

STABILITY OF DISJOINTNESS PRESERVING MAPPINGS

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ABSTRACT. Let X and Y be compact Hausdorff spaces and let $\varepsilon \geq 0$. A linear mapping $\Phi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is called ε -disjointness preserving if $fg = 0$ implies that $\|\Phi(f)\Phi(g)\| \leq \varepsilon\|f\|\|g\|$. If $\Phi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is a continuous or surjective ε -disjointness preserving linear mapping, we prove that there exists a disjointness preserving linear mapping $\Psi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ satisfying $\|\Phi(f) - \Psi(f)\| \leq 20\sqrt{\varepsilon}\|f\|$. We also prove that every unbounded ε -disjointness preserving linear functional on $\mathcal{C}(X)$ is disjointness preserving.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In the last few years stability problems for homomorphisms and isometries have attracted the attention of many mathematicians. Among the deepest achievements in this area are Johnson's results on linear maps between Banach algebras that are approximately multiplicative [4], [5] (see also [3]). Specializing to the case of function algebras, we treat the more general problem of stability of linear mappings that preserve only zero products. Following the terminology from Banach lattices we say that a linear mapping Φ between $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ is disjointness preserving if $fg = 0$ implies that $\Phi(f)\Phi(g) = 0$, where $f, g \in \mathcal{C}(X)$. Here and throughout the paper $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are algebras of \mathbb{F} -valued continuous functions defined on compact Hausdorff spaces X and Y , respectively, and \mathbb{F} is either the field of complex numbers \mathbb{C} or the field of real numbers \mathbb{R} . Suppose a mapping Φ satisfies the disjointness preserving property only approximately. Our purpose is to determine whether there exists a disjointness preserving mapping close to it.

Let $\varepsilon \geq 0$. A linear mapping $\Phi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is called ε -disjointness preserving if $\|\Phi(f)\Phi(g)\| \leq \varepsilon\|f\|\|g\|$ for any $f, g \in \mathcal{C}(X)$ satisfying $fg = 0$.

In our first theorem we show that the disjointness preserving property is stable for continuous linear functionals on $\mathcal{C}(X)$.

Theorem 1. *Let X be a compact Hausdorff space and $\varepsilon \geq 0$. Then for every continuous ε -disjointness preserving linear functional Φ on $\mathcal{C}(X)$ there exists a continuous disjointness preserving linear functional Ψ on $\mathcal{C}(X)$ satisfying*

$$\|\Phi(f) - \Psi(f)\| \leq 3\sqrt{\varepsilon}\|f\| \quad \text{for all } f \in \mathcal{C}(X).$$

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If Φ is a linear mapping from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$, then for every $y \in Y$ we define a linear functional $\Phi_y = \delta_y \circ \Phi$, where δ_y is evaluation at the point y . We will use the fact that Φ_y is ε -disjointness preserving if Φ is ε -disjointness preserving to obtain the following extension of Theorem 1.

Corollary. *Let X and Y be compact Hausdorff spaces and $\varepsilon \geq 0$. Then for every continuous ε -disjointness preserving linear mapping $\Phi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ there exists a continuous disjointness preserving linear mapping $\Psi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ satisfying*

$$\|\Phi(f) - \Psi(f)\| \leq 20\sqrt{\varepsilon}\|f\| \quad \text{for all } f \in \mathcal{C}(X).$$

For unbounded ε -disjointness preserving linear functionals we have a superstability phenomenon.

Theorem 2. *Let X be a compact Hausdorff space and Φ an unbounded linear functional on $\mathcal{C}(X)$. If there exists an $\varepsilon \geq 0$ such that Φ is ε -disjointness preserving, then Φ is disjointness preserving.*

Having this result, it is tempting to conjecture that the same superstability phenomenon occurs also for unbounded ε -disjointness preserving mappings between $\mathcal{C}(X)$ and $\mathcal{C}(Y)$. The following example shows that this is not the case. So, we face a completely different situation compared to Theorem 1 and its Corollary.

