

## CONTINUOUS SELECTIONS AND $C$ -SPACES

VALENTIN GUTEV AND VESKO VALOV

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ABSTRACT. A characterization of paracompact  $C$ -spaces via continuous selections avoiding  $Z_\infty$ -sets is given. The result is applied to prove a countable sum theorem for paracompact  $C$ -spaces, and to obtain a new partial solution of a question raised by E. Michael.

### 1. INTRODUCTION

We apply a recent characterization of paracompact  $C$ -spaces given by Uspenskij [16] to obtain a selection theorem of Michael's type characterizing this class of spaces. More precisely, the following theorem will be proved.

**Theorem 1.1.** *For a paracompact space  $X$  the following conditions are equivalent:*

- (a)  $X$  is a  $C$ -space.
- (b) Let  $Y$  be a Banach space and let  $\varphi : X \rightarrow \mathcal{F}_c(Y)$  be an l.s.c. mapping. Then, for every sequence of set-valued mappings  $\psi_n : X \rightarrow \mathcal{F}(Y)$  such that each  $\psi_n$  has a closed graph and  $\psi_n(x) \cap \varphi(x)$  is a  $Z_\infty$ -set in  $\varphi(x)$  for every  $x \in X$  and  $n \in \mathbb{N}$ , there exists a single-valued continuous map  $f : X \rightarrow Y$  with  $f(x) \in \varphi(x) \setminus \bigcup\{\psi_n(x) : n \in \mathbb{N}\}$  for each  $x \in X$ .
- (c) Let  $Y$  be a Banach space and let  $\varphi : X \rightarrow \mathcal{F}_c(Y)$  be an l.s.c. mapping. Then, for every closed  $F \subset Y$  there exists a single-valued continuous selection for  $\varphi$  avoiding  $F$ , provided that  $F \cap \varphi(x)$  is a  $Z_\infty$ -set in  $\varphi(x)$  for each  $x \in X$ .

Here,  $2^Y$  is the family of all nonempty subsets of  $Y$ ,  $\mathcal{F}(Y)$  that of all closed  $F \in 2^Y$ , and  $\mathcal{F}_c(Y)$  denotes the convex elements of  $\mathcal{F}(Y)$  (the same notation will be used when  $Y$  is a subset of a vector space). A set-valued mapping  $\theta : X \rightarrow 2^Y$  is *lower semi-continuous*, or l.s.c., if  $\theta^{-1}(U) = \{x \in X : \theta(x) \cap U \neq \emptyset\}$  is open in  $X$  for every open  $U \subset Y$ , and it has a *closed graph* if  $\text{Graph}(\theta) = \{(x, y) \in X \times Y : y \in \theta(x)\}$  is closed in  $X \times Y$ . A map  $f : X \rightarrow Y$  is a *selection* for  $\theta : X \rightarrow 2^Y$  if  $f(x) \in \theta(x)$  for every  $x \in X$ , and it *avoids* a set  $F \subset Y$  provided  $f(x) \notin F$  for every  $x \in X$ .

We adopt the following definition for a  $Z_\infty$ -set: a closed subset  $F$  of  $Y$  (possibly empty) is said to be a  $Z_n$ -set in  $Y$ ,  $n \in \mathbb{N}$ , if the set  $C(\mathbb{I}^n, Y \setminus F)$  is dense in  $C(\mathbb{I}^n, Y)$ , and we say that  $F$  is a  $Z_\infty$ -set in  $Y$  provided it is a  $Z_n$ -set in  $Y$  for all  $n \in \mathbb{N}$  (see [2] and [15]). The collection of all  $Z_\infty$ -sets (resp.,  $Z_n$ -sets) in  $Y$  will be denoted by  $\mathcal{Z}_\infty(Y)$  (resp.,  $\mathcal{Z}_n(Y)$ ).

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The  $C$ -space property was originally defined by W. Haver [6] for compact metric spaces. Addis and Gresham [1] reformulated Haver's definition for arbitrary spaces: A space  $X$  has property  $C$  (or  $X$  is a  $C$ -space) if for any sequence  $\{\omega_n : n \in \mathbb{N}\}$  of open covers of  $X$  there exists a sequence  $\{\gamma_n : n \in \mathbb{N}\}$  of open disjoint families in  $X$  such that each  $\gamma_n$  refines  $\omega_n$  and  $\bigcup\{\gamma_n : n \in \mathbb{N}\}$  is a cover of  $X$ . Note that, in fact, a  $C$ -space  $X$  is paracompact if and only if it is countably paracompact and normal. Every finite-dimensional paracompact space, as well as every countable-dimensional metric space, is a  $C$ -space [1], but there exists a compact metric  $C$ -space which is not countable-dimensional [14].

Our first application of Theorem 1.1 is a countable sum theorem for paracompact  $C$ -spaces, i.e. if a paracompact space  $X$  is a countable union of its closed  $C$ -subspaces, then  $X$  also has property  $C$  (see Theorem 4.1). Therefore, every strongly countable-dimensional paracompact space (a countable union of closed finite-dimensional subsets) has property  $C$ . Theorem 4.1 allows one to prove also that a paracompact space is a  $C$ -space provided it has locally property  $C$  (see Theorem 4.2). The latter result improves [1, Theorem 2.16] by dropping the assumption on the space under consideration to be hereditarily collectionwise normal. Let us point out here that Theorem 4.1 should be compared with another countable sum theorem [1, Theorem 2.7] which works without the assumption that the sets are closed but for hereditarily collectionwise normal spaces.

