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LEIBENZON'S BACKWARD SHIFT AND COMPOSITION OPERATORS

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ABSTRACT. We apply Leibenzon's backward shift to show that the composition operator on the unit ball of $\mathbb{C}^n$ always maps the weighted Hardy space H^2_{1-n} into the Hardy class H^2.

1. Introduction

Let $B = B_n$ be the open unit ball of $\mathbb{C}^n$. In what follows we assume that $n \geq 2$, and we use the symbol $\mathbb{D}$ to denote the unit disc of $\mathbb{C}$. Suppose $\varphi : B \to B$ is holomorphic. Then the composition operator C_φ is defined by $(C_\varphi f)(z) = f(\varphi(z))$, where $f : B \to \mathbb{C}$ is holomorphic, $z \in B$.

Classical results show that C_φ is bounded on the Hardy spaces $H^p(\mathbb{D})$, $0 < p \leq \infty$ (Littlewood's subordination principle). The same situation holds for the Bergman spaces $A^p(\mathbb{D})$, $0 < p < \infty$.

In contrast with the one variable case, C_φ may not induce a bounded operator on $H^p(B)$. This observation suggests two types of results. First, a characterization of all φ such that C_φ is bounded on $H^p(B)$ is obtained by B.D. MacCluer in [4]. Second, let $A^p_q(B)$, $q > -1$, denote a (standard) weighted Bergman space. Using Carleson measures considerations and a slice integration technique, B.D. MacCluer and P.R. Mercer prove in [5] that C_φ always maps $H^p(B_n)$ into $A^p_{n-2}(B_n)$. In fact, $A^p_{n-2}(B_n)$ is the "smallest" space with such a property. Further results in this direction show that $C_\varphi : A^p_q \to A^p_{q+n-1}(B_n)$ for all $q > -1$ (see [2]).

The elementary approach of the present note uses Leibenzon's backward shift operator. We fix a Hilbert target space Y, and we look for a largest function space X such that $C_\varphi : X \to Y$. In particular, we show that $C_\varphi : H^2_{1-n}(B_n) \to H^2(B_n)$, where H^2_{1-n} is a weighted Hardy space in the ball.

Notation. Normalized Lebesgue measure on the sphere $S = \partial B_n$ is σ. Respectively, ν is Lebesgue measure on B, $\nu(B) = 1$. So $L^p(S) = L^p(\sigma)$ and $L^p(B) = L^p(\nu)$.

The norm in the classical Hardy space $H^p(B)$ is

$$\|f\|^p_{H^p} = \sup_{0<r<1} \int_S |f(r\zeta)|^p \, d\sigma(\zeta) = \int_S |f^*(\zeta)|^p \, d\sigma(\zeta).$$

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Given $0 < p < \infty$ and $q > -1$, the weighted Bergman space $A^p_q(B)$ consists of functions f holomorphic in B and such that

$$\int_B |f(z)|^p (1-|z|^2)^q \, d\nu(z) < \infty.$$

We use the standard multi-index notation: if $\alpha = (\alpha_1, \dots, \alpha_n)$, then $|\alpha| = \sum_{j=1}^n \alpha_j$ and $\alpha! = \prod_{j=1}^n \alpha_j!$.

Definition. Let $q \geq 0$. A holomorphic function $f(z) = \sum_\alpha c_\alpha z^\alpha$, $z \in B$, belongs to the weighted Hardy space $H^2_q(B)$ if

$$\|f\|_q^2 = \sum_\alpha |c_\alpha|^2 \|z^\alpha\|^2_{L^2(S)} \binom{|\alpha|+q}{q}^{-1} < \infty,$$

where $\binom{a}{b} = \dfrac{\Gamma(a+1)}{\Gamma(a-b+1)\Gamma(b+1)}$ is "the binomial coefficient"; $a \geq b$. Respectively

$$\|f\|_{-q}^2 = \sum_\alpha |c_\alpha|^2 \|z^\alpha\|^2_{L^2(S)} \binom{|\alpha|+q}{q}.$$

Note that $H^2_0 = H^2$ (with equal norms) and $H^2_q \subset H^2_r$ if $q < r$.