Example. Let $X = \beta\mathbb{N}$ be the Stone-Ćech compactification of the set of all positive integers and let $Y = [0, 1]$ with the usual topology. Further, let G be the linear functional on $\mathcal{C}(X)$, defined by $G(f) = f(1) + f(2)$, and F any unbounded disjointness preserving linear functional on $\mathcal{C}(X)$ (the existence of such functionals will be proved in Lemma 3 in the next section). Then the linear mapping $\Phi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ given by

$$\Phi(f)(y) = \begin{cases} \sqrt{\varepsilon}(1 - 2y)G(f) & \text{if } 0 \leq y \leq \frac{1}{2}, \\ (2y - 1)F(f) & \text{if } \frac{1}{2} < y \leq 1, \end{cases}$$

is unbounded and ε -disjointness preserving (when checking this property one has to use the fact that $|G(f)G(g)| \leq \|f\|\|g\|$ whenever $fg = 0$) but it is not disjointness preserving.

In particular, it follows from Theorem 1 and Theorem 2 that the ε -disjointness property is stable for all linear functionals on $\mathcal{C}(X)$. Is this true also for linear mappings between $\mathcal{C}(X)$ and $\mathcal{C}(Y)$? More precisely, can we omit the continuity assumption in the Corollary? Our last theorem gives an affirmative answer under the additional assumption of surjectivity, which is automatically satisfied in the case of nonzero functionals.

Theorem 3. *Let X and Y be compact Hausdorff spaces and $\varepsilon \geq 0$. Then for every surjective ε -disjointness preserving linear mapping $\Phi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ there exists a disjointness preserving linear mapping $\Psi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ satisfying*

$$\|\Phi(f) - \Psi(f)\| \leq 20\sqrt{\varepsilon}\|f\| \quad \text{for all } f \in \mathcal{C}(X).$$

2. PROOFS

In order to prove Theorem 1 we will need the following lemma. For the sake of completeness we will present its proof which goes through in the same way as the proof of Theorem 4.1 in [4].

Lemma 1. *Let X be a compact Hausdorff space and $\varepsilon \geq 0$. If Φ is a positive continuous ε -disjointness preserving linear functional on $\mathcal{C}(X)$ satisfying $\|\Phi\| > 3\sqrt{\varepsilon}$, and if λ is a positive measure which represents Φ , then there exists a unique $x_0 \in X$ such that $\lambda(\{x_0\}) \geq \|\Phi\| - \frac{2\sqrt{\varepsilon}}{3}$ and*

$$\|\|\Phi\|\delta_{x_0} - \Phi\| \leq \frac{4}{3}\sqrt{\varepsilon}.$$

Proof of Lemma 1. Let us denote

$$\Phi_1 = \Phi / \|\Phi\|.$$

Since Φ_1 is a positive linear functional, there exists by the Riesz theorem a unique positive regular measure μ such that

$$\Phi_1(f) = \int_X f d\mu \quad \text{for all } f \in \mathcal{C}(X),$$

and $\mu(X) = \Phi_1(1_X) = \|\Phi_1\| = 1$. So μ is a probability measure. Let

$$\alpha = \sup\{\mu(\{x\}) : x \in X\}.$$

First, we will show that $\alpha \geq \frac{1}{3}$. Suppose, contrary to our claim, that $\alpha < \frac{1}{3}$. Since μ is a regular measure, there exists for every $x \in X$ an open set N_x containing x such that $\mu(N_x) < \frac{1}{3}$. The family $\{N_x : x \in X\}$ is a covering of the compact space X , so

there exists a finite subcovering $\{N_1, \dots, N_k\}$. Let us define $G_0 = \emptyset$, $G_i = \bigcup_{j=1}^i N_j$,