Another application of Theorem 1.1 concerns the following Michael's problem [11, Problem 396]: *Let  $X$  be a paracompact space,  $E$  a Banach space,  $Y \subset E$  a convex  $G_\delta$ -subset, and let  $\varphi : X \rightarrow \mathcal{F}_c(Y)$  be l.s.c. Does there exist a single-valued continuous selection for  $\varphi$ ?*

The first author has provided a positive answer to this question if  $X$  is either a countable-dimensional metric space [5, Corollary 1.2] or a strongly countable-dimensional paracompact space [5, Corollary 1.3] (actually, these results were proved under the less restrictive assumption that  $Y$  is  $G_\delta$  and  $\varphi$  is l.s.c. closed- and convex-valued in  $Y$ ). Using Theorem 1.1, we demonstrate that the restrictions on  $X$  can be weakened to " $X$  is a paracompact  $C$ -space" (see Theorem 4.4).

A few words about the paper. Section 2 is devoted to some preliminary results. The proof of Theorem 1.1 is given in Section 3. The final Section 4 contains applications of Theorem 1.1.

## 2. PRELIMINARIES

We agree on the following notation. For a metric space  $(Y, d)$ , a point  $y \in Y$  and a positive number  $\delta$ , the open ball in  $Y$  with center  $y$  and radius  $\delta$  is denoted by  $B_\delta(y)$ . If  $Y$  is a normed space, then  $d$  stands for the metric generated by the norm of  $Y$ .

In what follows, to a pair of set-valued mappings  $\varphi : X \rightarrow 2^Y$  and  $\psi : X \rightarrow 2^Y$  we associate a set-valued mapping  $\Delta_{(\varphi, \psi)} : X \times Y \rightarrow 2^{\mathbb{R}} \cup \{\emptyset\}$  such that, for every  $(x, y) \in X \times Y$ , the value  $\Delta_{(\varphi, \psi)}(x, y)$  consists of all  $\delta > 0$  for which there exists a neighbourhood  $U_\delta$  of  $x$  with the property that

$$(1) \quad B_\delta(y) \cap \varphi(z) \neq \emptyset \text{ and } B_\delta(y) \cap \psi(z) = \emptyset, \quad \text{whenever } z \in U_\delta.$$

Also, we associate another set-valued mapping  $\Phi_{(\varphi, \psi)} : X \rightarrow 2^Y \cup \{\emptyset\}$  defined by

$$(2) \quad \Phi_{(\varphi, \psi)}(x) = \{y \in Y : \Delta_{(\varphi, \psi)}(x, y) \neq \emptyset\}, \quad x \in X,$$

and functions  $u_{(\varphi,\psi)}, \ell_{(\varphi,\psi)} : \text{Graph}(\Phi_{(\varphi,\psi)}) \rightarrow \mathbb{R}$  defined for  $(x, y) \in \text{Graph}(\Phi_{(\varphi,\psi)})$  by

$$(3) \quad u_{(\varphi,\psi)}(x, y) = \inf \Delta_{(\varphi,\psi)}(x, y) \quad \text{and} \quad \ell_{(\varphi,\psi)}(x, y) = \sup \Delta_{(\varphi,\psi)}(x, y).$$

Finally, let us recall that a function  $r : X \rightarrow \mathbb{R}$  is *lower semi-continuous* (resp., *upper semi-continuous*), or merely lsc (usc), if  $r^{-1}(a, +\infty)$  (resp.,  $r^{-1}(-\infty, a)$ ) is open in  $X$  for every  $a \in \mathbb{R}$ .

**Lemma 2.1.** *Let  $(Y, d)$  be a metric space, and let  $\varphi : X \rightarrow 2^Y$  and  $\psi : X \rightarrow 2^Y$  be set-valued mappings such that  $\varphi$  is l.s.c.,  $\psi$  is with a closed graph and  $\varphi(x) \setminus \psi(x) \neq \emptyset$  for every  $x \in X$ . Then,*

- (i)  $\varphi(x) \setminus \psi(x) \subset \Phi_{(\varphi,\psi)}(x)$  for every  $x \in X$ ;
- (ii)  $u_{(\varphi,\psi)}(x, y) < \ell_{(\varphi,\psi)}(x, y)$  for every  $(x, y) \in \text{Graph}(\Phi_{(\varphi,\psi)})$ ;
- (iii)  $\text{Graph}(\Phi_{(\varphi,\psi)})$  is open in  $X \times Y$ ;
- (iv)  $u_{(\varphi,\psi)}$  is usc, while  $\ell_{(\varphi,\psi)}$  is lsc.

*Proof.* Let  $x \in X$  and  $y \in \varphi(x) \setminus \psi(x)$ . Since  $\psi$  has a closed graph, there exists a neighbourhood  $U$  of  $x$  and a neighbourhood  $V$  of  $y$  such that  $U \times V \cap \text{Graph}(\psi) = \emptyset$ . Then, let  $\delta > 0$  be such that  $B_\delta(y) \subset V$ . Finally, set  $U_\delta = \varphi^{-1}(B_\delta(y)) \cap U$  which is a neighbourhood of  $x$  because  $\varphi$  is l.s.c. It is now clear that, for every  $z \in U_\delta$ ,

$$B_\delta(y) \cap \varphi(z) \neq \emptyset \quad \text{while} \quad B_\delta(y) \cap \psi(z) = \emptyset.$$

Therefore  $\delta \in \Delta_{(\varphi,\psi)}(x, y)$ , i.e.  $y \in \Phi_{(\varphi,\psi)}(x)$  which is (i).

