

NUMERICAL BOUNDS FOR CRITICAL EXPONENTS OF CROSSING BROWNIAN MOTION

MARIO V. WÜTHRICH

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ABSTRACT. We consider d -dimensional crossing Brownian motion in a truncated Poissonian potential conditioned to reach a fixed hyperplane at distance L from the starting point. The transverse fluctuation of the path is expected to be of order L^ξ . We prove that for $d \geq 2$: $\xi \leq 3/4$. As a second critical exponent we introduce $\chi^{(2)}$, which describes the fluctuations of naturally defined distance functions for crossing Brownian motion. The numerical bound we obtain is an improvement of Corollary 3.1 in *Scaling identity for crossing Brownian motion in a Poissonian potential* (Probab. Theory Related Fields **112** (1998), 299–319), resulting in $\chi^{(2)} \geq 1/5$ if $d = 2$ and if the killing rate λ is strictly positive ($\lambda > 0$).

0. INTRODUCTION AND RESULTS

In this note we continue the work started in [11, 12]. Therefore we try to keep the description of the model as short as possible. Let \mathbb{P} stand for the Poissonian law with fixed intensity $\nu > 0$ on the space Ω of simple pure locally finite point measures on \mathbb{R}^d , $d \geq 2$. For $M > 0$, $\omega = \sum_i \delta_{x_i} \in \Omega$ and $x \in \mathbb{R}^d$, we define the truncated Poissonian potential as

$$(0.1) \quad V(x, \omega) = \left(\sum_i W(x - x_i) \right) \wedge M = \left(\int_{\mathbb{R}^d} W(x - y) \omega(dy) \right) \wedge M,$$

where the shape function $W \geq 0$ is measurable, bounded, compactly supported, not a.e. equal to 0 and *rotationally invariant*. We denote by P_x the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting at $x \in \mathbb{R}^d$, and by Z its canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. For $L > 0$ we define the half-space $\Lambda_L = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d; x_1 \geq L\}$. For $\lambda \geq 0$, $L > 0$ and $\omega \in \Omega$ the new path measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ is then defined by

$$(0.2) \quad d\hat{P}_0^{\partial\Lambda_L} = \frac{1}{e_\lambda(0, \partial\Lambda_L, \omega)} \exp \left\{ - \int_0^{H(\partial\Lambda_L)} (\lambda + V)(Z_s, \omega) ds \right\} dP_0,$$

where $e_\lambda(0, \partial\Lambda_L, \omega)$ is the normalizing constant and $H(\partial\Lambda_L) = \inf\{s \geq 0, Z_s \in \Lambda_L\}$ is the entrance time of $Z(w)$ into the half-space Λ_L .

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As in [12] we define transverse fluctuation as follows: We consider $l_0 = \{(\alpha, 0, \dots, 0) \in \mathbb{R}^d; \alpha \in \mathbb{R}\}$ the first coordinate axis. The truncated cylinder of radius L^γ and symmetry axis l_0 is defined as $C(L, \gamma) = \{z \in \Lambda_{-L^\gamma}; \text{dist}(z, l_0) \leq L^\gamma\}$. $A_0(L, \gamma)$ is the event that the perturbed Brownian path starting at the origin with goal $\partial\Lambda_L$ does **not** leave the cylinder $C(L, \gamma)$, i.e.,

$$(0.3) \quad A_0(L, \gamma) = \{w \in C(\mathbb{R}_+, \mathbb{R}^d); Z_s(w) \in C(L, \gamma) \text{ for all } s \leq H(\partial\Lambda_L)\}.$$

The critical exponent for transverse fluctuation is then defined as follows:

$$(0.4) \quad \xi^{(2)} = \inf \left\{ \gamma \geq 0; \limsup_{L \rightarrow \infty} \mathbb{E} \left[\hat{P}_0^{\partial\Lambda_L} [A_0(L, \gamma)] \right] = 1 \right\}.$$

