

THE PRODUCT OF A GÂTEAUX DIFFERENTIABILITY SPACE
AND A SEPARABLE SPACE IS A GÂTEAUX
DIFFERENTIABILITY SPACE

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ABSTRACT. This paper shows that the product of a Gâteaux differentiability space and a separable Banach space is again a Gâteaux differentiability space.

A Banach space is called a *Gâteaux differentiability space* if every convex continuous function on it is Gâteaux differentiable at the points of a dense set. For more information about this class and its relatives—weak Asplund and Asplund spaces—see [LP], [NP], [A], [P], [F], [DGZ], [G]. Relatively little is known about the stability of the class of the Gâteaux differentiability spaces. In the late eighties, the second named author showed that if X is a Gâteaux differentiability space, then so is the product of X and the real line; see [P, Proposition 6.5]. The aim of this note is to extend this fact to separable spaces.

Theorem. *Let X be a Gâteaux differentiability space and let Y be an arbitrary separable Banach space. Then $X \times Y$ is also a Gâteaux differentiability space.*

Proof. We shall elaborate Ioffe’s penalty method used in the proof of [P, Proposition 6.5], while incorporating some ideas from Klee [K].

Let B_X and B_Y denote the closed unit balls in X and Y respectively. Let $f : X \times Y \rightarrow \mathbb{R}$ be a convex continuous function. Let Ω be any nonempty open set in $X \times Y$. We have to find a point in Ω where f is Gâteaux differentiable. By a suitable shift of variables we can arrange things so that $(0, 0) \in \Omega$. Since f is continuous at $(0, 0)$, we can find $\Delta > 0$ so that Ω contains $S = 2\Delta B_X \times 2\Delta B_Y$ and that f is bounded on this “square”. Let $M > 0$ be a constant such that $|f(x, y)| < M$ whenever $(x, y) \in 2\Delta B_X \times 2\Delta B_Y$. Let $\{y_i; i \in \mathbb{N}\}$ be a dense subset in ΔB_Y . Define $g : 2\Delta B_X \rightarrow \mathbb{R}$ as

$$g(x) = \sup \left\{ f \left(x, \sum 2^{-i} \alpha_i y_i \right) - \tan \left(\sum \alpha_i^2 \right); \alpha_i \in \mathbb{R}, \sum \alpha_i^2 < \frac{\pi}{2} \right\}, \quad x \in 2\Delta B_X.$$

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Here and below the symbol ‘ \sum ’ will stand for summation for i from one to infinity. Note that for every $x \in 2\Delta B_X$ we have

$$-M < f(x, 0) \leq g(x) \leq \sup \left\{ f \left(x, \sum 2^{-i} \alpha_i y_i \right); \alpha_i \in \mathbb{R}, \sum \alpha_i^2 < \frac{\pi}{2} \right\} \leq M.$$

Hence g is a convex bounded function on $2\Delta B_X$. It follows by [P, Proposition 1.6] that g is Lipschitz on ΔB_X . Then it is well known that g , restricted to ΔB_X , has an extension to a convex Lipschitz function defined on all of X , which we also denote by g .

Since X is a Gâteaux differentiability space, there is $x_0 \in X$, with $\|x_0\| < \Delta$, where the function g is Gâteaux differentiable. Let $\alpha_{i,n} \in \mathbb{R}$, $i, n \in \mathbb{N}$, be such that $\sum_{i=1}^\infty \alpha_{i,n}^2 < \frac{\pi}{2}$ for every $n \in \mathbb{N}$ and

$$f \left(x_0, \sum 2^{-i} \alpha_{i,n} y_i \right) - \tan \left(\sum \alpha_{i,n}^2 \right) \rightarrow g(x_0) \quad \text{as } n \rightarrow \infty.$$

As f is bounded on S we derive that

$$\liminf_{n \rightarrow \infty} \tan \left(\sum \alpha_{i,n}^2 \right) < +\infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \sum \alpha_{i,n}^2 < \frac{\pi}{2}.$$

