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LADDER SYSTEMS ON TREES

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ABSTRACT. We formulate the notion of uniformization of colorings of ladder systems on subsets of trees. We prove that Suslin trees have this property and also Aronszajn trees in the presence of Martin's Axiom. As an application we show that if a tree has this property, then every countable discrete family of subsets of the tree can be separated by a family of pairwise disjoint open sets. Such trees are then normal and hence countably paracompact. As a dual result for special Aronszajn trees we prove that the weak diamond, Φ_ω, implies that no special Aronszajn tree can be countably paracompact.

§1. INTRODUCTION

Topological properties of trees have been extensively studied, particularly of trees with specific structure such as Suslin, almost Suslin and special Aronszajn trees. It was proved in [2] that Suslin trees are normal and that a tree is almost Suslin if and only if it is collectionwise Hausdorff. Starting from Martin's Axion (MA), Fleissner [4] proved that every special Aronszajn tree is normal and Hanazawa [6] pointed out that normality implies countable paracompactness in trees. In addition, Fleissner [5] further studied topological properties of trees and proved the equivalence of collection-wise normality to other properties of trees. In §2 we formulate a combinatorial property for ω_1-trees which, among other properties, incorporates in it normality and hence countable paracompactness. We then prove that Suslin trees have this property and also Aronszajn trees if MA + ¬CH holds.

On the other hand starting from $\diamondsuit_S(\forall$ stationary $S \subseteq \omega_1)$, Watson [8] established that no special Aronszajn tree can be countably paracompact. In §3 we recall the notion of a weak diamond Φ_n $(2 \leq n \leq \omega)$ studied in [7] and prove that under Φ_ω the property of §2 fails for special Aronszajn trees by proving that Φ_ω implies that no special Aronszajn tree can be countably paracompact. Thus, in light of the fact that $\diamondsuit_S(\forall$ stationary $S \subseteq \omega_1)$ is strictly stronger than $\diamondsuit$ which in turn is strictly stronger than Φ_ω, we also improve the result of Watson stated above.

We now formulate some basic notions and establish the notation which will be used in this paper. If α and β are ordinals, then $\alpha^\beta = \{f : f : \beta \to \alpha\}$ and $\alpha^{<\beta} = \bigcup_{\nu<\beta} \alpha^\nu$. For a set A, $[A]^{<\omega}$ denotes the collection of all finite subsets of A and $[A]^\kappa$ the collection of all subsets of A of size κ. A tree, $\mathbb{T} = \langle T, \leq_T\rangle$, is a partial order in the strict sense such that for each $t \in T$ the set $\hat{t} = \{s \in T : s <_T t\}$ is well ordered by $\leq_T$. If $t \in T$, the height of t in $\mathbb{T}$, $\mathrm{ht}(t, \mathbb{T})$, is the ordinal α which is the order

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type of $\hat{t}$. Let $T' = \{t \in T : \mathrm{ht}(t, \mathbb{T})$ is a limit ordinal$\}$. For each ordinal α, the αth level of $\mathbb{T}$, $\mathrm{Lev}_\alpha(\mathbb{T})$, is the set $\{t \in T : \mathrm{ht}(t, \mathbb{T}) = \alpha\}$ and $T_\alpha = \bigcup_{\nu<\alpha} \mathrm{Lev}_\nu(\mathbb{T})$. The height of $\mathbb{T}$, $\mathrm{ht}(\mathbb{T})$, is the least ordinal α such that $\mathrm{Lev}_\alpha(\mathbb{T}) = \emptyset$. If $A \subseteq T$ and $C \subseteq \omega_1$, then $A \restriction C = \{t \in A : \mathrm{ht}(t, \mathbb{T}) \in C\}$. And $A \subseteq T$ is stationary iff $\{\mathrm{ht}(t, \mathbb{T}) : t \in A\}$ is stationary in ω_1, i.e. has nonempty intersection with every closed unbounded subset of ω_1. An α-branch of $\mathbb{T}$ is a set $B \subseteq T$ such that B is well ordered by $\leq_T$ in order type α, and $\forall \beta < \alpha (B \cap \mathrm{Lev}_\beta(\mathbb{T}) \neq \emptyset)$. An anti-chain in $\mathbb{T}$ is a set $A \subseteq T$ such that $\forall s, t \in A(s \neq t \rightarrow s \not\leq_T t \wedge t \not\leq_T s)$. More generally, an anti-chain in a partial order is a set whose elements are pairwise incompatible (for trees this means incomparable). Compatibility of p and q in a partial order will be denoted by $p \sim q$ and their incompatibility by $p \not\sim q$.

We will only consider well pruned trees. A well pruned tree is a tree $\mathbb{T} = \langle T, \leq_T \rangle$ such that

(1) $|\mathrm{Lev}_0(\mathbb{T})| = 1$,

(2) $\forall \alpha < \beta < \mathrm{ht}(\mathbb{T}) \forall t \in \mathrm{Lev}_\alpha(\mathbb{T}) \exists s_1, s_2 \in \mathrm{Lev}_\beta(\mathbb{T})(s_1 \neq s_2 \wedge t \leq_T s_1, s_2)$,

(3) $\forall \alpha < \mathrm{ht}(\mathbb{T}) \forall s, t \in \mathrm{Lev}_\alpha(\mathbb{T})(\text{limit } \alpha \rightarrow (s = t \leftrightarrow \hat{s} = \hat{t}))$.

From this point on all trees are well pruned. An ω_1-tree is a well pruned tree $\mathbb{T} = \langle T, \leq_T \rangle$ such that $\mathrm{ht}(\mathbb{T}) = |T| = \omega_1$ and $\forall \alpha < \mathrm{ht}(\mathbb{T})(|\mathrm{Lev}_\alpha(\mathbb{T})| \leq \omega)$. With the exception of a few observations below, all the results in this paper deal with trees which do not have any ω_1-branches. Such trees are given specific names. Any ω_1-tree without ω_1-branch is called an Aronszajn tree. An Aronszajn tree is special if it is a union of countably many anti-chains. It is a consequence of MA + $\neg$CH that every Aronszajn tree is special. An ω_1-tree is Suslin if it does not contain any uncountable anti-chains; it is almost Suslin if it does not contain any stationary anti-chains. Almost Suslin trees were introduced in [2] where it was shown that if there is a Suslin tree, then there is an almost Suslin tree which is not Suslin. Thus, by countable completeness of the nonstationary ideal on ω_1, special Aronszajn trees are not almost Suslin. The existence of Suslin trees is denied by MA + $\neg$CH.

