 - \varepsilon)d^\gamma(u^+)^p$  for  $d \sim 0^+$  ( $d < \min\{\delta_0, \delta_1\}$ ). Since  $-\Delta u^+ = -A^+\alpha(\alpha + 1)d^{-\alpha-2} + O(d^{-\alpha-1})$ , this is a consequence of  $\alpha(\alpha + 1) \leq (C_0 - \varepsilon)(A^+)^{p-1} + O(d)$ , for  $d$  small enough, which is in turn implied by the choice of  $A^+$ .

Symmetric calculations show that  $u^- = A^-d^{-\alpha}$ ,  $A^- = (\alpha(\alpha + 1)/(C_0 + 2\varepsilon))^{1/(p-1)}$ , satisfies  $-\Delta u^- \leq \lambda(x)u^- - (C_0 + \varepsilon)d^\gamma(u^-)^p$  near  $\partial\Omega$ , therefore defining a local subsolution to (P) there.

Let us proceed now to estimate the blow-up rate of an arbitrary solution  $u$  to (P), under the form (2).

Let us ascertain under which conditions  $u^+ + K$  turns out to be a supersolution near  $\partial\Omega$ . The function  $\lambda(x)u - a(x)u^p$  is decreasing in  $u$  when  $\lambda(x) \leq 0$ , or if

$u > u_c(x) := \left( \frac{|\lambda^+|_\infty}{a(x)} \right)^{1/(p-1)}$ , with  $\lambda^+ = \max\{\lambda, 0\}$ . Taking into account that

$$u_c(x) \sim \left( \frac{\lambda}{C_0} \right)^{1/(p-1)} \left( \frac{1}{d} \right)^{\frac{\gamma}{p-1}}, \quad \text{while} \quad u^+(x) \sim A \left( \frac{1}{d} \right)^{\frac{\gamma+2}{p-1}},$$

we can ensure that for small  $d$  (say  $0 < d < \delta^*$ ),  $u^+(x) > u_c(x)$  holds. Thus,

$$-\Delta(u^+ + K) \geq \lambda(x)(u^+ + K) - a(x)(u^+ + K)^p,$$

for  $0 < d < \delta^*$ ,  $K > 0$ .

On the other hand, fix  $0 < \tau < \frac{\delta^*}{4}$  and introduce the region  $Q_\tau := \{\tau < d < \frac{\delta^*}{2}\}$ . Then

$$-\Delta(u^+(d - \tau, s) + K) \geq \lambda(x)(u^+(d - \tau, s) + K) - a(d, s)(u^+(d - \tau, s) + K)^p$$

in  $Q_\tau$ , where the functions involved have been expressed in local coordinates  $(d, s)$ . Moreover, this fact is uniformly valid, no matter what the value of  $\tau$  is.

Thus, for every  $\tau \in (0, \frac{\delta^*}{4})$ ,  $v^+ = u^+(d - \tau, s) + K$  is a supersolution to

$$(7) \quad \begin{cases} -\Delta v = \lambda(x)v - a(x)v^p, & x \in Q_\tau, \\ v = u, & x \in \partial Q_\tau, \end{cases}$$

with  $u$  an arbitrary fixed solution to (P), provided that  $K > 0$  is chosen so that  $u^+(\delta^*/2 - \tau, s) + K \geq u(\delta^*/2, s)$ , for every  $0 < \tau < \frac{\delta^*}{4}$ ,  $s \in \partial\Omega$ . In addition,

the auxiliary problem (7) has  $v = u$  as its unique solution. Since  $v_- = 0$  is a subsolution, we conclude

$$u(x) = u(d, s) \leq u^+(d - \tau, s) + K,$$

for every  $s \in \partial\Omega$ ,  $0 < \tau < d < \delta^*/2$  and  $0 < \tau < \delta^*/4$ . Letting  $\tau \rightarrow 0+$  we arrive at  $u(x) = u(d, s) \leq u^+(x) + K$  in  $d(x) < \delta^*/2$ , and we obtain

$$(8) \quad \overline{\lim}_{d \rightarrow 0+} \frac{u(x)}{Ad^{-\alpha}} \leq 1.$$

In what follows we will obtain the complementary estimate to (8). With an argument similar to the previous one, we conclude  $u \geq \theta u^-$  if  $0 < d(x) < \delta^*/4$ , for some  $0 < \theta < 1$ .