$i = 1, \dots, k$. Then $\emptyset = G_0 \subset G_1 \subset \dots \subset G_k = X$ and $\mu(G_i) - \mu(G_{i-1}) = \mu(G_i \setminus G_{i-1}) \leq \mu(N_i) < \frac{1}{3}$, so there exists an i_0 such that $\frac{1}{3} < \mu(G_{i_0}) < \frac{2}{3}$. Because the measure μ is regular, there exists a compact set $F_1 \subset G_{i_0}$ with $\mu(F_1) > \frac{1}{3}$. The compact set $F_2 = G_{i_0}^C$ also has the property that $\mu(F_2) = \mu(X) - \mu(G_{i_0}) > \frac{1}{3}$. Since F_1 and F_2 are disjoint closed sets and X is a compact Hausdorff space, there exist disjoint open sets $U_1 \supset F_1$, $U_2 \supset F_2$ and, by Urysohn's lemma, functions $h_1, h_2 \in \mathcal{C}(X)$ such that $0 \leq h_i \leq 1_X$, $h_i|_{U_i^c} \equiv 0$ and $h_i|_{F_i} \equiv 1$, $i = 1, 2$, so $h_1 h_2 = 0$. It follows on one side that

$$\Phi_1(h_1)\Phi_1(h_2) \leq \frac{\varepsilon}{\|\Phi\|^2} \|h_1\| \|h_2\| < \frac{1}{9},$$

since $\|\Phi\| > 3\sqrt{\varepsilon}$, but on the other side

$$\Phi_1(h_1)\Phi_1(h_2) \geq \mu(F_1)\mu(F_2) > \frac{1}{9},$$

which is a contradiction. So $\alpha \geq \frac{1}{3}$.

There exists an $x_0 \in X$ such that $\mu(\{x_0\}) = \alpha$. If not, there would be infinitely many $x \in X$ with $\mu(\{x\}) > c$ for any constant $0 < c < \frac{1}{3}$, which is not possible since $\mu(X) = 1$.

Let $\alpha < \alpha' < 1$. Then, because of the regularity of the measure μ , there exists an open neighborhood N_{x_0} of x_0 satisfying $\mu(N_{x_0}) < \alpha'$. The space X is compact Hausdorff, so there exist disjoint open sets $V_1 \supset \{x_0\}$, $V_2 \supset N_{x_0}^C$ and, by Urysohn's lemma, functions $h_1, h_2 \in \mathcal{C}(X)$ such that $0 \leq h_i \leq 1_X$, $h_i|_{V_i^c} \equiv 0$, $i = 1, 2$, and $h_1(x_0) = 1$, $h_2|_{N_{x_0}^C} \equiv 1$, so $h_1 h_2 = 0$. Then

$$\alpha(1 - \alpha') < \mu(\{x_0\})\mu(N_{x_0}^C) \leq \Phi_1(h_1)\Phi_1(h_2) \leq \frac{\varepsilon}{\|\Phi\|^2} \|h_1\| \|h_2\|$$

for all $\alpha' > \alpha$, and hence $\alpha(1 - \alpha) \leq \frac{\varepsilon}{\|\Phi\|^2}$. Let $\delta = \frac{\varepsilon}{\|\Phi\|^2}$. Because

$$\frac{1}{9} > \delta \geq \alpha(1 - \alpha) = \frac{1}{4} - \left(\alpha - \frac{1}{2}\right)^2$$

and $\alpha \geq \frac{1}{3}$, it follows that actually $\alpha > \frac{1}{2} + \frac{\sqrt{5}}{6}$, so

$$\alpha \geq \frac{1}{2} + \sqrt{\frac{1}{4} - \delta}.$$

And since $\frac{1}{4} - \delta \geq (\frac{1}{2} - 2\delta)^2$, we obtain that

$$\alpha \geq 1 - 2\delta.$$

Let $f \in \mathcal{C}(X)$ be such that $\|f\| \leq 1$. Then

$$\begin{aligned} |\delta_{x_0}(f) - \Phi_1(f)| &= \left| \int_X f(x_0) d\mu - \int_X f d\mu \right| = \left| \int_{X \setminus \{x_0\}} (f(x_0) - f) d\mu \right| \\ &\leq 2(1 - \alpha) \leq 4\delta \end{aligned}$$

and

$$(1) \quad \|\|\Phi\|\delta_{x_0} - \Phi\| \leq 4 \frac{\varepsilon}{\|\Phi\|} \leq \frac{4}{3} \sqrt{\varepsilon}.$$