Remark 1. A more general definition (cf. [1], Section 2.1) says that H^2_q is the weighted Hardy space $H^2(\beta)$ with $\beta(j) = \binom{j+q}{q}^{-1/2}$. In particular, if $q > 0$, then H^2_q is the weighted Bergman space A^2_{q-1}, with an equivalent norm; H^2_{-1} is a Dirichlet-type space (see [1] for details).

2. The embedding theorem

The main goal of the present section is to establish the following result.

Theorem 1. *Suppose that $\varphi : B_n \to B_n$ is holomorphic and $\varphi(0) = 0$. Then $\|C_\varphi f\|_0 \leq \|f\|_{1-n}$ for all $f \in H^2_{1-n}(B)$.*

Recall that the original proof of Littlewood's subordination principle uses the backward shift operator on ℓ^2 (see e.g. [6], Chapter 1). The sphere S is not a group, so there is no canonical analogue of the backward shift. Nevertheless, given a holomorphic function f, define

$$(L_j f)(z) = \int_0^1 \frac{\partial f}{\partial z_j}(tz)\, dt, \quad 1 \leq j \leq n,\ z \in B.$$

The "backward shifts" $L_j f$ were introduced by Leibenzon (see [3]) to solve the Gleason problem

$$f(z) - f(0) = \sum_{j=1}^n z_j (L_j f)(z). \tag{2.1}$$

As we show below, the shifts L_j are still useful in the study of composition operators.

Proof. Assume, without loss of generality, that f is a polynomial. Substitute z by $\varphi(z)$ in (2.1); then we have

$$C_\varphi f(z) = f(0) + \sum_{j=1}^{n} \varphi_j(z)(C_\varphi L_j f)(z).$$

Since $\varphi(0) = 0$, the terms on the right side are orthogonal in H_0^2. Hence

$$\begin{aligned} \|C_\varphi f\|_0^2 &= |f(0)|^2 + \left\| \sum_{j=1}^{n} \varphi_j \cdot C_\varphi L_j f \right\|_0^2 \\ &\le |f(0)|^2 + \int_S \left(\sum_{j=1}^{n} |\varphi_j|^2 \right) \left(\sum_{j=1}^{n} |C_\varphi L_j f|^2 \right) d\sigma \\ &\le |f(0)|^2 + \sum_{j=1}^{n} \|C_\varphi L_j f\|_0^2 . \end{aligned}$$

By induction, we obtain

$$\|C_\varphi f\|_0^2 \le \sum_{k=0}^{\deg f} \sum_{j_1,\dots,j_k} |(L_{j_1} \dots L_{j_k} f)(0)|^2 .$$

Let $f(z) = \sum_\alpha c_\alpha z^\alpha$. Then the above double sum is equal to

$$\sum_\alpha |c_\alpha|^2 \sum_{j_1,\dots,j_{|\alpha|}} \left|(L_{j_1} \dots L_{j_{|\alpha|}} z^\alpha)(0)\right|^2 .$$

Now, fix a multi-index α. Consider a sequence $J = \{j_1, \dots, j_{|\alpha|}\}$ and the corresponding operator $L_J = L_{j_1} \dots L_{j_{|\alpha|}}$. Observe that $(L_J z^\alpha)(0) \neq 0$ if and only if $\#\{k : j_k = m\} = \alpha_m$ for all $1 \le m \le n$. A simple combinatorial calculation shows that there are $|\alpha|!/\alpha!$ different sequences L_J such that $(L_J z^\alpha)(0) \neq 0$. Moreover, the value $(L_J z^\alpha)(0)$ is the same for all such J. Indeed, one has

$$(L_j z^\alpha)(\zeta) = \frac{\alpha_j}{|\alpha|} \zeta^{\alpha - e_j}, \quad \text{where } (e_j)_k = \delta_{jk},\ 1 \le k \le n.$$