Turning to the proof of (iii) and (iv), let us first observe that, by (1), the statement of (ii) is a consequence of the lower semi-continuity of  $\varphi$ . Indeed, take  $(x, y) \in \text{Graph}(\Phi_{(\varphi,\psi)})$ . Then,  $\Delta_{(\varphi,\psi)}(x, y) \neq \emptyset$ , so, by (1), there exists  $\delta > 0$  and a neighbourhood  $U_\delta$  of  $x$  with

$$B_\delta(y) \cap \varphi(z) \neq \emptyset \quad \text{and} \quad B_\delta(y) \cap \psi(z) = \emptyset, \quad \text{whenever } z \in U_\delta.$$

Since  $B_\delta(y) \cap \varphi(x) \neq \emptyset$ , there exists  $\nu > 0$  such that  $\nu < \delta$  and  $B_\nu(y) \cap \varphi(x) \neq \emptyset$ . Then,  $U_\nu = U_\delta \cap \varphi^{-1}(B_\nu(y))$  is a neighbourhood of  $x$  because  $\varphi$  is l.s.c., and clearly

$$B_\nu(y) \cap \varphi(z) \neq \emptyset \quad \text{and} \quad B_\nu(y) \cap \psi(z) = \emptyset, \quad \text{whenever } z \in U_\nu.$$

Therefore,  $\nu \in \Delta_{(\varphi,\psi)}(x, y)$  which finally implies that

$$u_{(\varphi,\psi)}(x, y) = \inf \Delta_{(\varphi,\psi)}(x, y) \leq \nu < \delta \leq \sup \Delta_{(\varphi,\psi)}(x, y) = \ell_{(\varphi,\psi)}(x, y).$$

To show now (iii) and (iv), take a point  $(x_0, y_0) \in \text{Graph}(\Phi_{(\varphi,\psi)})$  and an  $a \in \mathbb{R}$  with  $u_{(\varphi,\psi)}(x_0, y_0) < a < \ell_{(\varphi,\psi)}(x_0, y_0)$ . Also, take  $\delta_u, \delta_\ell \in \Delta_{(\varphi,\psi)}(x_0, y_0)$ , with  $\delta_u < a < \delta_\ell$ , and a neighbourhood  $U_0$  of  $x_0$  such that

$$(4) \quad B_{\delta_u}(y_0) \cap \varphi(z) \neq \emptyset \quad \text{and} \quad B_{\delta_\ell}(y_0) \cap \psi(z) = \emptyset, \quad z \in U_0.$$

Finally, let  $\rho = \min\{(\delta_\ell - a)/2, (a - \delta_u)/2\}$ . Take points  $x, z \in U_0$  and  $y \in B_\rho(y_0)$ . Then, by (4),

$$\emptyset \neq B_{\delta_u}(y_0) \cap \varphi(z) \subset B_{\delta_u + \rho}(y) \cap \varphi(z),$$

and

$$B_{\delta_\ell - \rho}(y) \cap \psi(z) \subset B_{\delta_\ell}(y_0) \cap \psi(z) = \emptyset.$$

Therefore  $\delta_u + \rho, \delta_\ell - \rho \in \Delta_{(\varphi,\psi)}(x, y)$  which implies  $U_0 \times B_\rho(y_0) \subset \text{Graph}(\Phi_{(\varphi,\psi)})$  and that  $u_{(\varphi,\psi)}(x, y) < a < \ell_{(\varphi,\psi)}(x, y)$  for every  $(x, y) \in U_0 \times B_\rho(y_0)$ .  $\square$

**Lemma 2.2.** *Let  $(Y, \|\cdot\|)$  be a normed space, and let  $\varphi : X \rightarrow 2^Y$  and  $\psi : X \rightarrow 2^Y$  be set-valued mappings such that  $\varphi$  is l.s.c.,  $\psi$  is with a closed graph and  $\varphi(x) \setminus \psi(x) \neq \emptyset$  for every  $x \in X$ . Then, for every  $x \in X$ , we have  $[y_0, y_1] \subset \Phi_{(\varphi, \psi)}(x)$  provided  $y_1 \in \Phi_{(\varphi, \psi)}(x)$  and  $y_0 \in B_{\delta_1}(y_1) \cap \varphi(x)$  for some  $\delta_1 \in \Delta_{(\varphi, \psi)}(x, y_1)$ .*

*Proof.* Let  $x \in X$ ,  $y_1 \in \Phi_{(\varphi, \psi)}(x)$ ,  $\delta_1 \in \Delta_{(\varphi, \psi)}(x, y_1)$  and  $y_0 \in B_{\delta_1}(y_1) \cap \varphi(x)$ . Note that, by (i) of Lemma 2.1,  $y_0 \in \Phi_{(\varphi, \psi)}(x)$ . Then, let  $y = (1-t)y_0 + ty_1$  for some  $t \in (0, 1)$ . It suffices to show that  $\Delta_{(\varphi, \psi)}(x, y) \neq \emptyset$ . For the purpose, let us show that  $\delta = t\delta_1 \in \Delta_{(\varphi, \psi)}(x, y)$ . Towards this end, let  $U_{\delta_1}$  be as in the definition of “ $\delta_1 \in \Delta_{(\varphi, \psi)}(x, y_1)$ ”, see (1). Next, note that  $B_\delta(y) \cap \varphi(x) \neq \emptyset$ . Indeed,

$$d(y, y_0) = \|y - y_0\| = t\|y_1 - y_0\| < t\delta_1 = \delta.$$

Then,  $U_\delta = \varphi^{-1}(B_\delta(y)) \cap U_{\delta_1}$  is a neighbourhood of  $x$  because  $\varphi$  is l.s.c. Thus, to show that  $\delta \in \Delta_{(\varphi, \psi)}(x, y)$ , it only remains to show that  $B_\delta(y) \cap \psi(z) = \emptyset$  for every  $z \in U_\delta$ . Suppose, if possible, that this fails for some point  $z \in U_\delta$ . Hence, we can find a point  $y_2 \in B_\delta(y) \cap \psi(z)$ . Then,  $d(y_1, y) = \|y_1 - y\| = (1-t)\|y_1 - y_0\|$  implies

$$d(y_1, y_2) \leq d(y_1, y) + d(y, y_2) < (1-t)\delta_1 + \delta = (1-t)\delta_1 + t\delta_1 = \delta_1.$$