If $\alpha = 1$, then $\delta_{x_0} = \Phi_1$, so $\|\Phi\|\delta_{x_0} = \Phi$ and (1) is true also in this case.

Since $\alpha > \frac{1}{2}$ and μ is a probability measure, there is only one $x_0 \in X$ such that $\mu(\{x_0\}) = \alpha$. Also, because $\lambda = \|\Phi\|\mu$, we have $\lambda(\{x_0\}) \geq \|\Phi\| - 2\|\Phi\|\delta \geq \|\Phi\| - \frac{2\sqrt{\varepsilon}}{3}$. \square

Proof of Theorem 1. Let Φ be a nontrivial bounded functional. Then there exists a positive linear functional $|\Phi|$ on $\mathcal{C}(X)$ called the modulus of Φ and defined by

$$|\Phi|(f) = \sup\{|\Phi(u)| : |u| \leq f\}$$

for every nonnegative real $f \in \mathcal{C}(X)$ (see the proof of Theorem 6.19 in [6] and also [7]). By the Riesz theorem there exists a unique measure μ such that

$$(2) \quad \Phi(f) = \int_X f d\mu \quad \text{for all } f \in \mathcal{C}(X),$$

and, by the argument given in the proof of Theorem 6.19 in [6], it follows that also

$$(3) \quad |\Phi|(f) = \int_X f \bar{w} d\mu = \int_X f d|\mu| \quad \text{for all } f \in \mathcal{C}(X),$$

where $w \in \mathcal{L}^1(|\mu|)$ with $|w| = 1$ a.e. $[|\mu|]$.

Let us prove that $|\Phi|$ is an ε -disjointness preserving functional if Φ is ε -disjointness preserving. If $f, g \geq 0$ and $fg = 0$, then note that for any $u, v \in \mathcal{C}(X)$ satisfying $|u| \leq f$, $|v| \leq g$ also $uv = 0$, and so

$$\begin{aligned} |\Phi|(f)|\Phi|(g) &= \sup\{|\Phi(u)| : |u| \leq f\} \sup\{|\Phi(v)| : |v| \leq g\} \\ &= \sup\{|\Phi(u)\Phi(v)| : |u| \leq f, |v| \leq g\} \leq \varepsilon \|f\| \|g\|. \end{aligned}$$

Since

$$|\Phi|(f) = \left| \int_X f d|\mu| \right| \leq \int_X |f| d|\mu| = |\Phi|(|f|)$$

for every $f \in \mathcal{C}(X)$, it follows that

$$|\Phi(f)|\Phi(g) \leq |\Phi(|f|)|\Phi(|g|) \leq \varepsilon\|f\|\|g\|$$

if f and g are continuous functions with $fg = 0$.

Since $|\Phi|$ is a positive linear functional, we have

$$(4) \quad \|\Phi\| = |\Phi|(1_X) = \sup\{|\Phi(u)| : |u| \leq 1_X\} = \|\Phi\|.$$

If $\|\Phi\| \leq 3\sqrt{\varepsilon}$, then Φ is near the zero functional.

It remains to consider the case $\|\Phi\| > 3\sqrt{\varepsilon}$. This will be done in the following lemma, which will also be used later on. \square

Lemma 2. *Let Φ be a continuous ε -disjointness preserving linear functional on $\mathcal{C}(X)$ and suppose $\|\Phi\| > 3\sqrt{\varepsilon}$. Then there exists a unique $x_0 \in X$ such that*

$$(5) \quad \|\Phi - w(x_0)\|\Phi\|\delta_{x_0}\| \leq 3\sqrt{\varepsilon},$$

where $|w(x_0)| = 1$.