We also consider trees as topological spaces. For a tree $\mathbb{T} = \langle T, \leq_T \rangle$, a basis for a topology on $\mathbb{T}$ is the following collection of sets:

$$\{\hat{t} : t \in T\} \cup \{\{u \in T : s <_T u <_T t\} : s, t \in T\}.$$

For $s, t \in T$, with $s \leq_T t$, we let $(s, t)_T = \{u \in T : s <_T u <_T t\}$ and such sets will be called intervals in $\mathbb{T}$. We also use the notation $[s, t)_T$, $(s, t]_T$, $[s, t]_T$ to specify whether or not the interval includes the end points.

All topological spaces in this paper are Hausdorff. Let X be one such space. X is countably paracompact if every countable open cover of X has a locally finite open refinement. All other topological notions appearing in this paper are standard and can be found in any text book on the subject. We point out that in a tree with topology as above any discrete set of points is also closed. Thus any reference below to a discrete set implies closed and discrete.

Then an ω_1-tree is normal, ... if it is normal, ... as a topological space with the topology described above. Suslin trees are normal and it was proved in [2] that a tree is almost Suslin iff it is collection-wise Hausdorff. Other topological properties of trees were mentioned in the earlier part of this section.

§2. Ladder systems

Devlin and Shelah [1] formulated the notion of uniformization of colorings of ladder systems on ω_1. We consider an analogous notion on ω_1-trees.

Definition 1. Let $\mathbb{T} = \langle T, \leq_T \rangle$ be an ω_1-tree and $A \subseteq T'$. A ladder system on A is a collection $\tilde{\eta}_A = \langle \eta_t : t \in A \rangle$ such that each η_t is a $<_T$-increasing sequence with limit t. $\tilde{\eta}_A$ is a full ladder system if for each $t \in A$ there is $r <_T t$ such that $\eta_t = [r, t)_T$. A coloring of $\tilde{\eta}_A$ is a collection $\tilde{c}_{\tilde{\eta}_A} = \langle c_t : t \in A \rangle$ such that $c_t : \eta_t \to \omega$ for each $t \in A$. $\tilde{c}_{\tilde{\eta}_A}$ is a nice coloring if it satisfies the following two conditions:

(1) $\forall t \in A\ \exists s <_T t\ \forall u \in A\ \exists v <_T u\ \forall a \in \eta_t \setminus \hat{s} \cap \eta_u \setminus \hat{v}(c_t(a) = c_u(a))$,
(2) $\forall t \in A\ \exists \varphi_t(\varphi_t$ is a function $\wedge\ \varphi_t : \omega \to \eta_t\ \wedge\ \forall m, n < \omega(m < n \to \varphi_t(m) <_T \varphi_t(n))\ \wedge\ \sup_T\{\varphi_t(n) : n < \omega\} = t\ \wedge\ \forall n < \omega(|\ \{c_t(a) : \varphi_t(n) \leq_T a <_T \varphi_t(n+1)\}\ | = 1))$.

A function $f : T \to \omega$ uniformizes the coloring $\tilde{c}_{\tilde{\eta}_A}$ if $\forall t \in A \exists s <_T t \forall a \in \eta_t \setminus \hat{s}(c_t(a) = f(a))$.

We point out that condition (1) is a nontrivial assertion only in the case when $\mathrm{ht}(t, \mathbb{T})$ is a limit of limit ordinals and there is a $B \subseteq A \cap \hat{t}$, which is $\leq_T$-unbounded in $\hat{t}$, such that $\forall u \in B(\sup_T(\eta_t \cap \eta_u) = u)$. Otherwise, it is easy to see that there is always an $s <_T t$ such that $\forall u \in A \exists v <_T u(\eta_t \setminus \hat{s} \cap \eta_u \setminus \hat{v} = \emptyset)$ so that condition (1) holds vacuously. It is also easy to see that if a coloring does not satisfy condition (1), then it cannot be uniformized. Thus, condition (1) is a necessary condition for a uniformizing function to exist. Condition (2) above states that for each $t \in A$ and $s \in T$, with $\varphi_t(0) \leq_T s <_T t$, c_t has only finitely many color changes on the interval $[\varphi_t(0), s)_T$. It is a nontrivial assertion only in the case when $\mathrm{ht}(t, \mathbb{T})$ is a limit of limit ordinals. Otherwise, there is always a function φ_t with the desired properties.

The next example justifies the inclusion of condition (2) in the above definition. In fact, the example shows that it is also a necessary requirement on colorings of ladder systems on subsets of Aronszajn trees for uniformizing functions to exist under MA$+\neg$CH. It is easy to see in condition (2) that for each $n < \omega$ if we require that domain of φ_t is $\omega \cdot n$ instead of ω and retain the other requirements, then we get an equivalent condition. The next example shows that we cannot require that domain φ_t is $\omega \cdot \omega$ for trees which are not almost Suslin such as, for example, special Aronszajn trees. It is an open question if there is an analogous example for Suslin or almost Suslin trees. Since condition (2) seems to play an essential role in the proof of Theorem 2 below, this might be evidence that there are similar examples for such trees as well.

Let $\mathbb{T} = \langle T, \leq_T \rangle$ be an ω_1-tree which is not almost Suslin and let A be a stationary anti-chain in T. Since $\mathbb{T}$ is well pruned we may assume that the height of each element of A is a limit of limit ordinals. For each $t \in A$ fix an $s_t <_T t$ and let $\eta_t = [s_t, t)_T$. Then $\tilde{\eta}_A = \langle \eta_t : t \in A \rangle$ is a full ladder system on A. For each $t \in A$ let $\varphi_t : \omega \cdot \omega \to \eta_t$ be such that $\varphi_t(0) = s_t$, $\zeta < \xi < \omega \cdot \omega \to \varphi_t(\zeta) <_T \varphi_t(\xi)$, and $\sup_T\{\varphi_t(\xi) : \xi < \omega \cdot \omega\} = t$. Let $B \in [\omega^\omega]^{\omega_1}$ and let $\{B_t : t \in A\}$ be a partition of B with $B_t = \{f_t^i : i < \omega\}$ for each $t \in A$ where $i \neq j \vee t \neq s \to f_t^i \neq f_s^j$. We now define coloring $\tilde{c}_{\tilde{\eta}_A} = \langle c_t : t \in A \rangle$. Fix $t \in A$. For each $i, n < \omega$ and $s \in [\varphi_t(\omega \cdot n + i), \varphi_t(\omega \cdot n + i + 1))_T$ let $c_t(s) = f_t^n(i)$. This defines $\tilde{c}_{\tilde{\eta}_A}$ and it has the property that

$$\forall i, n < \omega(|\ \{c_t(a) : \varphi_t(\omega \cdot n + i) \leq_T a <_T \varphi_t(\omega \cdot n + i + 1)\}\ | = 1).$$