That is,  $y_2 \in B_{\delta_1}(y_1) \cap \psi(z)$  which is impossible because  $z \in U_{\delta_1}$ .  $\square$

**Lemma 2.3.** *Let  $F \in \mathcal{Z}_n(A)$  for some  $n \in \mathbb{N}$ , where  $A$  is a closed convex subset of a normed space  $Y$ . Also, let  $y \in A$  and  $H = \overline{A \cap B_\delta(y)}$  for some  $\delta > 0$ . Then  $F \cap H$  is a  $Z_n$ -set in  $H$ .*

*Proof.* Let  $f \in C(\mathbb{I}^n, H)$ . We have to show that, for a given  $\varepsilon > 0$ , there exists  $g \in C(\mathbb{I}^n, H \setminus F)$  with  $d(f(x), g(x)) < \varepsilon$  for every  $x \in \mathbb{I}^n$ . To this end, let  $\eta < 1$  be a positive number such that  $\eta(\delta + 1) < \varepsilon$ . Define  $h \in C(\mathbb{I}^n, A \cap B_\delta(y))$  by  $h(x) = (1-\eta)f(x) + \eta y$ ,  $x \in \mathbb{I}^n$ . Since  $U = A \cap B_\delta(y)$  is open in  $A$ ,  $F \cap U$  is a  $Z_n$ -set in  $U$ . Therefore, we can find  $g \in C(\mathbb{I}^n, U \setminus F)$  such that  $d(g(x), h(x)) < \eta$  for every  $x \in \mathbb{I}^n$ . This  $g$  is the required one.  $\square$

A space  $S$  is called *aspherical* [16] if every continuous map  $f : P \rightarrow S$  from a compact polyhedron  $P$  can be continuously extended over the cone of  $P$ . Note that  $S$  is aspherical if and only if any continuous image of an  $n$ -sphere,  $n \in \mathbb{N}$ , in  $S$  is contractible in  $S$ , i.e. when  $S$  is  $C^n$  for every  $n \in \mathbb{N}$ . Finally, we say that a set-valued map  $\Phi : X \rightarrow 2^Y$  is *aspherical* if all  $\Phi(x)$ ,  $x \in X$ , are aspherical subsets of  $Y$ .

We conclude this section with the following special case of a result actually proved by Uspenskij [16].

**Proposition 2.4** ([16]). *Let  $X$  be a paracompact  $C$ -space, let  $Y$  be a topological space, and let  $\Phi : X \rightarrow 2^Y$  be an aspherical mapping with an open graph. Then,  $\Phi$  has a single-valued continuous selection.*

### 3. PROOF OF THEOREM 1.1

We shall prove the implications (a)  $\Rightarrow$  (b) and (c)  $\Rightarrow$  (a). The implication (b)  $\Rightarrow$  (c) is trivial.

(a)  $\Rightarrow$  (b). First, we show that this implication holds if the sequence  $\{\psi_n : n \in \mathbb{N}\}$  is constant.

**Proposition 3.1.** *Let  $X$  be a paracompact  $C$ -space and let  $Y$  be a Banach space. If  $\varphi : X \rightarrow \mathcal{F}_c(Y)$  is l.s.c. and  $\psi : X \rightarrow \mathcal{F}(Y)$  is with a closed graph such that  $\psi(x) \cap \varphi(x)$  is a  $Z_\infty$ -set in  $\varphi(x)$  for every  $x \in X$ , then there exists a continuous  $f : X \rightarrow Y$  with  $f(x) \in \varphi(x) \setminus \psi(x)$  for every  $x \in X$ .*

*Proof.* Note that each  $\varphi(x) \setminus \psi(x) \neq \emptyset$  because  $\psi(x) \cap \varphi(x)$  is a  $Z_\infty$ -set in  $\varphi(x)$ . Then, let us consider the mapping  $\Phi_{(\varphi, \psi)} : X \rightarrow 2^Y$  defined as in (2).

*Claim 1.*  $\Phi_{(\varphi, \psi)}$  is aspherical.

Let  $x_0 \in X$ , and let  $k : P \rightarrow \Phi_{(\varphi, \psi)}(x_0)$  be a continuous map with  $P$  a compact polyhedron. Also, let  $u_{(\varphi, \psi)}$  and  $\ell_{(\varphi, \psi)}$  be defined as in (3). Then, by Lemma 2.1,  $u_{(\varphi, \psi)}(x_0, k(p))$ ,  $p \in P$ , is a usc function on  $P$ ,  $\ell_{(\varphi, \psi)}(x_0, k(p))$ ,  $p \in P$ , is an lsc function on  $P$ , and  $u_{(\varphi, \psi)}(x_0, k(p)) < \ell_{(\varphi, \psi)}(x_0, k(p))$  for every  $p \in P$ . Since  $P$  is paracompact, by a result of [3] (see, also, [4, 7]), this implies the existence of a continuous function  $r : P \rightarrow \mathbb{R}$  such that

$$(5) \quad u_{(\varphi, \psi)}(x_0, k(p)) < r(p) < \ell_{(\varphi, \psi)}(x_0, k(p)) \quad \text{for every } p \in P.$$

