PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 12, Pages 3757–3758
S 0002-9939(01)06330-4
Article electronically published on June 27, 2001

A PROBLEM OF PRESCRIBING GAUSSIAN CURVATURE ON S^2

SULBHA GOYAL AND VINOD GOYAL

(Communicated by David S. Tartakoff)

ABSTRACT. A class of functions $K(x) = K(x_1, x_2, x_3)$ and the corresponding solutions of

$$\Delta u + K(x)e^{2u} = 1$$

are obtained as a special case of the solutions of

$$\Delta^m u + K(x)e^{au} = f(x), \qquad x = (x_1, x_2, \ldots, x_n),$$

where Δ^m is defined as $\Delta(\Delta^{m-1})$.

On the two-sphere $S^2 = \{x \in R^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$ with metric $ds_0^2 = dx_1^2 + dx_2^2 + dx_3^2$, if the metric is conformally changed to $ds^2 = e^{2u} ds_0^2$, then the Gaussian curvature $K(x)$ of the new metric is determined by the equation

$$\Delta u + K(x)e^{2u} = 1, \qquad x \in S^2, \tag{1}$$

where Δ denotes the Laplacian relative to the metric ds_0^2.

The question raised by L. Nirenberg is: Which functions $K(x)$ can be prescribed so that (1) has a solution u?

Integrating (1) over the whole sphere, we get

$$\int_{S^2} Ke^{2u}\, d\mu = 4\pi,$$

where $d\mu$ denotes the surface measure on S^2. Thus, an obvious necessary condition is that K must be positive somewhere. Another necessary condition was found by Kazdan and Warner [3] via integration by parts. Moser [4] proved that if K is an even function on S^2, then (1) has a solution. In [1], Cheng and Smoller considered the case of rotationally symmetric functions K.

The purpose of this note is to show that a class of functions K and the corresponding solutions can be produced by means of elementary arguments. We first prove the following result:

Theorem 1. *If $a > 0$ is an arbitrary constant and the functions $K(x)$ and $f(x)$ are positive and such that*

$$\Delta^m \ln\left(\frac{K(x)}{f(x)}\right) = 0, \qquad x = (x_1, x_2, \ldots, x_n), \tag{2}$$

Received by the editors December 20, 2000.
2000 *Mathematics Subject Classification.* Primary 35J30, 35J60; Secondary 31B30.
Key words and phrases. Laplace operator, Gaussian curvature, conformally equivalent, metric.

©2001 American Mathematical Society

then the solution of

$$\Delta^m u + K(x)e^{au} = f(x) \tag{3}$$

is given by

$$u = \frac{1}{a}\ln\left(\frac{f(x)}{K(x)}\right). \tag{4}$$

Proof. We divide (3) by $f(x)$ and write it as

$$\frac{1}{f(x)}\Delta^m u + e^{au+\ln\left(\frac{K(x)}{f(x)}\right)} = 1. \tag{5}$$

Now by the change of variables

$$au + \ln\left(\frac{K(x)}{f(x)}\right) = \nu \tag{6}$$

and with the help of (2), we obtain

$$\Delta^m u = \frac{1}{a}\Delta^m \nu. \tag{7}$$

Thus the equation (5) reduces to

$$\frac{1}{af(x)}\Delta^m \nu + e^{\nu} = 1. \tag{8}$$

Clearly, $\nu = 0$ is a solution of (8). We then get from (6)

$$u = \frac{1}{a}\ln\left(\frac{f(x)}{K(x)}\right).$$

This completes the proof. □

If in Theorem 1 we choose $m = 1,\ a = 2,\ n = 3$ and $f(x) = 1$, then we have

Theorem 2. *If the function $K(x) = K(x_1, x_2, x_3)$ is positive and $\ln(K(x))$ is harmonic, then $u = \frac{1}{2}\ln(\frac{1}{K(x)})$ is a solution of (1).*

Remark. Evidently $u = \frac{1}{a}\ln(\frac{f(x)}{K(x)})$ is also a solution of the nonlinear Dirichlet problem $\Delta u + K(x)e^{au} = f(x)$ in D and $u = 0$ on ∂D as long as $f(x) = K(x)$ on ∂D and $\ln(\frac{K(x)}{f(x)})$ is harmonic in D.

References

1. K. Cheng and J. Smoller, *Conformal metric with prescribed Gaussian curvature on S^2*, UAB International Conference on Differential Equations and Mathematical Physics (Abstracts), March 15–21, 1990, p. 48; Trans. Amer. Math. Soc. **336** (1993), 219–251. MR **93e:**53044
2. V. B. Goyal, *Remark on a paper of Cheng and Smoller*, Proc. Amer. Math. Soc. **113** (1991), 795–797. MR **92b:**58243
3. J. Kazdan and F. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) **99** (1974), 14–47. MR **49:**7949
4. J. Moser, *On a non-linear problem in differential geometry*, Dynamical Systems, M. Peixoto (ed.), Academic Press, New York, 1973. MR **49:**4018

Department of Mathematics, Tuskegee University, Tuskegee, Alabama 36088

Department of Mathematics, Tuskegee University, Tuskegee, Alabama 36088