We show that this coloring cannot be uniformized. By way of contradiction assume $f : T \to \omega$ is a uniformizing function. For each $t \in A$ let $n_t < \omega$ be such that f and c_t agree on $[\varphi_t(\omega \cdot n_t), t)_T$. Since A is stationary and the nonstationary ideal $\mathcal{I}$

on ω_1 is countably complete we may assume that $\forall t \in A(n_t = n)$ for some fixed n. Now by Fodor's Lemma (i.e. Pressing Down Lemma), there is a stationary $A' \subseteq A$ such that

$$\forall t, s \in A'(\mathrm{ht}(\varphi_t(\omega \cdot n), \mathbb{T}) = \mathrm{ht}(\varphi_s(\omega \cdot n), \mathbb{T})).$$

Using countable completeness of $\mathcal{I}$ again, since levels of $\mathbb{T}$ are countable we may, in fact, assume that

$$\forall t, s \in A'(\varphi_t(\omega \cdot n) = \varphi_s(\omega \cdot n)).$$

Using Fodor's Lemma and countable completeness of $\mathcal{I}$ one more time we get a stationary $A'' \subseteq A'$ such that

$$\forall t, s \in A''(\varphi_t(\omega \cdot (n+1)) = \varphi_s(\omega \cdot (n+1))).$$

Then

$$\forall t, s \in A''([\varphi_t(\omega \cdot n), \varphi_t(\omega \cdot (n+1)))_T = [\varphi_s(\omega \cdot n), \varphi_s(\omega \cdot (n+1)))_T).$$

Now $f_t^n \neq f_s^n$ whenever $t, s \in A''$ with $t \neq s$ so that whenever $t \neq s$ in A'' there is an

$$r \in [\varphi_t(\omega \cdot n), \varphi_t(\omega \cdot (n+1)))_T$$

such that $c_t(r) \neq c_s(r)$. But then either $f(r) \neq c_t(r)$ or $f(r) \neq c_s(r)$ which contradicts the assumption that f is a uniformizing function for the coloring $\tilde{c}_{\tilde{\eta}_A}$. This shows that the requirement that domain$(\varphi_t) = \omega$ in Definition 1(2) is an optimal requirement for a coloring to have a uniformizing function under MA+$\neg$CH in the case of special Aronszajn trees.

In the following observations, with the exception of (c), $\mathbb{T}$ may have ω_1-branches.

(a) If $\mathbb{T}$ is an ω_1-tree and $A \subseteq T'$ is countable, then any nice coloring of a ladder system on A can be uniformized.

(b) If $\mathbb{T}$ is an ω_1-tree and $A \subseteq T'$ is nonstationary, then any nice coloring of a ladder system on A can be uniformized.

(c) If $\mathbb{T}$ is almost Suslin and $A \subseteq T'$ is discrete, then any coloring of a ladder system on A can be uniformized.

(d) If MA + $\neg$CH holds, $\mathbb{T}$ is an ω_1-tree, $A \subseteq T'$, $\tilde{\eta}_A = \langle \eta_t : t \in A \rangle$ is a ladder system on A such that each η_t has order type ω, and $\tilde{c}_{\tilde{\eta}_A}$ is any coloring of $\tilde{\eta}_A$ (hence nice), then there is a uniformizing function.

To prove (a) enumerate A in order type ω and proceed inductively using the fact that the coloring is nice. For (b) let $C \subseteq \omega_1$ be closed unbounded such that $T \restriction C$ is disjoint from A. Now proceed inductively by using (a) above to define a uniformizing function on the elements of T whose heights lie between consecutive points of C. For (c) use the fact that almost Suslin trees are collection-wise Hausdorff. The proof of Theorem 5.2 of [1] can easily be modified to yield (d).

The next theorem shows that normal almost Suslin trees have the uniformization property. But first two lemmas.

Lemma 2. *Let $\mathbb{T} = \langle T, \leq_T \rangle$ be an almost Suslin tree and $B \subseteq T$ a discrete set. Then B is not stationary.*

Proof. By way of contradiction assume B is stationary. Since B is discrete and $\mathbb{T}$ is almost Suslin, hence collection-wise Hausdorff, for each $t \in B$ there is a $p_t <_T t$ such that if $t \neq s$ in B, then $(p_t, t]_T \cap (p_s, s]_T = \emptyset$. By Fodor's Lemma, there is a stationary $B' \subseteq B$ and an $\alpha < \omega_1$ such that $\forall t \in B'(\mathrm{ht}(p_t, \mathbb{T}) = \alpha)$. Since $\mathrm{Lev}_\alpha(\mathbb{T})$

is countable, by countable completeness of the nonstationary ideal on ω_1, there is a $p \in \mathrm{Lev}_\alpha(\mathbb{T})$ and a stationary $B'' \subseteq B'$ such that $\forall t \in B''(p_t = p)$. For each $t \in B''$ let $r_t \in (p,t]_T \cap \mathrm{Lev}_{\alpha+1}(\mathbb{T})$. Since $(p,s]_T \cap (p,t]_T = \emptyset$ whenever $s \neq t$ in B'' it follows that $\forall s,t \in B''(s \neq t \rightarrow r_s \neq r_t)$. But $\mathrm{Lev}_{\alpha+1}(\mathbb{T})$ is countable, so B'' is also countable, hence it cannot be stationary, a contradiction. □

Lemma 3. *Let $\mathbb{T} = \langle T, \leq_T\rangle$ be a normal almost Suslin tree and $B \subseteq T$ discrete. Then there is a discrete family $\{U_b : b \in B\}$ of closed and open sets and a club $C \subseteq \omega_1$ such that $\forall b \in B(b \in U_b \subseteq T)$ and $T \restriction C \cap \bigcup_{b\in B} U_b = \emptyset$.*