An important property of  $r$  is that each  $B_{r(p)}(k(p)) \cap \varphi(x_0)$ ,  $p \in P$ , is non-empty and its closure doesn't meet  $\psi(x_0)$ . Further,  $\theta(p) = \overline{B_{r(p)}(k(p)) \cap \varphi(x_0)}$ ,  $p \in P$ , defines an l.s.c. mapping  $\theta : P \rightarrow \mathcal{F}_c(Y)$  because  $r$  is continuous. Then, according to the Michael's selection theorem [8, Theorem 3.2''],  $\theta$  has a continuous selection  $h$ . Let

$$\text{Cone}(P) = \{tv + (1-t)p : p \in P \text{ and } t \in [0, 1]\}$$

be the cone of  $P$  with a vertex  $v$ . Then we have  $\text{Cone}(P) = H_1 \cup H_2$ , where  $H_1 = \{tv + (1-t)p : p \in P, t \in [0, 2^{-1}]\}$  and  $H_2 = \{tv + (1-t)p : p \in P, t \in [2^{-1}, 1]\}$ . Define  $k_1 \in C(H_1, Y)$  by  $k_1(tv + (1-t)p) = (1-2t)k(p) + 2th(p)$ . Note that, for every  $p \in P$ , (5) implies  $h(p) \in B_{\delta(p)}(k(p)) \cap \varphi(x_0)$  for some  $\delta(p) \in \Delta_{(\varphi, \psi)}(x_0, k(p))$ . Since  $k : P \rightarrow \Phi_{(\varphi, \psi)}(x_0)$ , it now follows by Lemma 2.2 that  $k_1(z) \in \Phi_{(\varphi, \psi)}(x_0)$  for every  $z \in H_1$ . Next observe that  $k_1|_P = k$  while  $\varphi(x_0) \setminus \psi(x_0)$  is an  $AR$  because  $\psi(x_0) \cap \varphi(x_0) \in \mathcal{Z}_\infty(\varphi(x_0))$  (see [16, Proposition 6.4]; this also follows from [15]). Therefore  $k_1|_H$  can be extended to a continuous map  $k_2 : H_2 \rightarrow \varphi(x_0) \setminus \psi(x_0)$ , where  $H = H_1 \cap H_2$ . Finally, we define a continuous extension  $\bar{k} : \text{Cone}(P) \rightarrow \Phi_{(\varphi, \psi)}(x_0)$  of  $k$  by  $\bar{k}|_{H_1} = k_1$  and  $\bar{k}|_{H_2} = k_2$ .

We now proceed to the rest of the proof. By Proposition 2.4,  $\Phi_{(\varphi, \psi)}$  has a continuous selection  $g : X \rightarrow Y$  because, by Claim 1, it is aspherical and, by Lemma 2.1, it has an open graph. Relying once again on Lemma 2.1 and a result of [3, 4, 7], there exists a continuous function  $r : X \rightarrow \mathbb{R}$  such that, for every  $x \in X$ ,

$$(6) \quad d(g(x), \varphi(x)) \leq u_{(\varphi, \psi)}(x, g(x)) < r(x) < \ell_{(\varphi, \psi)}(x, g(x)) \leq d(g(x), \psi(x)).$$

Therefore,  $\theta(x) = \overline{B_{r(x)}(g(x)) \cap \varphi(x)}$  defines an l.s.c. mapping  $\theta : X \rightarrow \mathcal{F}_c(Y)$ . Hence, applying again the Michael selection theorem [8, Theorem 3.2''], we get a continuous selection  $f$  for  $\theta$ . According to (6),  $f$  is as required.  $\square$

We finally accomplish the implication (a)  $\Rightarrow$  (b) of Theorem 1.1 with the following proposition.

**Proposition 3.2.** *Let  $X$  be a paracompact  $C$ -space, let  $Y$  be a Banach space, and let  $\varphi : X \rightarrow \mathcal{F}_c(Y)$  be an l.s.c. mapping. Then, for every sequence of mappings*

$\psi_n : X \rightarrow \mathcal{F}(Y)$  such that each  $\psi_n$  has a closed graph and  $\psi_n(x) \cap \varphi(x)$  is a  $Z_\infty$ -set in  $\varphi(x)$  for every  $x \in X$  and  $n \in \mathbb{N}$ , there is a single-valued continuous map  $f : X \rightarrow Y$  with  $f(x) \in \varphi(x) \setminus \bigcup \{\psi_n(x) : n \in \mathbb{N}\}$  for every  $x \in X$ .

*Proof.* By Proposition 3.1,  $\varphi$  admits a continuous selection  $f_1$  with  $f_1(x) \notin \psi_1(x)$  for each  $x \in X$ . Then, as in the proof of Proposition 3.1, there exists a continuous function  $\delta_1 : X \rightarrow \mathbb{R}$  with

$$u_{(\varphi, \psi_1)}(x, f_1(x)) < \delta_1(x) < \ell_{(\varphi, \psi_1)}(x, f_1(x)), \quad x \in X,$$

i.e., a continuous selection for the set-valued mapping  $\Delta_{(\varphi, \psi_1)}(x, f_1(x))$ ,  $x \in X$ . Hence,  $r_1(x) = \min\{\delta_1(x), 1\}$ ,  $x \in X$ , defines a continuous function  $r_1 : X \rightarrow (0, 1]$  such that  $r_1(x) \leq d(f_1(x), \psi_1(x))$  for every  $x \in X$ . As in the proof of [5, Lemma 2.1], we shall construct a sequence of continuous selections  $f_n$  for  $\varphi$  and a sequence of continuous functions  $r_n : X \rightarrow (0, 1]$  such that, for every  $x \in X$  and  $n$ ,

$$(7) \quad r_n(x) \leq \min\{d(f_k(x), \psi_k(x)) : k \leq n\},$$

$$(8) \quad d(f_{n+1}(x), f_n(x)) \leq 3^{-n}r_n(x).$$

The above conditions will be sufficient. Indeed, by (8),  $\{f_n\}$  is a Cauchy sequence and the limit of this sequence defines a continuous map  $f : X \rightarrow Y$  which is a selection for  $\varphi$ . By (7) and [12, Lemma 6.1.1],  $f(x) \notin \bigcup \{\psi_n(x) : n \in \mathbb{N}\}$ .