In [12] formulas (0.10)-(0.12) we have obtained the following lower bounds:

$$(0.5) \quad \xi^{(2)} \geq 1/2 \quad \text{for } d \geq 3 \text{ or } \lambda = 0,$$

$$(0.6) \quad \xi^{(2)} \geq 3/5 \quad \text{for } d = 2 \text{ and } \lambda > 0.$$

Our first new result is an upper bound on $\xi^{(2)}$:

Theorem 0.1. *In all dimensions $d \geq 2$ and for $\lambda \geq 0$ we have*

$$(0.7) \quad \xi^{(2)} \leq 3/4.$$

We remark that Theorem 0.1 is the point-to-plane version of a result obtained in Theorem 1.1 in [10] (point-to-point model). The main improvement here is that we can show both an upper and a lower bound for $\xi^{(2)}$, whereas in the point-to-point model there is no interesting lower bound on the critical exponent for transverse fluctuation.

Analogously to the point-to-plane model we introduce the point-to-point crossing Brownian motion. The normalizing constant $e_\lambda(x, y, \omega)$ plays an important role in our considerations: for $\lambda \geq 0$, $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$ we define

$$(0.8) \quad e_\lambda(x, y, \omega) = E_x \left[\exp \left\{ - \int_0^{H(y)} (\lambda + V)(Z_s, \omega) ds \right\}, H(y) < \infty \right],$$

where $H(y)$ denotes the entrance time of the Brownian motion into the closed ball $\bar{B}(y, 1)$. Symmetrizing the logarithm of the normalizing constant,

$$(0.9) \quad d_\lambda(x, y, \omega) = \max \left\{ - \inf_{\bar{B}(x, 1)} \log e_\lambda(\cdot, y, \omega); - \inf_{\bar{B}(y, 1)} \log e_\lambda(\cdot, x, \omega) \right\},$$

we obtain, \mathbb{P} -a.s., a distance function on \mathbb{R}^d which induces the usual topology (see [8] (1.7)). Sznitman's shape theorem (see [9], Theorem 5.2.5) gives a first result on the asymptotic behaviour of $d_\lambda(0, y, \omega)$ for $|y| \rightarrow \infty$. Our second goal is to get finer asymptotics, therefore we define the critical exponent for distance fluctuations:

$$(0.10) \quad \chi^{(2)} = \inf \left\{ \kappa \geq 0; \lim_{r \rightarrow \infty} \mathbb{P} \left[\sup_{x \in \bar{B}(0, r)} |d_\lambda(0, x, \omega) - M_\lambda(0, x)| \leq r^\kappa \right] = 1 \right\},$$

where the median $M_\lambda(\cdot, \cdot)$ of d_λ is chosen such that it is rotationally and shift invariant (this can be done thanks to our assumptions on the model). In Proposition 0.1 of [11] we have shown that $\chi^{(2)} \leq 1/2$. Here we improve the result obtained in Corollary 3.1 of [11]:

Theorem 0.2. For $d = 2$ and $\lambda > 0$,

$$(0.11) \quad \chi^{(2)} \geq 1/5.$$

To my knowledge there is only one specific related model where one can (at the moment) explicitly calculate ξ and χ in dimension $d = 2$, namely in the model of maximal increasing subsequences on the plane (see Baik-Deift-Johansson [1] and Johansson [4]). Since one is presently not able to generalize their results to other related models for growing interfaces (see Krug-Spohn [5]), it is of great interest to develop similar results for other models. In all these models it is conjectured that if $d = 2$, then $\xi = 2/3$ and $\chi = 1/3$ (see [5]) whereas for higher dimensions there are conflicting predictions (see discussion in Licea-Newman-Piza [6], p. 561). But at least one expects that for $d \geq 3$, $\xi \geq 1/2$. Usually (in the lattice models such as standard first-passage percolation on \mathbb{Z}^d , see Newman-Piza [7] and Licea-Newman-Piza [6]) difficulties arise by the lack of rotational invariance of the model and one is not able to provide both lower and upper bounds for the same exponents (often one can prove an upper bound for one definition of transverse (or distance) fluctuation, but one can only prove a lower bound for a slightly different definition of that exponent). Here we are able to prove both upper and lower bounds for the same definition of the exponents. Recently there has been developed a rotationally invariant version of first-passage percolation (generated by Poissonian clouds, see Howard-Newman [2, 3]) where one should also be able to show results similar to (0.5)-(0.7) and (0.11).