Proof. By the previous lemma, B is not stationary, so let $C \subseteq \omega_1$ be a club such that $B \cap T \restriction C = \emptyset$. Without loss of generality we may assume that $0 \in C$ and that each $\alpha \in C \setminus \{0\}$ is a limit ordinal. Since both B and $T \restriction C$ are closed, by normality of $\mathbb{T}$, let $U \subseteq T$ be open such that $B \subseteq U$ and $T \restriction C \cap \bar{U} = \emptyset$. Let $\{\alpha_\xi : \xi < \omega_1\}$ be the increasing enumeration of C and for $\xi < \omega_1$ let $\{b_i^\xi : i < \omega\}$ and $\{x_i^\xi : i < \omega\}$ be enumerations of $B \cap (T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi})$ and $\bar{U} \cap (T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi}) \setminus B$ respectively. For each $\xi < \omega_1$ we shall construct, by induction, collections $\{U_i^\xi : i < \omega\}$ and $\{V_i^\xi : i < \omega\}$ of sets such that

(1) $\forall i < \omega($ U_i^ξ and V_i^ξ are closed and open),

(2) $\forall i < \omega(U_i^\xi \subseteq U \cap (T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi}) \wedge V_i^\xi \subseteq (T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi}))$,

(3) $\forall i < \omega(b_i^\xi \in U_i^\xi \wedge x_i^\xi \in V_i^\xi)$,

(4) $\forall i,j < \omega(i \neq j \rightarrow U_i^\xi \cap U_j^\xi = \emptyset)$,

(5) $\forall i < \omega(\exists j < \omega(V_i^\xi \cap U_j^\xi \neq \emptyset) \rightarrow V_i^\xi = U_j^\xi)$.

So, fix $\xi < \omega_1$ and $n < \omega$ and suppose $\{U_i^\xi : i < n\}$ and $\{V_i^\xi : i < n\}$ have been constructed satisfying (1)–(5) above and also

(6) $\bigcup_{i<n}(U_i^\xi \cup V_i^\xi) \cap \{b_i^\xi : i \geq n\} = \emptyset$,

(7) $\forall i < n(V_i^\xi \cap \bigcup_{j<n} U_j^\xi = \emptyset \rightarrow V_i^\xi \cap B = \emptyset)$.

We consider three cases.

Case 1: $\exists i < n(x_n^\xi \in U_i^\xi)$.

Since U_i^ξ's and V_i^ξ's are closed, so is $\bigcup_{i<n}(U_i^\xi \cup V_i^\xi)$ and is disjoint from $B \setminus \{b_i^\xi : i < n\}$. So let $s <_T b_n^\xi$ be such that $(s, b_n^\xi]_T \subseteq U \cap (T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi})$ and $(s, b_n^\xi]_T \cap (\bigcup_{i<n}(U_i^\xi \cup V_i^\xi) \cup (B \setminus \{b_n^\xi\})) = \emptyset$. In this case let $U_n^\xi = (s, b_n^\xi]_T$ and $V_n^\xi = U_i^\xi$ and note that (1)–(7) above are satisfied.

Case 2: $\exists i < n(x_n^\xi \in V_i^\xi \wedge \forall j < n(V_i^\xi \cap U_j^\xi = \emptyset))$.

Since U_i^ξ's and V_i^ξ's are closed, so is $\bigcup_{i<n}(U_i^\xi \cup V_i^\xi)$ and is disjoint from $B \setminus \{b_i^\xi : i < n\}$. So let $s <_T b_n^\xi$ be such that $(s, b_n^\xi]_T \subseteq U \cap (T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi})$ and $(s, b_n^\xi]_T \cap (\bigcup_{i<n}(U_i^\xi \cup V_i^\xi) \cup (B \setminus \{b_n^\xi\})) = \emptyset$. In this case let $U_n^\xi = (s, b_n^\xi]_T$ and $V_n^\xi = V_i^\xi$ and note that (1)–(7) above are satisfied.

Case 3: $x_n^\xi \notin \bigcup_{i<n}(U_i^\xi \cup V_i^\xi)$.

Using the hypothesis on U_i^ξ's and V_i^ξ's and the fact that $\mathbb{T}$ is Hausdorff and B is discrete let $s <_T b_n^\xi$ and $r <_T x_n^\xi$ be such that $(s, b_n^\xi]_T \subseteq U \cap (T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi})$, $(r, x_n^\xi]_T \subseteq T_{\alpha_{\xi+1}} \setminus T_{\alpha_\xi}$, $(s, b_n^\xi]_T \cap (r, x_n^\xi]_T = \emptyset$, $(s, b_n^\xi]_T \cap (\bigcup_{i<n}(U_i^\xi \cup V_i^\xi) \cup (B \setminus \{b_n^\xi\})) = \emptyset$, $(r, x_n^\xi]_T \cap \bigcup_{i<n}(U_i^\xi \cup V_i^\xi \cup B) = \emptyset$. In this case let $U_n^\xi = (s, b_n^\xi]_T$ and $V_n^\xi = (r, x_n^\xi]_T$ and note that (1)–(7) above are satisfied.

It is now easy to see that $\{U_i^\xi : i < \omega\}$ and $\{V_i^\xi : i < \omega\}$ satisfy (1)–(5) above. For each $b \in B$, $b = b_n^\xi$ for some $\xi < \omega_1$ and $n < \omega$. Let $U_b = U_n^\xi$. Then each U_b

is closed and open, $b \in U_b$ and $T \restriction C \cap \bigcup_{b\in B} U_b = \emptyset$ since $\forall b \in B(U_b \subseteq U)$ and $\bar{U} \cap T \restriction C = \emptyset$. In addition $\{U_b : b \in B\}$ is a pairwise disjoint family. In fact it is a discrete family. To see this let $t \in T$. If $t \in \bar{U}$, then $t = b^\xi_n$ or $t = x^\xi_n$ for some $\xi < \omega_1$ and $n < \omega$. Then U^ξ_n or V^ξ_n witnesses that t has an open neighborhood which has a nonempty intersection with at most one element in $\{U_b : b \in B\}$. And if $t \notin \bar{U}$, then $T \setminus \bar{U}$ is an open neighborhood of t which is disjoint from $\bigcup_{b\in B} U_b$. This proves the lemma. □

Theorem 4. *Let $\mathbb{T}$ be a normal almost Suslin tree and suppose that $A \subseteq T'$ is closed in T. Then every nice coloring of a full ladder system on A can be uniformized.*

Proof. Let $\tilde{\eta}_A = \langle \eta_t : t \in A\rangle$ be a full ladder system on A and $\tilde{c}_{\tilde{\eta}_A} = \langle c_t : t \in A\rangle$ a nice coloring of $\tilde{\eta}_A$. Let $B = \{t \in A : c_t$ is not eventually constant$\}$. Then B is discrete in T. To see this, by way of contradiction, assume B has an accumulation point. Since A is closed this point is in A. But $\tilde{c}_{\tilde{\eta}_A}$ is nice so for all t in some neighborhood of this accumulation point, c_t is eventually constant, and this is a contradiction by the definition of B. Then by Lemma 2, B is not stationary in T.