So, it remains to define these  $f_n$ 's and  $r_n$ 's. Suppose that  $f_1, \dots, f_n$  and  $r_1, \dots, r_n$  have been already defined. Consider the l.s.c. mapping  $\varphi_{n+1} : X \rightarrow \mathcal{F}_c(Y)$  defined by  $\varphi_{n+1}(x) = \overline{B_{3^{-n}r_n(x)}(f_n(x)) \cap \varphi(x)}$ ,  $x \in X$ . Applying Lemma 2.3 for any  $x$  (with  $A = \varphi(x)$ ,  $F = \psi_{n+1}(x) \cap \varphi(x)$ ,  $y = f_n(x)$  and  $\delta = 3^{-n}r_n(x)$ ), we obtain that  $\psi_{n+1}(x) \cap \varphi_{n+1}(x)$  is a  $Z_\infty$ -set in  $\varphi_{n+1}(x)$ . Hence, by Proposition 3.1 (applied with the mappings  $\varphi_{n+1}$  and  $\psi_{n+1}$ ), there exists a continuous selection  $f_{n+1}$  for  $\varphi_{n+1}$  such that  $f_{n+1}(x) \notin \psi_{n+1}(x)$ ,  $x \in X$ . For  $r_{n+1}$ , as before, take a continuous selection  $\delta_{n+1}$  for  $\Delta_{(\varphi, \psi_{n+1})}(x, f_{n+1}(x))$ ,  $x \in X$ . Then, merely set  $r_{n+1}(x) = \min\{\delta_{n+1}(x), r_n(x)\}$ ,  $x \in X$ .  $\square$

Recall that if  $(Y, d)$  is a metric space,  $X$  is a space and  $f \in C(X, Y)$ , then the *fine topology* on  $C(X, Y)$  is the topology in which the family of all sets

$$V(f, \alpha) = \{g \in C(X, Y) : d(f(x), g(x)) < \alpha(x) \text{ for each } x \in X\},$$

where  $\alpha$  is a positive continuous function on  $X$ , is a local base at  $f$  (see [13]). The next proposition is an analogue of Lemma 3.2 from [10] and demonstrates that, in Proposition 3.1, there are many selections for  $\varphi$  avoiding  $\psi$ . Also, it can be used (instead of Proposition 3.2) to prove the final step of implication (a)  $\Rightarrow$  (b) in Theorem 1.1.

**Proposition 3.3.** *Let  $X, Y, \varphi$  and  $\psi$  be as in Proposition 3.1. Also, let  $\mathcal{S}$  be the set of all continuous selections for  $\varphi$ , endowed with the fine topology. Then  $\mathcal{S}$  is a Baire space and  $\mathcal{U} = \{f \in \mathcal{S} : f \text{ avoids } \psi\}$  is open and dense in  $\mathcal{S}$ .*

*Proof.* As in the proof of [10, Lemma 3.2], one can show that  $\mathcal{S}$  is a Baire space and  $\mathcal{U} \subset \mathcal{S}$  is open. It remains to prove that  $\mathcal{U} \subset \mathcal{S}$  is dense. So, let  $f \in \mathcal{S}$  and  $\alpha : X \rightarrow (0, \infty)$  be continuous, and let us show that  $\mathcal{U} \cap V(f, \alpha) \neq \emptyset$ . Consider the l.s.c. mapping  $\theta : X \rightarrow \mathcal{F}_c(Y)$  defined by  $\theta(x) = \overline{\varphi(x) \cap B_{2^{-1}\alpha(x)}(f(x))}$ . By Lemma 2.3,  $\psi(x) \cap \theta(x)$  is a  $Z_\infty$ -set in  $\theta(x)$  for every  $x \in X$ . Hence, by Proposition 3.1,  $\theta$  has a continuous selection  $g$  avoiding  $\psi$ . Then  $g \in \mathcal{U} \cap V(f, \alpha)$ .  $\square$

(c)  $\Rightarrow$  (a). We follow the proof of [16, Theorem 1.3, implication  $S_2 \Rightarrow C$ ]. Take a sequence  $\{\gamma_n : n = 1, 2, \dots\}$  of open covers of  $X$ . By definition, we must prove that there exists a sequence  $\{\mu_n : n = 1, 2, \dots\}$  of disjoint families of open sets in  $X$  such that each  $\mu_n$  refines  $\gamma_n$  and the union  $\bigcup\{\mu_n : n = 1, 2, \dots\}$  is a cover of  $X$ . To this end, we proceed as in [16] with a few modifications. Namely, first note that, for every  $n$ , there exists a metric space  $(M_n, d_n)$  and a continuous map  $f_n : X \rightarrow M_n$  such that

$$(9) \quad d_n \leq 3 \text{ (i.e., } d_n(z, t) \leq 3 \text{ for all } z, t \in M_n);$$

$$(10) \quad \{f_n^{-1}(B_2(z)) : z \in M_n\} \text{ is an open cover of } X \text{ refining } \gamma_n.$$

Let  $M = \bigsqcup\{M_n : n = 1, 2, \dots\}$  be the disjoint union of our spaces  $M_n$  while  $d$  is the metric on  $M$  defined as

$$(11) \quad d|M_n \times M_n = d_n, \text{ and } d(z, t) = 3 \text{ provided } z \in M_n, t \in M_k \text{ and } n \neq k.$$