Let us briefly describe the methods we use to prove the results. The main strategy follows that in [7, 11] to obtain upper bounds on $\xi^{(2)}$. We explicitly calculate the costs of the paths performing too large transverse fluctuations. This together with the numerical bounds obtained in [11, 12] implies the results.

1. PROOFS OF THEOREMS 0.1 AND 0.2

Our main goal is to investigate the fluctuations of $d_\lambda(0, y, \cdot)$ for $|y| \rightarrow \infty$. By Sznitman’s shape theorem ([9], Theorem 5.2.5) we know that there exists a deterministic norm $\alpha_\lambda(\cdot)$ on \mathbb{R}^d (which in our case is proportional to Euclidean norm) such that, \mathbb{P} -a.s.,

$$(1.1) \quad \lim_{y \rightarrow \infty} \frac{1}{|y|} |-\log e_\lambda(0, y, \omega) - \alpha_\lambda(y)| = 0.$$

Formula (1.1) also holds if we replace $-\log e_\lambda(0, y, \omega)$ by $d_\lambda(0, y, \omega)$.

We define the following sets: Choose $\gamma \in (0, 1)$ fixed. \mathcal{C}_L is a finite covering of $\partial C(L, \gamma) \setminus \Lambda_L$ if $\mathcal{C}_L \subset \partial C(L, \gamma) \setminus \Lambda_L$, $|\mathcal{C}_L| \leq c_1 L^{d-1}$ and $\bigcup_{z \in \mathcal{C}_L} B(z, 1) \supset \partial C(L, \gamma) \setminus \Lambda_L$. We denote by $z_L = z_L(z) = (-L^\gamma, z_2, \dots, z_d)$ the projection of $z \in \mathcal{C}_L$ onto $\partial \Lambda_{-L^\gamma}$. Analogously we define $\mathcal{B}(x, r)$ to be a finite covering of $\partial B(x, r)$ with $|\mathcal{B}(x, r)| \leq c_2 r^{d-1}$ for all $x \in \mathbb{R}^d$, $r > 1$.

Lemma 1.1. For $d \geq 2$, $\lambda \geq 0$ and $\gamma \in (0, 1)$ there exists $c_3 > 0$ such that with \mathbb{P} -probability to 1 as $L \rightarrow \infty$,

$$(1.2) \quad \hat{P}_0^{\partial \Lambda_L} [A_0(L, \gamma)^c] \leq c_3 L^{2d-1} \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} \exp \{ d_\lambda(0, y_L, \omega) + d_\lambda(z_L, z, \omega) - d_\lambda(0, z, \omega) - d_\lambda(z_L, \bar{y}_L, \omega) \},$$

where $y_L = (L + 1, 0, \dots, 0)$.

Proof of Lemma 1.1. Choose $\gamma \in (0, 1)$ fixed. Using the strong Markov property (see [9], formula (5.2.6)) we find for $L > 1$, $z \in \mathcal{C}_L$,

$$(1.3) \quad e_\lambda(z_L, \partial\Lambda_L, \omega) \geq e_\lambda(z_L, z, \omega) \inf_{\bar{B}(z, 1)} e_\lambda(\cdot, \partial\Lambda_L, \omega).$$

Using $H(y_L) \geq H(\partial\Lambda_L)$, P_0 -a.s., we have for $L > 0$,

$$(1.4) \quad \begin{aligned} e_\lambda(0, \partial\Lambda_L, \omega) &= E_0 \left[\exp \left\{ - \int_0^{H(\partial\Lambda_L)} (\lambda + V)(Z_s, \omega) ds \right\} \right] \\ &\geq E_0 \left[\exp \left\{ - \int_0^{H(y_L)} (\lambda + V)(Z_s, \omega) ds \right\}, H(y_L) < \infty \right] \\ &= e_\lambda(0, y_L, \omega). \end{aligned}$$