Let $C \subseteq \omega_1$ be closed unbounded such that $T \restriction C$ is disjoint from B. We may assume that $0 \in C$ and that each $\alpha \in C \setminus \{0\}$ is a limit ordinal. Let $\{U_b : b \in B\}$ be as in Lemma 3 and $V = T \setminus \bigcup_{b\in B} U_b$. Then, since $\{U_b : b \in B\}$ is a discrete family of closed and open sets, V is also closed and open in T and $T \restriction C \subseteq V$. Now for each $t \in A \cap V$, c_t is eventually constant. So for $i < \omega$ let $A_i = \{t \in A \cap V : c_t$ is eventually equal to $i\}$. Since $A \cap V$ is closed in T and $\tilde{c}_{\tilde{\eta}_A}$ is nice it follows that each A_i is closed in T, $A_i \cap A_j = \emptyset$ for $i \neq j$, and that $\{A_i : i < \omega\}$ is a discrete family. Since normality implies countable collection-wise normality there are pairwise disjoint open sets V_i such that $A_i \subseteq V_i \subseteq V$ for $i < \omega$. We may assume that $\bigcup_{i<\omega} V_i = V$. Now we define a uniformizing $f : T \to \omega$. If $t \in A_i$, then for $s \in \eta_t \cap V_i$ let $f(s) = i$. This defines f on $V \cap \bigcup_{t\in A} \eta_t$. We define now f on $(T\setminus V) \cap \bigcup_{t\in A} \eta_t$. Let $\{\alpha_\nu : \nu < \omega_1\}$ be the monotone enumeration of C. We proceed by induction on ν. Since $A \cap (T_{\alpha_{\nu+1}} \setminus (T_{\alpha_\nu} \cup V))$ is countable, we use observation (a) to define f on $(T_{\alpha_{\nu+1}} \setminus (T_{\alpha_\nu} \cup V)) \cap \bigcup_{t\in A} \eta_t$. Note that $T_{\alpha_{\nu+1}} \setminus (T_{\alpha_\nu} \cup V)$ is closed and open in T so $A \cap (T_{\alpha_{\nu+1}} \setminus (T_{\alpha_\nu} \cup V))$ is closed, hence contains all of its accumulation points so there is no danger of conflict with the previously defined values of f. This defines f on each η_t ($t \in A$). Now extend f arbitrarily to cover all of T. Clearly f uniformizes $\tilde{c}_{\tilde{\eta}_A}$ and the theorem is proved. □

An immediate corollary is that if $\mathbb{T}$ is Suslin and $A \subseteq T'$ is closed, then every nice coloring of a full ladder system on A can be uniformized.

Our next task is to show that if MA $+ \neg$CH holds, $\mathbb{T}$ is Aronszajn and $A \subseteq T'$, then every nice coloring of a full ladder system on A can be uniformized. Let $\tilde{\eta}_A = \langle \eta_t : t \in A\rangle$ be a ladder system on A and $\tilde{c}_{\tilde{\eta}_A} = \langle c_t : t \in A\rangle$ a nice coloring of $\tilde{\eta}_A$. Let $t \in A$ and let $s <_T t$ be as in Definition 1(1) and φ_t as in Definition 1(2). For the rest of this section fix a $p_t \in T$ such that $s, \varphi_t(0) \leq_T p_t <_T t$. If $\mathrm{ht}(t, \mathbb{T})$ is not a limit of limit ordinals (it is always a limit ordinal since $t \in A \subseteq T'$), then also require that p_t succeeds the immediate predecessor of t in T'. For each $x \in [A \times T]^{<\omega}$ also let $x^0 = \{t : \exists s((t, s) \in x)\}$. We now define a partial order intended to uniformize $\tilde{c}_{\tilde{\eta}_A}$.

Definition 5. Let $\mathbb{T}$, A, $\tilde{\eta}_A$, $\tilde{c}_{\tilde{\eta}_A}$, $p_t (t \in A)$ be as above. Let

$$\mathbb{P}_{A\tilde{\eta}\tilde{c}} = \{(x,g) : x \in [A \times T]^{<\omega} \wedge \forall (t,s) \in x (p_t \leq_T s <_T t) \wedge \\ g \text{ is a function } \wedge g : \bigcup_{(t,s)\in x} \eta_t \setminus \hat{s} \to \omega \wedge \forall (t,s) \in x \forall a \in \eta_t \setminus \hat{s} (g(a) = c_t(a))\}$$

where $(x_2, g_2) \leq (x_1, g_1)$ iff $x_2 \supseteq x_1$ and $g_2 \supseteq g_1$.

Lemma 6. *Let $\mathbb{T} = \langle T, \leq_T \rangle$ be a special Aronszajn tree, $A \subseteq T'$, $\tilde{\eta}_A = \langle \eta_t : t \in A \rangle$ a full ladder system on A, and $\tilde{c}_{\tilde{\eta}_A} = \langle c_t : t \in A \rangle$ a nice coloring of $\tilde{\eta}_A$. Then $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$ has the ccc.*

Proof. By way of contradiction, let $\{(x_\alpha, g_\alpha) : \alpha < \omega_1\}$ be an uncountable anti-chain in $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$. As $\tilde{\eta}_A$ is a full ladder system, $\forall \alpha < \omega_1 \forall (t,s) \in x_\alpha (\eta_t \setminus \hat{s} = [s,t)_T)$. By the Δ-system lemma we may assume that $\{x^0_\alpha : \alpha < \omega_1\}$ forms a Δ-system with root r and after another thinning process we may assume $r = \emptyset$. Since $\forall \alpha, \beta < \omega_1 (\alpha \neq \beta \to (x_\alpha, g_\alpha) \not\sim (x_\beta, g_\beta))$ we have

$$(\diamond) \quad \forall \alpha, \beta < \omega_1 (\alpha \neq \beta \to \exists (t,s) \in x_\alpha \exists (t', s') \in x_\beta \\ \exists a \in [s,t)_T \cap [s', t')_T (g_\alpha(a) \neq g_\beta(a))).$$