Thus, in fact, we may regard each  $f_n$  as a continuous map from  $X$  into the metric space  $(M, d)$ . Embed  $(M, d)$  isometrically into a Banach space  $(E, d)$ , where  $E = C(K)$  is the space of all continuous functions on a zero-dimensional compact space  $K$  and  $d$  is the metric on  $E$  generated by the norm  $\|\cdot\|$  of  $E$ . Next, as in [16], set  $A = \{h \in C(K) : h(x) \in \mathbb{Z} \text{ for some } x \in K\}$ , where  $\mathbb{Z}$  is the set of all integers. Then,  $A$  is a closed and nowhere dense subset of  $E$  which is the union of hyperplanes in  $E$  of the form  $\{h \in E : h(x) = n\}$ ,  $x \in K$  and  $n \in \mathbb{Z}$ . As a result, these hyperplanes provide a partition of  $E$  into open cells of diameter  $\leq 1$ . It now suffices to construct a sequence  $\{g_n : n = 1, 2, \dots\}$  of continuous maps  $g_n : X \rightarrow E$  such that

$$(12) \quad d(f_n(x), g_n(x)) \leq 1 \text{ for all } x \in X \text{ and } n;$$

$$(13) \quad \text{For every } x \in X \text{ there exists an } n \text{ with } g_n(x) \notin A.$$

Indeed, if  $\lambda$  is the family of all components of  $E \setminus A$ , then  $\mu_n = g_n^{-1}(\lambda)$ ,  $n = 1, 2, \dots$ , is a sequence of disjoint families of open subsets of  $X$ . By (10) and (12), each  $\mu_n$  refines  $\gamma_n$  while, by (13), the union  $\bigcup\{\mu_n : n = 1, 2, \dots\}$  is a cover of  $X$ . That is,  $\{\mu_n : n = 1, 2, \dots\}$  is the required one.

Turning to the construction of these  $g_n$ 's, for every  $n$ , we consider the l.s.c. mapping  $\varphi_n : X \rightarrow \mathcal{F}_c(E)$  defined by  $\varphi_n(x) = \overline{B_1(f_n(x))}$ ,  $x \in X$ . Also, let  $Z$  be the linear subspace of  $E^\omega$  defined as

$$Z = \left\{ (y_n) \in E^\omega : \sum_{n=1}^{\infty} 2^{-n} \|y_n\| < +\infty \right\}.$$

Note that  $Z$  is a normed space because  $\|(y_n)\|_\omega = \sum_{n=1}^{\infty} 2^{-n} \|y_n\|$  defines a norm on  $Z$ , and the topology of  $(Z, \|\cdot\|_\omega)$  coincides with the topology of  $Z$  as a subspace of  $E^\omega$ . Let  $\rho$  be the metric on  $Z$  generated by the norm  $\|\cdot\|_\omega$ , and let  $\varphi : X \rightarrow \mathcal{F}_c(E^\omega)$  be the mapping defined by

$$\varphi(x) = \prod \{\varphi_n(x) : n = 1, 2, \dots\}, \quad x \in X.$$

It is easily seen that  $\varphi$  is l.s.c.

*Claim 2.* Each  $\varphi(x)$  is a  $\rho$ -complete subset of  $Z$ .

First, note that by (9) and (11), the diameter of  $M$  as a subset of  $(E, d)$  is finite. Therefore, according to the definition of the mappings  $\varphi_n$ , there exists a closed ball  $B$  over the origin  $0$  of  $E$  such that  $\varphi_n(x) \subset B$  for all  $x \in X$  and  $n \in \mathbb{N}$ . Then, the above claim follows from the fact that  $B^\omega \subset Z$ .

We complete the proof as follows. Let  $Y$  be the completion of  $Z$ . Thus, we get a Banach space and, by Claim 2, an l.s.c. mapping  $\varphi : X \rightarrow \mathcal{F}_c(Y)$ . Further, let  $F$  be the closure in  $Y$  of the set  $A^\omega \cap Z$ . Then  $F \cap \varphi(x) = A^\omega \cap \varphi(x)$  for every  $x \in X$ . Since each  $A^\omega \cap \varphi(x)$  is the product  $\prod\{A \cap \overline{B_1(f_n(x))} : n \in \mathbb{N}\}$ , and  $\overline{B_1(f_n(x))} \cap A$  is a proper subset of  $\overline{B_1(f_n(x))}$ , we finally obtain that each  $F \cap \varphi(x)$  is a  $Z_\infty$ -set in  $\varphi(x)$  (we use the well known fact that, if  $A_n$  is a proper closed subset of  $B_n$ , then  $\prod\{A_n : n \in \mathbb{N}\}$  is a  $Z_\infty$ -set in  $\prod\{B_n : n \in \mathbb{N}\}$ ; see for instance [16, Proposition 6.2]). Therefore, according to (c), there exists a continuous selection  $g : X \rightarrow Z$  for  $\varphi$  avoiding the set  $A^\omega$ . Let  $g_n : X \rightarrow E$  be such that the corresponding diagonal map  $\Delta\{g_n : n = 1, 2, \dots\}$  coincides with  $g$ . These  $g_n$ 's satisfy both (12) and (13).