For $z \in \mathcal{C}_L$ we define

$$H_L^\gamma(z_L(z)) = \inf\{s \geq 0 : Z_s \in \partial B(z_L(z), L + L^\gamma)\}.$$

Then we have for $L > 1$ (using $H(\partial\Lambda_L) \geq H_L^\gamma(z_L)$, P_{z_L} -a.s., and $\text{dist}(z_L(z), \partial\Lambda_L) = L + L^\gamma \leq 2L$):

$$(1.5) \quad \begin{aligned} e_\lambda(z_L, \partial\Lambda_L, \omega) &= E_{z_L} \left[\exp \left\{ - \int_0^{H(\partial\Lambda_L)} (\lambda + V)(Z_s, \omega) ds \right\} \right] \\ &\leq E_{z_L} \left[\exp \left\{ - \int_0^{H_L^\gamma(z_L)} (\lambda + V)(Z_s, \omega) ds \right\} \right] \\ &\leq \sum_{\bar{y}_L \in \mathcal{B}(z_L, L + L^\gamma)} E_{z_L} \left[\exp \left\{ - \int_0^{H(\bar{y}_L)} (\lambda + V)(Z_s, \omega) ds \right\}, H(\bar{y}_L) \leq H_L^\gamma(z_L) < \infty \right] \\ &\leq c_2 2^{d-1} L^{d-1} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L + L^\gamma)} e_\lambda(z_L, \bar{y}_L, \omega). \end{aligned}$$

If $\text{dist}(z, \partial\Lambda_L) > 4$ we use the point-to-plane version of Harnack's inequality (see (1.2) of [11]), resp., if $\text{dist}(z, \partial\Lambda_L) \leq 4$ we simply use that $e_\lambda(z, \partial\Lambda_L, \omega)$ can be bounded from above and below by positive constants uniformly in ω (see (1.1) of [11]) to obtain: there exists $c_4 > 0$ such that for all $z \in \mathbb{R}^d$, $L > 0$, $\omega \in \Omega$,

$$(1.6) \quad \frac{\sup_{B(z, 1)} e_\lambda(\cdot, \partial\Lambda_L, \omega)}{\inf_{B(z, 1)} e_\lambda(\cdot, \partial\Lambda_L, \omega)} \leq c_4.$$

Hence using the strong Markov property, we obtain as in (2.12) of [10] for $L > 1$

$$\begin{aligned}
 (1.7) \quad & \hat{P}_0^{\partial\Lambda_L}[A_0(L, \gamma)^c] \leq \sum_{z \in \mathcal{C}_L} \hat{P}_0^{\partial\Lambda_L}[H(z) \leq H(\partial\Lambda_L)] \\
 & = \sum_{z \in \mathcal{C}_L} \frac{1}{e_\lambda(0, \partial\Lambda_L, \omega)} E_0 \left[\exp \left\{ - \int_0^{H(\partial\Lambda_L)} (\lambda + V)(Z_s, \omega) ds \right\}, \right. \\
 & \qquad \qquad \qquad \left. H(z) \leq H(\partial\Lambda_L) \right] \\
 & \leq \frac{1}{e_\lambda(0, \partial\Lambda_L, \omega)} \sum_{z \in \mathcal{C}_L} e_\lambda(0, z, \omega) \sup_{B(z, 1)} e_\lambda(\cdot, \partial\Lambda_L, \omega) \\
 (1.4), (1.6) \quad & \leq \frac{c_4}{e_\lambda(0, y_L, \omega)} \sum_{z \in \mathcal{C}_L} e_\lambda(0, z, \omega) \inf_{B(z, 1)} e_\lambda(\cdot, \partial\Lambda_L, \omega) \\
 (1.3) \quad & \leq \frac{c_4}{e_\lambda(0, y_L, \omega)} \sum_{z \in \mathcal{C}_L} e_\lambda(0, z, \omega) \frac{e_\lambda(z_L, \partial\Lambda_L, \omega)}{e_\lambda(z_L, z, \omega)} \\
 (1.5) \quad & \leq c_3 L^{2(d-1)} \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} \frac{e_\lambda(0, z, \omega) e_\lambda(z_L, \bar{y}_L, \omega)}{e_\lambda(0, y_L, \omega) e_\lambda(z_L, z, \omega)},
 \end{aligned}$$