For all $\alpha < \omega_1$ and each $(t,s) \in x_\alpha$ there is r_t, with $s \leq_T r_t \leq_T t$, such that $\bigcup_{(t,s)\in x_\alpha} [s,t)_T = \bigcup_{t \in x^0_\alpha} [r_t, t)_T$ and $\forall t, t' \in x^0_\alpha (t \neq t' \to [r_t, t)_T \cap [r_{t'}, t')_T = \emptyset)$. For each $\alpha < \omega_1$ let $y_\alpha = \{(t, r_t) : t \in x^0_\alpha \wedge r_t <_T t\}$. Then, since $\mathrm{dom}(g_\alpha) = \bigcup_{(t,s)\in x_\alpha} [s,t)_T = \bigcup_{(t,r)\in y_\alpha} [r,t)_T$ for each $\alpha < \omega_1$, we have $\forall \alpha < \omega_1 ((y_\alpha, g_\alpha) \in \mathbb{P}_{A\tilde{\eta}\tilde{c}})$. In addition, $(\diamond)$ implies

$$(\circ) \quad \forall \alpha, \beta < \omega_1 (\alpha \neq \beta \to \exists (t,s) \in y_\alpha \exists (t', s') \in y_\beta \\ \exists a \in [s,t)_T \cap [s', t')_T (g_\alpha(a) \neq g_\beta(a))).$$

Thus $\{(y_\alpha, g_\alpha) : \alpha < \omega_1\}$ is an uncountable anti-chain in $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$.

From now on we work with the uncountable anti-chain $\{(y_\alpha, g_\alpha) : \alpha < \omega_1\}$ toward a contradiction. We may assume that $\forall \alpha < \omega_1 (| y_\alpha |= n)$ for some $n < \omega$. So for each $\alpha < \omega_1$ let $y_\alpha = \{(t^\alpha_i, r^\alpha_i) : i < n\}$ and let $\varphi_{t^\alpha_i}$ be as in Definition 1(2). For each $\alpha < \omega_1$ and $i < n$ we define $\psi^\alpha_i : \omega \to [r^\alpha_i, t^\alpha_i)_T$ as follows. Let l^α_i be the least such that $r^\alpha_i <_T \varphi_{t^\alpha_i}(l^\alpha_i)$. Let $\psi^\alpha_i(0) = r^\alpha_i$ and $\psi^\alpha_i(j) = \varphi_{t^\alpha_i}(l^\alpha_i + j - 1)$ for $0 < j < \omega$. By another thinning process we may assume

$$\forall \beta < \omega_1 (\sup\{\mathrm{ht}(t^\alpha_i, \mathbb{T}) : i < n \wedge \alpha < \beta\} < \min\{\mathrm{ht}(\psi^\beta_i(m_\beta), \mathbb{T}) : i < n\})$$

for some $m_\beta < \omega$. We may further assume that $\forall \alpha < \omega_1 (m_\alpha = m)$ for some $m < \omega$. Now $\mathbb{T} = \langle T, \leq_T \rangle$ is a special Aronszajn tree, so let $T = \bigcup_{l<\omega} A_l$ where each A_l $(l < \omega)$ is an anti-chain in $\mathbb{T}$. By a further thinning process we may assume

$$\forall i < n \forall k \leq m \exists l < \omega (\{\psi^\alpha_i(k) : \alpha < \omega_1\} \subseteq A_l).$$

And by another thinning process we may assume

$$\forall \alpha, \beta < \omega_1 \forall i < n \forall k < m \\ \forall a \in [\psi^\alpha_i(k), \psi^\alpha_i(k+1))_T \forall b \in [\psi^\beta_i(k), \psi^\beta_i(k+1))_T (c_{t^\alpha_i}(a) = c_{t^\beta_i}(b)).$$

This in turn implies

$$(\star) \quad \forall \alpha, \beta < \omega_1 \forall i < n ((\{(t^\alpha_i, r^\alpha_i)\}, g_\alpha \restriction [r^\alpha_i, t^\alpha_i)_T) \sim (\{(t^\beta_i, r^\beta_i)\}, g_\beta \restriction [r^\beta_i, t^\beta_i)_T)).$$

Let $\mathcal{U}$ be a uniform ultrafilter on ω_1. Then ($\circ$) implies that for each $\alpha < \omega_1$ there are $i(\alpha)$, $j(\alpha) < n$ such that

$$B_\alpha = \{\beta : \alpha < \beta < \omega_1 \wedge \exists a \in [r^\alpha_{i(\alpha)}, t^\alpha_{i(\alpha)})_T \cap [r^\beta_{j(\alpha)}, t^\beta_{j(\alpha)})_T (g_\alpha(a) \neq g_\beta(a))\} \in \mathcal{U}.$$

Furthermore, there must be $i, j < n$ such that

$$B = \{\alpha : i(\alpha) = i \wedge j(\alpha) = j\} \in \mathcal{U}.$$

Note that by ($\star$) we have $i \neq j$. Let $\alpha \in B$ and note that $B \cap B_\alpha \in \mathcal{U}$. Let $\beta \in B \cap B_\alpha$ and note that $B \cap B_\alpha \cap B_\beta \in \mathcal{U}$. Let $\gamma \in B \cap B_\alpha \cap B_\beta$ and note that $\alpha < \beta < \gamma$. We also have that

$$\exists a \in [r^\alpha_i, t^\alpha_i)_T \cap [r^\beta_j, t^\beta_j)_T (g_\alpha(a) \neq g_\beta(a)),$$

$$\exists a \in [r^\alpha_i, t^\alpha_i)_T \cap [r^\gamma_j, t^\gamma_j)_T (g_\alpha(a) \neq g_\gamma(a)),$$

$$\exists a \in [r^\beta_i, t^\beta_i)_T \cap [r^\gamma_j, t^\gamma_j)_T (g_\beta(a) \neq g_\gamma(a)).$$

But this implies

$$[r^\alpha_i, t^\alpha_i)_T \cap [r^\beta_j, t^\beta_j)_T \neq \emptyset,$$

$$[r^\alpha_i, t^\alpha_i)_T \cap [r^\gamma_j, t^\gamma_j)_T \neq \emptyset,$$

$$[r^\beta_i, t^\beta_i)_T \cap [r^\gamma_j, t^\gamma_j)_T \neq \emptyset.$$

This in turn implies $[r^\beta_i, t^\beta_i)_T \cap [r^\beta_j, t^\beta_j)_T \neq \emptyset$ which contradicts the earlier assumption that $[r^\beta_i, t^\beta_i)_T \cap [r^\beta_j, t^\beta_j)_T = \emptyset$. Therefore $\{(y_\alpha, g_\alpha) : \alpha < \omega_1\}$ cannot be an uncountable anti-chain in $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$ and hence $\{(x_\alpha, g_\alpha) : \alpha < \omega_1\}$ cannot be an uncountable anti-chain in $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$. This proves the lemma. □