Proof of Lemma 2. Since $|\Phi|$ is a positive continuous ε -disjointness preserving linear functional, it follows by Lemma 1 and (4) that there exists a unique $x_0 \in X$ such that

$$(6) \quad \|\|\Phi\|\delta_{x_0} - |\Phi|\| \leq \frac{4}{3}\sqrt{\varepsilon}.$$

Let us prove that $w(x_0)\|\Phi\|\delta_{x_0}$ is near the linear functional Φ . It follows by (2), (3), (6) and Lemma 1 that

$$\begin{aligned} |\Phi(f) - w(x_0)\|\Phi\|f(x_0)| &\leq |\Phi(f) - w(x_0)|\Phi(f)| \\ &\quad + |w(x_0)|\Phi(f) - w(x_0)\|\Phi\|f(x_0)| \\ &\leq \left| \int_X f(w - w(x_0))d|\mu| \right| + \frac{4}{3}\sqrt{\varepsilon} \\ &\leq 2\|f\| \int_{X \setminus \{x_0\}} d|\mu| + \frac{4}{3}\sqrt{\varepsilon} \\ &\leq \frac{8}{3}\sqrt{\varepsilon} \end{aligned}$$

for $f \in \mathcal{C}(X)$ satisfying $\|f\| \leq 1$, and so the proof is completed. \square

Proof of the Corollary. Let $\varepsilon > 0$ and let Φ be a continuous ε -disjointness preserving linear mapping from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$. Clearly, for every $y \in Y$ the linear functional Φ_y is ε -disjointness preserving. For any $K \geq 0$ we define the set $Y_K = \{y \in Y : \|\Phi_y\| > K\}$. Take any $y \in Y_K$. There exists a function $f \in \mathcal{C}(X)$, $\|f\| \leq 1$, satisfying $|\Phi(f)(y)| = |\Phi_y(f)| > K$ and, since $\Phi(f) \in \mathcal{C}(Y)$, a neighborhood U of y such that $|\Phi(f)(z)| > K$ for all $z \in U$. It follows that $\|\Phi_z\| \geq |\Phi(f)(z)| > K$ for all $z \in U$, so $U \subset Y_K$ and the set Y_K is open. By Lemma 2 there exists for any continuous ε -disjointness preserving linear functional Υ on $\mathcal{C}(X)$ with $\|\Upsilon\| > 3\sqrt{\varepsilon}$ a unique $x_\Upsilon \in X$ satisfying condition (5). Define a mapping $\phi: Y_{6\sqrt{\varepsilon}} \rightarrow X$ by

$$(7) \quad \phi(y) = x_{\Phi_y}.$$

Let us show that ϕ is continuous. Assume, on the contrary, that there exist a $y \in Y_{6\sqrt{\varepsilon}}$ and an open neighborhood V of $\phi(y)$ such that every neighborhood U of y contains a point $z \in Y_{6\sqrt{\varepsilon}}$ with $\phi(z) \notin V$. Since X is a compact Hausdorff

space, there exists a function $f \in \mathcal{C}(X)$ satisfying $\text{supp } f \subset V$, $0 \leq f \leq 1_X$ and $f(\phi(y)) = 1$. Then by Lemma 2

$$|\Phi_y(f) - w_y(\phi(y))\|\Phi_y\|f(\phi(y))| = |\Phi(f)(y) - w_y(\phi(y))\|\Phi_y\|| \leq 3\sqrt{\varepsilon}.$$

Because $|w_y(\phi(y))| = 1$ and $\|\Phi_y\| > 6\sqrt{\varepsilon}$,

$$|\Phi(f)(y)| \geq |w_y(\phi(y))\|\Phi_y\| - 3\sqrt{\varepsilon} > 3\sqrt{\varepsilon}.$$

Let $U \subset Y_{6\sqrt{\varepsilon}}$ be a neighborhood of y and let $z \in U$ be such that $\phi(z) \notin V$. Then by Lemma 2

$$|\Phi(f)(z)| = |\Phi(f)(z) - w_z(\phi(z))\|\Phi_z\|f(\phi(z))| \leq 3\sqrt{\varepsilon},$$

which is a contradiction since $\Phi(f) \in \mathcal{C}(Y)$. So, ϕ is continuous.