A diagonal process allows us to assume that $\alpha_{i,n} \rightarrow \alpha_i$ as $n \rightarrow \infty$ for every $i \in \mathbb{N}$. Then

$$\sum \alpha_i^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^N \alpha_i^2 = \lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^N \alpha_{i,n}^2 \leq \liminf_{n \rightarrow \infty} \sum_{i=1}^\infty \alpha_{i,n}^2 < \frac{\pi}{2}.$$

Setting $y_0 = \sum 2^{-i} \alpha_i y_i$, we observe that $y_0 \in \Delta B_Y$. Now, for any $N \in \mathbb{N}$ we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left\| \sum_{i=1}^\infty 2^{-i} \alpha_{i,n} y_i - y_0 \right\| &\leq \limsup_{n \rightarrow \infty} \left\| \sum_{i=1}^N 2^{-i} (\alpha_{i,n} - \alpha_i) y_i \right\| \\ &\quad + \limsup_{n \rightarrow \infty} \left\| \sum_{i=N+1}^\infty 2^{-i} (\alpha_{i,n} - \alpha_i) y_i \right\| \\ &\leq \limsup_{n \rightarrow \infty} \left(\sum_{i=N+1}^\infty 2^{-2i} \right)^{1/2} \left(\sum_{i=N+1}^\infty (\alpha_{i,n} - \alpha_i)^2 \right)^{1/2} \\ &\leq \frac{1}{\sqrt{3}} 2^{-N} \limsup_{n \rightarrow \infty} \left(2 \sum_{i=N+1}^\infty \alpha_{i,n}^2 + 2 \sum_{i=N+1}^\infty \alpha_i^2 \right)^{1/2} < \sqrt{\frac{2\pi}{3}} 2^{-N}, \end{aligned}$$

and hence $\| \sum_{i=1}^\infty 2^{-i} \alpha_{i,n} y_i - y_0 \| \rightarrow 0$ as $n \rightarrow \infty$. Thus, using the continuity of f we have

$$\begin{aligned} g(x_0) &= \lim_{n \rightarrow \infty} \left[f \left(x_0, \sum 2^{-i} \alpha_{i,n} y_i \right) - \tan \left(\sum \alpha_{i,n}^2 \right) \right] \\ &= f(x_0, y_0) - \tan \left(\liminf_{n \rightarrow \infty} \sum \alpha_{i,n}^2 \right) \\ &\leq f(x_0, y_0) - \tan \left(\sum \alpha_i^2 \right) \leq g(x_0). \end{aligned}$$

Now, fix arbitrary $h \in X$ and $j \in \mathbb{N}$. Then for $t > 0$ small enough we have

$$\sum_{i \neq j} \alpha_i^2 + (\alpha_j \pm 2^j t)^2 < \frac{\pi}{2}$$

and hence

$$\begin{aligned} 0 &\leq f(x_0 + th, y_0 + ty_j) + f(x_0 - th, y_0 - ty_j) - 2f(x_0, y_0) \\ &\leq g(x_0 + th) + \tan\left(\sum_{i \neq j} \alpha_i^2 + (\alpha_j + 2^j t)^2\right) \\ &\quad + g(x_0 - th) + \tan\left(\sum_{i \neq j} \alpha_i^2 + (\alpha_j - 2^j t)^2\right) \\ &\quad - 2g(x_0) - 2 \tan\left(\sum \alpha_i^2\right) = o(t) + o(t) \end{aligned}$$

since g is Gâteaux differentiable at x_0 . (This is true also if $\sum \alpha_i^2 = 0$ since $\tan \tau < 2\tau$ for small $\tau > 0$.)

Finally, recalling that $\{y_i; i \in \mathbb{N}\}$ is a dense set in ΔB_Y , and using the local Lipschitzness of f [P, Proposition 1.6], we have that

$$f(x_0 + th, y_0 + tk) + f(x_0 - th, y_0 - tk) - 2f(x_0, y_0) = o(t)$$

for every $h \in X$ and every $k \in Y$. Since f is convex, this is equivalent to the Gâteaux differentiability of f at (x_0, y_0) . \square

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