Lemma 7. *Let $\mathbb{T}$, A, $\tilde{\eta}_A$, $\tilde{c}_{\tilde{\eta}_A}$ be as in the previous lemma. For each $t \in A$, the sets $D_t = \{(x, g) \in \mathbb{P}_{A\tilde{\eta}\tilde{c}} : t \in x^0\}$ are dense in $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$.*

Proof. Fix $t \in A$ and let $(x, g) \in \mathbb{P}_{A\tilde{\eta}\tilde{c}}$. If $t \in x^0$, there is nothing to show. So suppose $t \notin x^0$. Using Definition 1(1) we choose $s \in T$, with $p_t \leq_T s <_T t$, such that

$$\forall (u, v) \in x \forall a \in [s, t)_T \cap [v, u)_T (c_t(a) = c_u(a)).$$

Let $y = x \cup \{(t, s)\}$, $\mathrm{dom}(h) = \mathrm{dom}(g) \cup [s, t)_T$, $h \restriction \mathrm{dom}(g) = g$, and $\forall a \in [s, t)_T (h(a) = c_t(a))$. Clearly $(y, h) \in \mathbb{P}_{A\tilde{\eta}\tilde{c}}$, $(y, h) \in D_t$, and $(y, h) \leq (x, g)$ so that D_t is a dense subset of $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$. □

Theorem 8. *Assume* MA + $\neg$CH. *If $\mathbb{T} = \langle T, \leq_T \rangle$ is an Aronszajn tree, $A \subseteq T'$, $\tilde{\eta}_A$ is a full ladder system on A, and $\tilde{c}_{\tilde{\eta}_A}$ is a nice coloring of $\tilde{\eta}_A$, then $\tilde{c}_{\tilde{\eta}_A}$ can be uniformized.*

Proof. Since MA + $\neg$CH holds, $\mathbb{T}$ is a special Aronszajn tree. Let $\tilde{\eta}_A = \langle \eta_t : t \in A \rangle$ and $\tilde{c}_{\tilde{\eta}_A} = \langle c_t : t \in A \rangle$ be as in the statement of the theorem. Let $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$ and D_t ($t \in A$) be as above. Then $\mathbb{P}_{A\tilde{\eta}\tilde{c}}$ has the *ccc* and D_t ($t \in A$) is dense. By MA + $\neg$CH let $G \subseteq \mathbb{P}_{A\tilde{\eta}\tilde{c}}$ be a filter which meets each D_t ($t \in A$). Let $\tilde{f} = \bigcup\{g : \exists x((x, g) \in G)\}$. Clearly, for each $t \in A$ there is $s <_T t$ such that $\forall a \in [s, t)_T (\tilde{f}(a) = c_t(a))$. But $\tilde{f}$ may not be a total function on T. To rectify this, define $f : T \to \omega$ by letting $f \restriction \mathrm{dom}(\tilde{f}) = \tilde{f}$ and $f(a) = 0$ for all $a \in T \setminus \mathrm{dom}(\tilde{f})$. Then f is a uniformizing function for the coloring $\tilde{c}_{\tilde{\eta}_A}$ and the theorem is proved. □

In Theorem 4 normality played a crucial role in proving that the uniformization property for $\mathbb{T}$ holds. And since MA + $\neg$CH implies that all Aronszajn trees are normal, normality is also present in trees as in Theorem 8. This suggests that normality and the uniformization property in trees are closely related. In fact, normality is implied by the uniformization property, as the next result shows. In fact, together with Theorem 4, the next result shows that for almost Suslin trees, normality is equivalent to the existence of uniformizing functions for nice colorings of full ladder systems on closed sets. We state it in a somewhat more general form.

Theorem 9. *Suppose $\mathbb{T}$ is an ω_1-tree such that for each closed $A \subseteq T'$, each nice coloring of a full ladder system on A can be uniformized. Then any countable discrete family of subsets of $\mathbb{T}$ can be separated by pairwise disjoint open sets.*

Proof. Let $\{A_n : n < \omega\}$ be a countable discrete family of subsets of T. Then $\{\bar{A}_n : n < \omega\}$ is also discrete and let $A = \bigcup_{n<\omega} \bar{A}_n \cap T'$. We define a full ladder system, $\tilde{\eta}_A = \langle \eta_t : t \in A\rangle$, on A and a nice coloring, $\tilde{c}_{\tilde{\eta}_A} = \langle c_t : t \in A\rangle$, of $\tilde{\eta}_A$. Let $t \in A$ and choose n such that $t \in A_n$. Since $\{\bar{A}_i : i < \omega\}$ is discrete, there is $s_t <_T t$ such that $[s_t, t]_T \cap \bigcup_{i \neq n} \bar{A}_i = \emptyset$. Let $\eta_t = [s_t, t)_T$ and $c_t(a) = n$ for each $a \in \eta_t$. This defines $\tilde{\eta}_A$ and $\tilde{c}_{\tilde{\eta}_A}$ as required. Let f be a uniformizing function for $\tilde{c}_{\tilde{\eta}_A}$. For each $t \in A$ choose r_t with $s_t \leq_T r_t <_T t$ such that $\forall a \in [r_t, t)_T (f(a) = c_t(a))$. Then $\{A_n \cup \bigcup_{t \in A \cap A_n} (r_t, t)_T : n < \omega\}$ is a family of pairwise disjoint open subsets of T which separate $\{A_n : n < \omega\}$. This proves the theorem. □

The original version of Theorem 9 did not require in the hypothesis that A is closed. This strengthening was pointed out by the referee to whom we are grateful.

If $\mathbb{T}$ is as in the previous theorem, then $\mathbb{T}$ is also normal, hence countably paracompact, by a result of Hanazawa [6]. The next section presents a dual result.

§3. Weak diamonds

For each $2 \leq n \leq \omega$ let Φ_n denote the following assertion:

$$\forall F : \omega^{<\omega_1} \to n \ \exists g : \omega_1 \to n \ \forall f : \omega_1 \to \omega \ (\{\alpha < \omega_1 : F(f \restriction \alpha) = g(\alpha)\} \text{ is stationary}).$$

It is easily seen that for $2 \leq m \leq n \leq \omega$, $\Diamond \to \Phi_n \to \Phi_m$. Shelah [7] proved that the implications in the reversed directions do not hold and later showed that for $3 \leq n \leq \omega$, $\neg\Phi_n$ is consistent with GCH. We now state some properties of Φ_ω (see [1] for more details) which are needed in the proof of Theorem 9 below.