Our next objective is finding  $\delta > 0$  small enough,  $\tau_0 = \tau_0(\delta)$  and a constant  $K = K(\delta)$  in such a way that  $u^-(d + \tau, s) - K(\delta)$  defines a subsolution to  $-\Delta u = \lambda(x)u - a(x)u^p$  in the domain  $\Omega_\delta := \{0 < d(x) < \delta\}$  for every  $\tau \in (0, \tau_0(\delta))$ , also satisfying

$$(9) \quad u^-(d + \tau, s) - K(\delta) \leq u(d, s),$$

for  $d = \delta$  and every  $0 < \tau < \tau_0(\delta)$ . Inequality (9) is implied by  $u^-(\delta + \tau, s) - \theta u^-(\delta, s) \leq K(\delta)$ , where  $0 < \tau < \tau_0(\delta)$ . By virtue of the decreasing character of  $u^-$  with  $d$ , this last inequality is a consequence of

$$K(\delta) \sim (1 - \theta)A\delta^{-\alpha},$$

as  $\delta \rightarrow 0+$ .

On the other hand,  $u^-(d + \tau, s) - K(\delta)$  will be a subsolution if

$$(10) \quad u^-(d + \tau, s) - K(\delta) \geq \left( \frac{\lambda}{p(C_0 + \varepsilon)(d + \tau)^\gamma} \right)^{\frac{1}{p-1}},$$

for  $0 < d + \tau < \delta + \tau_0(\delta) = (1 + \zeta)\delta$  if  $\tau_0(\delta) = \zeta\delta$  is chosen for some fixed  $\zeta \in (0, 1)$ . For the validity of (10) it is enough to have

$$A \left( \frac{1}{d + \tau} \right)^\alpha - (1 - \theta)A \left( \frac{1}{\delta} \right)^\alpha > \left\{ \frac{\lambda}{pC_0} \right\}^{\frac{1}{p-1}} \left( \frac{1}{d + \tau} \right)^{\frac{\gamma}{p-1}},$$

for  $0 < d < (1 + \zeta)\delta$ , which in turn is true if  $\zeta, \delta$  are small enough.

We therefore obtain that

$$u^-(d + \tau, s) - K(\delta)$$

defines for  $0 < \tau < (1 + \zeta)\delta$  a family of finite subsolutions to problem  $-\Delta v = \lambda(x)v - a(x)v^p$  in  $\Omega_\delta$ , with  $v = u$  in  $d = \frac{\delta^*}{4}$  and  $v = +\infty$  in  $d = \delta$ . Then  $u^-(d + \tau, s) - K(\delta) \leq u(d, s)$  if  $0 < d < \delta$ , and letting  $\tau \rightarrow 0+$  we arrive at

$$u^-(d, s) - K(\delta) \leq u(s, d), \quad 0 < d < \delta,$$

which completes the proof of (2). □

*Proof of uniqueness.* Now let  $u$  and  $v$  be *positive* solutions to (P). By virtue of (2),  $u$  and  $v$  satisfy  $\lim_{d(x) \rightarrow 0+} u(x)/v(x) = 1$ . Thus, for every  $\varepsilon > 0$ , we can find  $\delta > 0$  (as small as we please) such that

$$(11) \quad (1 - \varepsilon)v(x) \leq u(x) \leq (1 + \varepsilon)v(x)$$

when  $0 < d(x) \leq \delta$ . On the other hand,  $w^- = (1 - \varepsilon)v(x)$  and  $w^+ = (1 + \varepsilon)v(x)$  are sub- and supersolutions to

$$\begin{cases} -\Delta w = \lambda(x)w - a(x)w^p, & x \in \Omega \setminus \Omega_\delta, \\ w = u, & x \in \partial(\Omega \setminus \Omega_\delta), \end{cases}$$

where  $\Omega_\delta = \{0 < d(x) < \delta\}$ . The unique solution to this problem is  $w = u$ . Then  $(1 - \varepsilon)v(x) \leq u(x) \leq (1 + \varepsilon)v(x)$  holds in  $\Omega \setminus \Omega_\delta$ , and (11) is true in  $\Omega$ . Letting  $\varepsilon \rightarrow 0+$  we arrive at  $u = v$ . □

### 3. ESTIMATES FOR THE SECOND TERM

We will now obtain estimate (4), and so we henceforth assume that the function  $a$  verifies (3).

In this case, with an argument similar to the one used to show estimate (2), we can obtain for every  $\varepsilon > 0$  a pair of sub- and supersolutions of the form

$$u^\pm = Ad^{-\alpha}(1 + B^\pm d), \quad 0 < d(x) < \delta,$$

where  $B^+ \geq B$ ,  $B^- \leq B$ ,  $B^\pm \rightarrow B$  as  $\varepsilon \rightarrow 0+$ ,  $B$  given by (5). It is worthy of mention to say that  $\Delta d(x) \rightarrow -(n - 1)H(s)$  when  $d(x) \rightarrow 0+$ , where  $H(s)$  is the mean curvature of  $\partial\Omega$  at  $s$  (cf. [11]).