where $c_3 = c_1 c_4 c_2 2^{d-1}$. So there remains to prove that

$$(1.8) \quad \lim_{L \rightarrow \infty} \mathbb{P} \left[\sup_{x, y \in B(0, 4L)} |-\log e_\lambda(x, y, \omega) - d_\lambda(x, y, \omega)| \leq (\log L)/4 \right] = 1.$$

For $d \geq 3$ or $\lambda > 0$ claim (1.8) follows from (1.1) and (1.4) in [11], which states that $|-\log e_\lambda(x, y, \omega) - d_\lambda(x, y, \omega)|$ can be bounded uniformly in $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$ by a constant $c_5 = c_5(d, \lambda)$. For $d = 2$ and $\lambda = 0$ we use in addition Lemma 1.1 of [11], which states that $\sup_{x, y \in B(0, 4L)} |-\log e_\lambda(x, y, \omega) - d_\lambda(x, y, \omega)|$ is of order $o(\log L)$ with \mathbb{P} -probability to one as $L \rightarrow \infty$. This finishes the proof of Lemma 1.1. \square

Our second lemma is a purely geometric one. It calculates the costs for a detour via the boundary.

Lemma 1.2. *For $d \geq 2$, $\lambda \geq 0$, $\kappa > \chi^{(2)}$ and $\gamma \in (0, 1)$: There exist positive constants c_6, c_7 such that for all large L , $y_L = (L+1, 0, \dots, 0)$, $z \in \mathcal{C}_L$, $z_L = z_L(z)$, and $\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)$,*

$$(1.9) \quad \alpha_\lambda(y_L) + \alpha_\lambda(z - z_L) - \alpha_\lambda(z) - \alpha_\lambda(\bar{y}_L - z_L) \leq -c_6 L^{2\gamma-1},$$

$$(1.10) \quad M_\lambda(0, y_L) + M_\lambda(z_L, z) - M_\lambda(0, z) - M_\lambda(z_L, \bar{y}_L) \leq -c_6 L^{2\gamma-1} + c_7 L^\kappa.$$

Proof of Lemma 1.2. First we prove (1.9). Since $\alpha_\lambda(\cdot)$ is proportional to the Euclidean norm it suffices to prove relation (1.9) for the Euclidean norm. If $z = z_L$, then

$$\begin{aligned}
 (1.11) \quad & |y_L| + |z - z_L| - |z| - |\bar{y}_L - z_L| = |y_L| - |z_L| - |\bar{y}_L - z_L| \\
 & \leq L + 1 - L^\gamma - (L + L^\gamma) = -2L^\gamma + 1.
 \end{aligned}$$

But in this case the claim follows because $\gamma < 1$. So there remains to consider the case $z \neq z_L$. For simplicity we define $\hat{y}_L = \partial B(z_L, L + L^\gamma) \cap \partial \Lambda_L$. Then

$$\begin{aligned}
 (1.12) \quad |y_L| + |z - z_L| - |z| - |\bar{y}_L - z_L| &= |y_L| + |z - z_L| - |z| - |z - z_L| - |\hat{y}_L - z| \\
 &= L + 1 - |z| - |\hat{y}_L - z| \leq L + 1 - |\hat{y}_L| \\
 &= 1 + L - \sqrt{L^2 + L^{2\gamma}} = 1 + L \left(1 - \sqrt{1 + L^{2\gamma-2}}\right) \\
 &= 1 + \frac{-L^{2\gamma-1}}{1 + \sqrt{1 + L^{2\gamma-2}}}.
 \end{aligned}$$

But then the claim of (1.9) follows for all large L , since $\gamma < 1$.