Recall that $\mathcal{I}$ is the ideal of nonstationary subsets of ω_1. It is well known that $\mathcal{I}$ is normal, i.e. if $\{I_\nu : \nu < \omega_1\} \subseteq \mathcal{I}$, then $\{\alpha < \omega_1 : \exists \nu < \alpha(\alpha \in I_\nu)\} \in \mathcal{I}$. And in particular, $\mathcal{I}$ is countably complete, i.e. if $\{I_\nu : \nu < \omega\} \subseteq \mathcal{I}$, then $\bigcup_{\nu<\omega} I_\nu \in \mathcal{I}$. We say that $S \subseteq \omega_1$ is small if

$$\exists F : \omega^{<\omega_1} \to \omega \ \forall g : \omega_1 \to \omega \ \exists f : \omega_1 \to \omega(\{\alpha \in S : F(f \restriction \alpha) = g(\alpha)\} \in \mathcal{I}).$$

Let $\mathcal{S}$ be a collection of all small subsets of ω_1. Clearly, $\Phi_\omega \leftrightarrow \omega_1 \notin \mathcal{S}$. For the next theorem we recall a result from [1].

Lemma 10. *$\mathcal{S}$ is a normal ideal on ω_1.*

Theorem 11. *If Φ_ω holds, then no special Aronszajn tree is countably paracompact.*

Proof. Let $\mathbb{T} = \langle T, \leq_T \rangle$ be a special Aronszajn tree. By way of contradiction assume $\mathbb{T}$ is countably paracompact. We may assume that $T = \omega_1$, $\mathrm{ht}(\alpha, \mathbb{T}) = \alpha$ for each limit α, $\alpha \leq_T \beta \rightarrow \alpha \leq \beta$, and $\mathrm{ht}(\alpha, \mathbb{T}) < \mathrm{ht}(\beta, \mathbb{T})$ for each $\alpha < \beta$ with limit β.

Let A_n $(n < \omega)$ be pairwise disjoint anti-chains of $\mathbb{T}$ with $T = \bigcup_{n<\omega} A_n$. Since $\mathcal{S}$ is countably complete and $\omega_1 \notin \mathcal{S}$, there is an m such that $A_m \notin \mathcal{S}$. Let $E = \{\alpha \in A_m : \text{ limit } \alpha\}$ and note that $E \notin \mathcal{S}$.

We define $F : \omega^{<\omega_1} \rightarrow \omega$ as follows: If α is a limit ordinal and $f \in \omega^\alpha$ is bounded on $(\beta, \alpha)_T$ for some $\beta <_T \alpha$, let n be the largest value in the range of $f \restriction (\beta, \alpha)_T$ whose set of preimages is $\leq_T$-unbounded in $(\beta, \alpha)_T$. In this case let $F(f) = n$. Let $F(f) = 0$ otherwise.

Since $E \notin \mathcal{S}$, there is a $g \in \omega^{\omega_1}$ such that for all $f \in \omega^{\omega_1}$, $\{\alpha \in E : F(f \restriction \alpha) = g(\alpha)\}$ is stationary in ω_1. Let $E_n = \{\alpha \in E : g(\alpha) = n\}$. Since E is an anti-chain in $\mathbb{T}$, it is closed and discrete so that $\{U_n = (T \setminus E) \cup E_n : n < \omega\}$ is an open cover of T. By countable paracompactness, let $\{V_n : n < \omega\}$ be a locally finite open refinement of $\{U_n : n < \omega\}$ such that $E_n \subseteq V_n \subseteq U_n$.

We use the cover $\{V_n : n < \omega\}$ to define $f : \omega_1 \rightarrow \omega$. Let $\alpha \in T$. If $\mathrm{ht}(\alpha, \mathbb{T})$ is a successor ordinal, then $\{\alpha\}$ is open in T and by countable paracompactness only finitely many V_n intersect $\{\alpha\}$. In this case let $f(\alpha) = \max\{n + 1 : \alpha \in V_n\}$. Now suppose $\mathrm{ht}(\alpha, \mathbb{T})$ is a limit ordinal and $f(\beta)$ has been defined for each $\beta <_T \alpha$. By countable paracompactness again, there is $\beta <_T \alpha$ such that $(\beta, \alpha]_T$ intersects only finitely many V_n and f is bounded on $(\beta, \alpha)_T$. Let $f(\alpha)$ be the largest value in the range of $f \restriction (\beta, \alpha)_T$ whose set of preimages is $\leq_T$-unbounded in $(\beta, \alpha)_T$. This defines f.

We make an observation: For any $n < \omega$ and any $\alpha \in E_n$, $n < f(\alpha)$.

And now since $\{\alpha \in E : F(f \restriction \alpha) = g(\alpha)\}$ is stationary, $E = \bigcup_{i<\omega} E_i$ and $\mathcal{I}$ is countably complete, there is $n < \omega$ such that $K = \{\alpha \in E_n : F(f \restriction \alpha) = g(\alpha)\}$ is stationary, hence nonempty. Let $\alpha \in K$. Then $F(f \restriction \alpha) = g(\alpha)$. On the other hand, by definition of E_n, $g(\alpha) = n$ and by the above observation, $f(\alpha) > n$ so that $f(\alpha) > g(\alpha)$. Furthermore, since $\mathrm{ht}(\alpha, \mathbb{T}) = \alpha$ is a limit ordinal, by definition of f, there is $\beta <_T \alpha$ such that f is bounded on $(\beta, \alpha)_T$ and $f(\alpha)$ is the largest value that f assumes on the interval $(\beta, \alpha)_T$ whose set of preimages is $\leq_T$-unbounded in $(\beta, \alpha)_T$. So by the definition of F, $F(f \restriction \alpha) = f(\alpha)$ and, together with $f(\alpha) > g(\alpha)$, we get $F(f \restriction \alpha) > g(\alpha)$. But this contradicts $F(f \restriction \alpha) = g(\alpha)$ and the theorem is proved. □

As a corollary of this theorem and Theorem 9 we get that if Φ_ω holds, then the uniformization fails for special Aronszajn trees. In fact, it was shown in [3] that a weaker assumption than Φ_ω, namely $2^\omega < 2^{\omega_1}$, implies that normality fails in special Aronszajn trees, hence by Theorem 9, the uniformization also fails for such trees.

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