#### 4. SOME APPLICATIONS

**Theorem 4.1** (Countable sum theorem). *If a paracompact space  $X$  is a countable union of closed  $C$ -subsets, then  $X$  is itself a  $C$ -space.*

*Proof.* Let  $X = \bigcup\{X_n : n \in \mathbb{N}\}$  with each  $X_n \subset X$  a closed  $C$ -space. By Theorem 1.1, it suffices to show that  $X$  satisfies condition (c). To this end, let  $Y$ ,  $\varphi$  and  $F$  be as in (c) of Theorem 1.1. To show that  $\varphi$  has a continuous selection avoiding  $F$ , we construct by induction a sequence  $\{f_n\}$  of continuous selections for  $\varphi$  and open sets  $U_n \subset X$  such that

$$(14) \quad \bigcup\{X_i : 1 \leq i \leq n\} \subset U_n, \quad n \in \mathbb{N},$$

$$(15) \quad f_n(x) = f_{n+1}(x) \notin F \quad \text{for all } x \in \overline{U_n} \text{ and } n \in \mathbb{N}.$$

Suppose that  $f_n$  and  $U_n$  have been already defined. Consider the l.s.c. mapping  $\varphi_{n+1} : X_{n+1} \rightarrow \mathcal{F}_c(Y)$  defined by  $\varphi_{n+1}(x) = \{f_n(x)\}$  if  $x \in X_{n+1} \cap \overline{U_n}$  and  $\varphi_{n+1}(x) = \varphi(x)$  otherwise. Since  $X_{n+1}$  is a paracompact  $C$ -space, by (c) of Theorem 1.1, there is a continuous selection  $g_{n+1} : X_{n+1} \rightarrow Y$  of  $\varphi_{n+1}$  avoiding the set  $F$ . Define another l.s.c. mapping  $\theta_{n+1} : X \rightarrow \mathcal{F}_c(Y)$  by  $\theta_{n+1}(x) = \{f_n(x)\}$  if  $x \in \overline{U_n}$ ,  $\theta_{n+1}(x) = \{g_{n+1}(x)\}$  if  $x \in X_{n+1}$  and  $\theta_{n+1}(x) = \varphi(x)$  if  $x \notin X_{n+1} \cup \overline{U_n}$ . According to the Michael's selection theorem [8, Theorem 3.2''],  $\theta_{n+1}$  admits a continuous selection  $f_{n+1}$ . Obviously,  $V = \{x \in X : f_{n+1}(x) \notin F\}$  is an open subset of  $X$  containing  $X_{n+1} \cup \overline{U_n}$ . Choose the open set  $U_{n+1} \subset X$  such that  $X_{n+1} \cup \overline{U_n} \subset U_{n+1} \subset \overline{U_{n+1}} \subset V$ . This completes the construction.

Finally, by (14) and (15), setting  $f(x) = f_n(x)$  if  $x \in X_n$ ,  $n \in \mathbb{N}$ , we obtain a continuous selection  $f$  of  $\varphi$  avoiding  $F$ .  $\square$

Having already established Theorem 4.1, we can prove that property  $C$  is a local property. The next theorem was proved in [1, Theorem 2.16] under the additional requirement on  $X$  to be hereditarily collectionwise normal. The same proof remains valid in the present situation.

**Theorem 4.2.** *A paracompact space  $X$  has the property  $C$  if and only if every point of  $X$  has a neighbourhood which has property  $C$ .*

To prepare for our next application, we need the following lemma.

**Lemma 4.3.** *Let  $E$  be a Banach space and let  $Y \subset E$  be a convex  $G_\delta$ -subset of  $E$ . If  $H \subset E$  is closed and  $H \cap Y \in \mathcal{Z}_\infty(Y)$ , then*

$$H \cap \overline{Y} \in \mathcal{Z}_\infty(\overline{Y}).$$

*Proof.* It suffices to show that, for a given  $\delta > 0$  and  $f \in C(K, \overline{Y})$ , where  $K$  is a finite-dimensional metrizable compactum, there exists  $g \in C(K, Y)$  such that  $d(f(x), g(x)) \leq \delta$  for each  $x \in K$ . To this end consider the set-valued mapping  $\varphi : K \rightarrow \mathcal{F}_c(Y)$  defined by  $\varphi(x) = \overline{Y \cap B_\delta(f(x))}^Y$ ,  $x \in K$ . Since  $f$  is continuous,  $\varphi$  is l.s.c. Obviously,  $\varphi(x)$ ,  $x \in K$ , are closed convex subsets of  $Y$ . Then, by [5, Corollary 1.2] (or [9, Theorem 1.2 and Example 2.5]),  $\varphi$  admits a continuous selection  $g$ .  $\square$

We are now ready to prove the following result providing a partial answer to the Michael's problem [11, Problem 396].

**Theorem 4.4.** *Let  $X$  be a paracompact  $C$ -space, let  $E$  be a Banach space, and let  $Y$  be a  $G_\delta$ -subset of  $E$ . Then every l.s.c. mapping  $\varphi : X \rightarrow \mathcal{F}_c(Y)$  admits a continuous selection.*

*Proof.* Let  $E \setminus Y = \bigcup \{F_n : n \in \mathbb{N}\}$  with each  $F_n \subset E$  closed. Consider the l.s.c. mapping  $\overline{\varphi} : X \rightarrow \mathcal{F}_c(E)$  defined by  $\overline{\varphi}(x) = \overline{\varphi(x)}$ ,  $x \in X$ . Since, by Lemma 4.3,  $\overline{\varphi}(x) \cap F_n$  is a  $\mathcal{Z}_\infty$ -set in  $\overline{\varphi}(x)$  for all  $x \in X$  and  $n \in \mathbb{N}$ , we can apply (b) of Theorem 1.1 to obtain a continuous selection  $f$  of  $\overline{\varphi}$  avoiding  $\bigcup \{F_n : n \in \mathbb{N}\}$ . Then  $f$  is a selection of  $\varphi$ .  $\square$

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SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, FACULTY OF SCIENCE, UNIVERSITY OF NATAL, KING GEORGE V AVENUE, DURBAN 4041, SOUTH AFRICA

*E-mail address:* `gutev@nu.ac.za`

DEPARTMENT OF MATHEMATICS, NIPISSING UNIVERSITY, 100 COLLEGE DRIVE, P. O. BOX 5002, NORTH BAY, ONTARIO, CANADA P1B 8L7

*E-mail address:* `veskov@unipissing.ca`