Our next claim is that the closure of the open set $Y_{10\sqrt{\varepsilon}}$ is a subset of the open set $Y_{6\sqrt{\varepsilon}}$, that is,

$$(8) \quad \overline{Y_{10\sqrt{\varepsilon}}} \subset Y_{6\sqrt{\varepsilon}}.$$

Suppose, contrary to our claim, that there is a $y \in \overline{Y_{10\sqrt{\varepsilon}}}$ such that $\|\Phi_y\| \leq 6\sqrt{\varepsilon}$. Then there exists a neighborhood U of y such that

$$|\Phi(1_X)(z)| \leq 7\sqrt{\varepsilon} \quad \text{for all } z \in U.$$

Because $y \in \overline{Y_{10\sqrt{\varepsilon}}}$, there exists a $z_0 \in Y_{10\sqrt{\varepsilon}} \cap U$, so by Lemma 2

$$|\Phi_{z_0}(1_X)| \geq \|\Phi_{z_0}\| - 3\sqrt{\varepsilon} > 7\sqrt{\varepsilon},$$

which is impossible.

It follows that $\overline{Y_{10\sqrt{\varepsilon}}^C} \cup Y_{6\sqrt{\varepsilon}} = Y$. The space Y is compact Hausdorff, so there exists a function $h \in \mathcal{C}(Y)$ satisfying $\text{supp } h \subset Y_{6\sqrt{\varepsilon}}$, $0 \leq h \leq 1_Y$ and $h(y) = 1$ for all $y \in \overline{Y_{10\sqrt{\varepsilon}}}$. Pick any $x \in X$ and extend the mapping (7) to $\phi: Y \rightarrow X$ by $\phi(y) = x$ for all $y \in Y \setminus Y_{6\sqrt{\varepsilon}}$. Let us define a linear mapping Ψ on $\mathcal{C}(X)$ by

$$(9) \quad \Psi(f)(y) = h(y)\Phi(1_X)(y)f(\phi(y)), \quad f \in \mathcal{C}(X), y \in Y.$$

Since $\text{supp } h \subset Y_{6\sqrt{\varepsilon}}$ and the mappings h , $\Phi(1_X)$, ϕ are continuous on $Y_{6\sqrt{\varepsilon}}$, Ψ maps from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$. The mapping (9) is obviously linear and disjointness preserving and it is also continuous because for any $f \in \mathcal{C}(X)$

$$\|\Psi(f)\| = \sup_{y \in Y} |h(y)\Phi(1_X)(y)f(\phi(y))| \leq \|\Phi\|\|f\|.$$

It remains to prove that Ψ is near Φ . In order to do that take $f \in \mathcal{C}(X)$ with $\|f\| \leq 1$. If $y \in Y \setminus Y_{10\sqrt{\varepsilon}}$, then

$$(10) \quad |\Phi(f)(y) - \Psi(f)(y)| \leq |\Phi_y(f)| + |\Phi_y(1_X)| \leq 20\sqrt{\varepsilon}.$$