Moreover,  $u^+ + K$  and  $u^- - K(\delta)$  are still sub- and supersolutions with a convenient choice of  $K$  and  $K(\delta)$ . However, this only leads to estimate (4) in the case  $\alpha > 1$ , that is, for  $p < \gamma + 3$ . This is why we need an alternative procedure for the complementary range  $p \geq \gamma + 3$ . The following reasoning is indeed valid if  $p \geq 2$ .

We are looking for a constant  $K > 0$  such that

$$-\Delta(AKd^{2-\alpha}) \geq \lambda(x)AKd^{2-\alpha} + a(x)(u^+)^p - a(x)(u^+ + AKd^{2-\alpha})^p.$$

This inequality, together with  $-\Delta u^+ \geq \lambda(x)u^+ - a(x)(u^+)^p$ , will imply that  $u^+ + AKd^{2-\alpha}$  is a supersolution to (P) near  $\partial\Omega$ .

Since for  $t \geq 0$ ,  $1 - (1 + t)^p \leq -pt$ , it is enough to prove  $\Delta d^{2-\alpha} \leq -\lambda(x)d^{2-\alpha} + p a(x)(u^+)^{p-1}d^{2-\alpha}$ , which is independent of  $K$ . Now  $\Delta d^{2-\alpha} = (2 - \alpha)(1 - \alpha)d^{-\alpha} + O(d^{-\alpha+1})$ , and  $a(x) \geq C_0d^\gamma(1 - \varepsilon)$  for small  $d$ , so this is a consequence of

$$(2 - \alpha)(1 - \alpha) \leq p \alpha(\alpha + 1)(1 - \varepsilon) + O(d),$$

which in turn holds if  $(2 - \alpha)(1 - \alpha) < p \alpha(\alpha + 1)$ . This inequality holds no matter what the values of  $p$  and  $\gamma$  are, thus we have a supersolution  $v^+ = u^+ + AKd^{2-\alpha}$ , for every  $K > 0$ .

If  $u$  is an arbitrary solution to (P) and we take  $K$  to have  $v^+ \geq u$  in  $d = \delta$ , we obtain, with an argument similar to the one used in section 2, that  $v^+ \geq u$  in  $0 < d(x) < \delta$ . Thus letting  $d \rightarrow 0+$  and then  $\varepsilon \rightarrow 0+$ , we conclude

$$\overline{\lim}_{d \rightarrow 0+} \frac{1}{d} \left( \frac{u(x)}{Ad^{-\alpha}} - 1 \right) \leq B.$$

The lower estimate is *not* obtained in a completely similar way. We are now bound to find  $K$  so that

$$-\Delta(-AKd^{2-\alpha}) \leq -\lambda(x)AKd^{2-\alpha} + a(x)(u^-)^p - a(x)(u^- - AKd^{2-\alpha})^p,$$

and we will get that  $v^- = u^- - AKd^{2-\alpha}$  is a subsolution.

If  $p \geq 2, t \geq 0$ , we have  $1 - (1 - t)^p \geq pt - p(p - 1)t^2/2$ , and it is then sufficient to show

$$\Delta d^{2-\alpha} \leq p a(x)(u^-)^{p-1} \left( d^{2-\alpha} - \frac{p-1}{2} \frac{AKd^{4-2\alpha}}{u^-} \right) - \lambda(x)d^{2-\alpha},$$

which is implied by

$$(2 - \alpha)(1 - \alpha) \leq p a(x) \frac{\alpha(\alpha + 1)}{C_0 d^\gamma} (1 + B^- d)^{p-1} \left( 1 - \frac{p-1}{2} \frac{Kd^2}{1 + B^- d} \right) + O(d).$$

Since in addition

$$\frac{Kd^2}{1 + B^- d} \leq \frac{K\delta^2}{1 - |B^-|_\infty \delta},$$

if we choose  $K\delta^2 = 2\theta(1 - |B^-|_\infty \delta)/(p - 1), 0 < \theta < 1$ , it is enough to have

$$(12) \quad (2 - \alpha)(1 - \alpha) < p \alpha(\alpha + 1)(1 - \varepsilon)(1 - \theta).$$

We can certainly take  $\varepsilon$  and  $\theta$  small enough to satisfy (12). With this choice,  $v^- = u^- - AKd^{2-\alpha}$  is a subsolution.

Now let  $u$  be an arbitrary solution to (P). We need to choose  $K$  so that  $v^- \leq u$  if  $d = \delta$ . Since  $u \geq (1 - \varepsilon)Ad^{-\alpha}$ , it suffices to have

$$A\delta^{-\alpha}(1 + B^- \delta) - AK\delta^{2-\alpha} \leq (1 - \varepsilon)A\delta^{-\alpha}.$$

This last inequality is a consequence of  $\theta > (p - 1)\varepsilon/2$ , which we can achieve by diminishing  $\varepsilon$  if necessary.