Therefore $|(L_J z^\alpha)(0)| = \alpha!/|\alpha|!$, and

$$\sum_J |(L_J z^\alpha)(0)|^2 = \alpha!/|\alpha|!.$$

Recall that

$$\|z^\alpha\|_{L^2(S_n)}^2 = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}.$$

So, finally we obtain

$$\|C_\varphi f\|_0^2 \le \sum_\alpha |c_\alpha|^2 \frac{\alpha!}{|\alpha|!} = \sum_\alpha |c_\alpha|^2 \|z^\alpha\|_{L^2(S)}^2 \binom{|\alpha| + n - 1}{|\alpha|} = \|f\|_{1-n}^2 .$$

□

Corollary 2. *For all $\varphi : B_n \to B_n$, the composition operator C_φ maps $H_{1-n}^2(B_n)$ into $H^2(B_n)$.*

Proof. Let ψ be a holomorphic automorphism of B_n. It is well known that $C_\psi : H^2 \to H^2$. On the other hand, the automorphisms of the ball act transitively. □

Remark 2. Observe that Theorem 1 is optimal in the scale of weighted Hardy (or weighted Dirichlet) spaces. Namely, if $q > 1 - n$, then there exists φ, $\varphi(0) = 0$, such that C_φ does not map $H^2_q(B_n)$ into $H^2(B_n)$.

For example, assume $n = 2$ and $q > 1 - n = -1$. Let $I : B_2 \to \mathbb{D}$, $I(0) = 0$, be an inner function, that is, $|I^*| = 1$ σ-a.e. Define $\varphi(z) = (I(z), 0)$ and consider $f_k(z) = \sqrt{k^{1+q}}\, z_1^k$, $k \in \mathbb{N}$. Then $\|f_k\|_q^2 \asymp 1$ for all k. On the other hand, we have $\|f_k(\varphi(z))\|_0^2 = k^{1+q}\|I^k\|_0^2 = k^{1+q}$. In other words, C_φ does not map H^2_q into H^2.

Proposition 3. *Suppose that $\varphi : B_n \to B_n$ is holomorphic, $\varphi(0) = 0$, and $q \geq 0$. Then $C_\varphi : H^2_{q+1-n}(B_n) \to H^2_q(B_n)$.*

Proof. If $q = 0$, then we have Theorem 1. To avoid ugly calculations, consider only an illustrative case $q = 1$. In other words, let the target space be $H^2_1 = A^2$. Recall that

$$C_\varphi f(z) = f(0) + \sum_{j=1}^{n} \varphi_j(z)(C_\varphi L_j f)(z).$$

Since $\varphi(0) = 0$, the terms on the right side are orthogonal in $L^2(B, |z|^{2k}\, d\nu(z))$ for all $k \in \mathbb{Z}_+$. On the other hand, by the Schwarz lemma in the ball,

$$\sum_{j=1}^{n} |\varphi_j(z)|^2 \leq |z|^2.$$

Therefore, by induction,

$$\begin{aligned} \|C_\varphi f\|_1^2 &\leq \sum_{k\geq 0} \sum_{j_1,\ldots,j_k} \|\, |z|^{2k} \|_{L^2(B)}\, |(L_{j_1} \ldots L_{j_k} f)(0)|^2 \\ &= \sum_\alpha |c_\alpha|^2 \frac{\alpha!}{|\alpha|!} \frac{n}{|\alpha| + n} \leq 2\|f\|^2_{2-n}. \end{aligned}$$

□

Corollary 4. *Let $\varphi : B_n \to B_n$ and $q \geq 0$. Then $C_\varphi : H^2_{q+1-n} \to H^2_q$.*

Proof. Let ψ be an automorphism of B_n. It is well known that $C_\psi : H^2_q \to H^2_q$ for all $q \geq 0$. □

The appropriate choice of q (cf. Remark 1) yields

Corollary 5. *Let $\varphi : B_n \to B_n$. Then $C_\varphi : H^2 \to A^2_{n-2}$ and $C_\varphi : A^2_r \to A^2_{r+n-1}$ for all $r > -1$.*

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