The proof of (1.10) is similar to the proof of (1.9), but here we have (in addition) to use formulas (2.7), (2.11), (2.20) and (1.1) of [11]. For the reader's convenience we prove the claim for $z = z_L$ ($z \neq z_L$ goes analogously). Choose $\kappa > \chi^{(2)}$. If $z = z_L$, then $d_\lambda(z_L, z, \omega) = 0$; hence $M_\lambda(z_L, z) = 0$. Therefore we obtain for large L (using Lemmas 2.1, 2.2 of [11] and the rotational and shift invariance of M_λ):

$$\begin{aligned}
 (1.13) \quad M_\lambda(0, y_L) + M_\lambda(z_L, z) - M_\lambda(0, z) - M_\lambda(z_L, \bar{y}_L) \\
 &= M_\lambda(0, y_L) - M_\lambda(0, z_L) - M_\lambda(z_L, \bar{y}_L) \\
 &\leq M_\lambda(0, y_L) - M_\lambda(z_L, \bar{y}_L) \\
 &= M_\lambda(0, y_L) - M_\lambda\left(0, \frac{L + L^\gamma}{L + 1} y_L\right) \\
 &\stackrel{(2.7) \text{ of [11]}}{\leq} -M_\lambda\left(y_L, \frac{L + L^\gamma}{L + 1} y_L\right) + c_8 L^\kappa \\
 &\stackrel{(2.20) \text{ of [11]}}{\leq} -c_9 \left(\frac{L + L^\gamma}{L + 1} - 1\right) |y_L| + c_8 L^\kappa = -c_9(L^\gamma - 1) + c_8 L^\kappa.
 \end{aligned}$$

But now the claim follows as in (1.11). This finishes the proof of Lemma 1.2. \square

Theorem 0.2 is an easy consequence of (0.6) and the following proposition:

Proposition 1.3. *For $d = 2$ and $\lambda \geq 0$ we have*

$$(1.14) \quad \chi^{(2)} \geq 2\xi^{(2)} - 1.$$

We remark that Proposition 1.3 is the point-to-plane version of Theorem 0.2 in [11].

Proof of Proposition 1.3. We already know that $\chi^{(2)} \leq 1/2$, hence we choose κ and γ such that

$$(1.15) \quad \frac{\chi^{(2)} + 1}{2} < \frac{\kappa + 1}{2} < \gamma < 1.$$

We want to prove that for all these γ 's

$$(1.16) \quad \mathbb{E} \left[\hat{P}_0^{\partial \Lambda_L} [A_0(L, \gamma)] \right] \rightarrow 1 \text{ as } L \rightarrow \infty,$$

hence $\gamma \geq \xi^{(2)}$, from which the claim of the proposition follows. To prove this we apply a version of Lemma 1.1 where this time $z_L = z_L(z)$ is defined to be

$z_L = (-L^\gamma, -L^\gamma)$ if $z \in \mathcal{C}_L$ has negative second coordinate and $z_L = (-L^\gamma, L^\gamma)$ otherwise. Then on a set $\Omega_L^1 \subset \Omega$ with $\mathbb{P}[\Omega_L^1] \rightarrow 1$ as $L \rightarrow \infty$ we obtain

$$(1.17) \quad \hat{P}_0^{\partial\Lambda_L} [A_0(L, \gamma)^c] \leq c_3 L^3 \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} \exp \{ d_\lambda(0, y_L, \omega) + d_\lambda(z_L, z, \omega) - d_\lambda(0, z, \omega) - d_\lambda(z_L, \bar{y}_L, \omega) \}.$$