To summarize, we arrive at  $u \geq v^-$  in  $0 < d < \delta$ , which in turn implies

$$\liminf_{d \rightarrow 0^+} \frac{1}{d} \left( \frac{u(x)}{Ad^{-\alpha}} - 1 \right) \geq B,$$

and estimate (4) is proved. □

#### 4. PERTURBED PROBLEMS

This section is devoted to the proof of Theorem 2. To begin with the existence notice that  $(P)_g$  admits arbitrarily large supersolutions. Indeed, given  $\varepsilon > 0$ , we have  $g(x, u) \geq -\varepsilon u^p$  if  $u \geq u_0(\varepsilon) > 0$ . So, denoting by  $v$  the unique solution to the problem  $-\Delta u = \lambda(x)u - (1 - \varepsilon)a(x)u^p, u_{\partial\Omega} = +\infty$  and choosing  $\theta > 1$  to have  $\theta v \geq u_0$ , it turns out that  $u^+ = \theta v$  is a supersolution to  $(P)_g$ . On the other hand, it is possible to construct an increasing sequence  $\{u_n\}, u = u_n$  a solution of:

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)(u^p + g(x, u)), & x \in \Omega, \\ u = n, & x \in \partial\Omega, \end{cases}$$

as follows:  $u_0 = 0, u_{n+1}$  obtained using  $u_n$  as a subsolution and  $u^+$  as a supersolution. The arguments in section 2 lead to  $u_n \rightarrow u, u \in C^{2,\alpha}(\Omega)$  a solution to  $(P)_g$ .

To achieve uniqueness we will first prove estimate (2) for all solutions to  $(P)_g$ . Accordingly, let  $u$  be any solution to  $(P)_g$  and choose  $\varepsilon > 0, u_0(\varepsilon)$  as above. Taking  $\delta$  small to have  $u \geq u_0$  in  $\Omega_\delta = \{0 < d(x) < \delta\}$ , it follows that  $u$  is a subsolution to

$$(13) \quad \begin{cases} -\Delta v = \lambda(x)v - a(x)(1 - \varepsilon)v^p, & x \in \Omega_\delta, \\ v = u, & x \in \partial\Omega_\delta. \end{cases}$$

It is easy to prove that  $\theta u$  is a supersolution to (13) if  $\theta$  is large. Since uniqueness holds for problem (13) (Remark 1 c), then Lemma 4 and Remark 3 lead to

$$\overline{\lim}_{d \rightarrow 0^+} \frac{u}{Ad^{-\alpha}} \leq (1 - \varepsilon)^{-1/(p-1)}.$$

Letting  $\varepsilon \rightarrow 0$ , we obtain  $\overline{\lim} u/Ad^{-\alpha} \leq 1$ . The lower estimate is performed in a similar way. Thus (2) is proved.

Now let us undertake the uniqueness assertion. Assume  $g$  satisfies  $(H)_{g'}$ . We claim the existence of  $c > 0$  such that every positive solution  $u$  to  $(P)_g$  verifies  $u \geq c$  in  $\Omega$ . Thus, defining  $M_g = \max\{0, \sup_{u \geq c} \frac{g(x,u)}{u^p}\}$ , we conclude that every solution  $u$  to  $(P)_g$  satisfies  $u \geq w$ , where  $w$  uniquely solves the problem

$$(P)' \quad \begin{cases} -\Delta w = \lambda(x)w - (1 + M_g)|a|_{\infty} w^p, & x \in \Omega, \\ w = +\infty, & x \in \partial\Omega. \end{cases}$$

On the other hand, problem  $(P)'$  can be estimated from below by the one-dimensional version of (6) on intervals  $(-R, R)$ , with  $2R \geq L_{\Omega}$ ,  $L_{\Omega}$  being the minimum distance between parallel hyperplanes enclosing  $\Omega$  (see [10] for more details). Hence, from Lemma 3 we have the estimate

$$(14) \quad \inf_{\Omega} w \geq \frac{2B_0^2}{(p+1)(1+M_g)L_{\Omega}^2|a|_{\infty}}.$$

Thus if  $|a|_{\infty}$  is small enough we have that all positive solutions  $u$  to  $(P)_g$  fall within the range where  $u^{p-1} + g(x, u)/u$  is increasing. The proof of uniqueness then follows word-for-word the one given in section 2.

Finally let us show the claim. If  $\lambda(x) > 0$  in  $\overline{\Omega}$ , direct checking reveals that  $c$  can be chosen as any positive value such that  $u^{p-1} + g(x, u)/u < \inf_{\Omega}(\lambda/a)$  for  $x \in \overline{\Omega}$  and  $0 \leq u \leq c$ . In this case, (14) shows that uniqueness holds provided that  $(1 + M_g)L_{\Omega}^2|a|_{\infty}$  is small. In the case where  $\lambda(x)$  changes sign, the argument in Lemma 3.2 of [10] leads to the desired lower estimate for solutions to  $(P)_g$ .

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