For $\kappa > \chi^{(2)}$ we find a set $\Omega_L^2 \subset \Omega$ with $\mathbb{P}[\Omega_L^2] \rightarrow 1$ as $L \rightarrow \infty$ on which for all $z \in \mathcal{C}_L$ and $\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)$ we have

$$(1.18) \quad \begin{aligned} d_\lambda(0, y_L, \omega) - M_\lambda(0, y_L) &\leq L^\kappa, & d_\lambda(z_L, z, \omega) - M_\lambda(z_L, z) &\leq L^\kappa, \\ d_\lambda(0, z, \omega) - M_\lambda(0, z) &\geq -L^\kappa, & d_\lambda(z_L, \bar{y}_L, \omega) - M_\lambda(z_L, \bar{y}_L) &\geq -L^\kappa. \end{aligned}$$

Hence with \mathbb{P} -probability to 1 as $L \rightarrow \infty$

$$(1.19) \quad \hat{P}_0^{\partial\Lambda_L} [A_0(L, \gamma)^c] \leq c_3 L^3 \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} \exp \{ M_\lambda(0, y_L) + M_\lambda(z_L, z) - M_\lambda(0, z) - M_\lambda(z_L, \bar{y}_L) + 4L^\kappa \}.$$

With Lemma 1.2, the remark that for $z \in \mathcal{C}_L \cap \partial\Lambda_{-L^\gamma}$ $|z - z_L| \leq L^\gamma \leq |z|$ and (1.15) we finish the proof of Proposition 1.3. \square

Proof of Theorem 0.1. Choose $\gamma \in (3/4, 1)$. We claim that

$$(1.20) \quad \mathbb{E} \left[\hat{P}_0^{\partial\Lambda_L} [A_0(L, \gamma)] \right] \rightarrow 1 \text{ as } L \rightarrow \infty,$$

which implies that $\xi^{(2)} \leq 3/4$. The proof of Theorem 0.1 is similar to the proof of Proposition 1.3: Using Lemma 1.1 we obtain on a set $\Omega_L^1 \subset \Omega$ with $\mathbb{P}[\Omega_L^1] \rightarrow 1$ as $L \rightarrow \infty$: for all large L

$$(1.21) \quad \begin{aligned} \hat{P}_0^{\partial\Lambda_L} [A_0(L, \gamma)^c] &\leq c_3 L^{2d-1} \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} \exp \{ d_\lambda(0, y_L, \omega) + d_\lambda(z_L, z, \omega) \\ &\quad - d_\lambda(0, z, \omega) - d_\lambda(z_L, \bar{y}_L, \omega) \} \\ &\stackrel{(1.22)}{\leq} c_3 L^{2d-1} \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} \exp \{ \alpha_\lambda(y_L) + \alpha_\lambda(z - z_L) - \alpha_\lambda(z) \\ &\quad - \alpha_\lambda(\bar{y}_L - z_L) + F_{\lambda, \omega}(y_L, z_L, z, \bar{y}_L) \} \\ &\stackrel{(1.9)}{\leq} c_3 L^{2d-1} \exp \left\{ -c_6 L^{2\gamma-1} + \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} F_{\lambda, \omega}(y_L, z_L, z, \bar{y}_L) \right\}, \end{aligned}$$

where

$$(1.22) \quad \begin{aligned} F_{\lambda, \omega}(y_L, z_L, z, \bar{y}_L) &= |d_\lambda(0, y_L, \omega) - \alpha_\lambda(y_L)| + |d_\lambda(z_L, z, \omega) - \alpha_\lambda(z - z_L)| \\ &\quad + |d_\lambda(0, z, \omega) - \alpha_\lambda(z)| + |d_\lambda(z_L, \bar{y}_L, \omega) - \alpha_\lambda(\bar{y}_L - z_L)|. \end{aligned}$$

Once we have shown Lemma 1.4 below, we know that there exists a set $\Omega_L^3 \subset \Omega$ with $\mathbb{P}[\Omega_L^3] \rightarrow 1$ as $L \rightarrow \infty$ such that on Ω_L^3 we have that $F_{\lambda, \omega}(y_L, z_L, z, \bar{y}_L) \leq 4L^{1/2} \log^3 L$. But then the claim of Theorem 0.1 follows since $2\gamma - 1 > 1/2$. \square

For $L > 0$ we define the following subsets of Ω :

$$\begin{aligned} A_1(L) &= \left\{ \omega; \sup_{z \in \mathcal{C}_L} |d_\lambda(z_L, z, \omega) - \alpha_\lambda(z - z_L)| \leq L^{1/2} \log^3 L \right\}, \\ A_2(L) &= \left\{ \omega; \sup_{z \in \mathcal{C}_L} \sup_{\bar{y}_L \in \mathcal{B}(z_L, L+L^\gamma)} |d_\lambda(z_L, \bar{y}_L, \omega) - \alpha_\lambda(\bar{y}_L - z_L)| \leq L^{1/2} \log^3 L \right\}, \\ A_3(L) &= \left\{ \omega; \sup_{z \in \mathcal{C}_L} |d_\lambda(0, z, \omega) - \alpha_\lambda(z)| \leq L^{1/2} \log^3 L \right\}, \\ A_4(L) &= \left\{ \omega; |d_\lambda(0, y_L, \omega) - \alpha_\lambda(y_L)| \leq L^{1/2} \log^3 L \right\}. \end{aligned}$$

Lemma 1.4. *Choose $\gamma \in (3/4, 1)$; then we have*

$$(1.23) \quad \lim_{L \rightarrow \infty} \mathbb{P}[A_1 \cap A_2 \cap A_3 \cap A_4] = 1.$$

Proof of Lemma 1.4. It suffices to show that $\lim_{L \rightarrow \infty} \mathbb{P}[A_i] = 1$ for all $i = 1, \dots, 4$. We have

$$(1.24) \quad \mathbb{P}[A_1(L)^c] \leq \sum_{z \in \mathcal{C}_L} \mathbb{P}\left[|d_\lambda(z_L, z, \omega) - \alpha_\lambda(z - z_L)| > L^{1/2} \log^3 L\right].$$

Using Theorems 2.1, 2.5 and Corollary 3.4 of [8] we see that for $z \in \mathcal{C}_L$ (L large)

$$(1.25) \quad \begin{aligned} &\mathbb{P}\left[|d_\lambda(z_L, z, \omega) - \alpha_\lambda(z - z_L)| > L^{1/2} \log^3 L\right] \\ &\leq \mathbb{P}\left[|d_\lambda(z_L, z, \omega) - \mathbb{E}[d_\lambda(z_L, z)]| + |\mathbb{E}[d_\lambda(z_L, z)] - \alpha_\lambda(z - z_L)| > L^{1/2} \log^3 L\right] \\ &\leq \mathbb{P}\left[|d_\lambda(z_L, z, \omega) - \mathbb{E}[d_\lambda(z_L, z)]| > L^{1/2} \log^3 L - c_{10} L^{1/2} \log^2 L\right] \\ &\leq c_{11} \exp\{-c_{12} \log^2 L\}. \end{aligned}$$

But since $|\mathcal{C}_L| \leq c_1 L^{d-1}$ we obtain

$$(1.26) \quad \mathbb{P}[A_1(L)^c] \leq c_1 c_{11} \exp\{(d-1) \log L - c_{12} \log^2 L\},$$

which converges to 0 as $L \rightarrow \infty$. The proof for $i = 2, 3, 4$ goes analogously (for $i = 2$ one has also to take into account that $|\mathcal{B}(0, L + L^\gamma)| \leq c_2 2^{d-1} L^{d-1}$ for $L > 1$). \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NIJMEGEN, TOERNOOIVELD 1, NL-6525 ED NIJMEGEN, THE NETHERLANDS

E-mail address: `mario.wuethrich@winterthur.ch`

Current address: Winterthur Insurance, Roemerstrasse 17, P.O. Box 357, CH-8401 Winterthur, Switzerland