And if $y \in Y_{10\sqrt{\varepsilon}}$, then by Lemma 2

$$(11) \quad \begin{aligned} |\Phi(f)(y) - \Psi(f)(y)| &\leq |\Phi_y(f) - w_y(\phi(y))\|\Phi_y\|f(\phi(y))| \\ &\quad + |w_y(\phi(y))\|\Phi_y\|f(\phi(y)) - h(y)\Phi_y(1_X)f(\phi(y))| \\ &\leq 3\sqrt{\varepsilon} + |w_y(\phi(y))\|\Phi_y\| - \Phi_y(1_X)| |f(\phi(y))| \\ &\leq 3\sqrt{\varepsilon} + |w_y(\phi(y))\|\Phi_y\|1_X(\phi(y)) - \Phi_y(1_X)| \\ &\leq 6\sqrt{\varepsilon}. \end{aligned}$$

So,

$$\|\Phi - \Psi\| \leq 20\sqrt{\varepsilon}.$$

□

Proof of Theorem 2. Let $\varepsilon > 0$ and let Φ be ε -disjointness preserving. First, we will show that $\Phi(f)\Phi(g) = 0$ whenever $\text{supp } f \cap \text{supp } g = \emptyset$. Suppose this is false. Then we can find $f, g \in \mathcal{C}(X)$ satisfying $\|f\|, \|g\| \leq 1$ and $\text{supp } f \cap \text{supp } g = \emptyset$, for which $\Phi(f) \neq 0$ and $\Phi(g) \neq 0$. If we define $U_1 = (\text{supp } g)^C$ and $U_2 = (\text{supp } f)^C$, then $U_1 \cup U_2 = X$. So, since X is a compact Hausdorff space, there exist $h_1, h_2 \in \mathcal{C}(X)$ such that $0 \leq h_i \leq 1_X$, $\text{supp } h_i \subset U_i$, $i = 1, 2$, and $h_1 + h_2 = 1_X$. Because Φ is an unbounded functional, there exists a function $k \in \mathcal{C}(X)$ with $\|k\| \leq 1$ and

$$|\Phi(k)| > \frac{\varepsilon}{|\Phi(f)|} + \frac{\varepsilon}{|\Phi(g)|}.$$

Since $fk h_2 = 0$, we have

$$|\Phi(k h_2)| \leq \frac{\varepsilon}{|\Phi(f)|},$$

so

$$|\Phi(k h_1)| \geq |\Phi(k)| - \frac{\varepsilon}{|\Phi(f)|} > \frac{\varepsilon}{|\Phi(g)|}.$$

Also, because $gk h_1 = 0$, it follows that

$$|\Phi(k h_1)| \leq \frac{\varepsilon}{|\Phi(g)|},$$

a contradiction.

Further, suppose $|\Phi(f)\Phi(g)| \neq 0$, where $f, g \in \mathcal{C}(X)$ with $fg = 0$. Then $F_1 = \{x : |f(x)| \geq \frac{|\Phi(f)\Phi(g)|}{\varepsilon\|g\|}\}$ and $F_2 = \text{supp } g$ are disjoint closed sets, so there exists an $h \in \mathcal{C}(X)$ such that $0 \leq h \leq 1_X$, $h|_{F_1} \equiv 1$ and $\text{supp } h \subset (\text{supp } g)^C$. Since $\text{supp } hf \cap \text{supp } g = \emptyset$, it follows that $\Phi(hf) = 0$ and $|\Phi(f - hf)| = |\Phi(f)|$. We also know that $(f - hf)g = 0$ and $\|f - hf\| < \frac{|\Phi(f)\Phi(g)|}{\varepsilon\|g\|}$, so

$$|\Phi(f - hf)\Phi(g)| = |\Phi(f)\Phi(g)| > \varepsilon\|f - hf\|\|g\|,$$

which contradicts the assumption that Φ is ε -disjointness preserving. \square

It should be mentioned that the idea of the proof of the following lemma is similar to the one used by Abramovich [1, Example 1] (see also Jarosz [2, Example]).

Lemma 3. *There exists an unbounded disjointness preserving linear functional on $\mathcal{C}(X)$.*

Proof of Lemma 3. Let \mathbb{N} be the space of all positive integers with the discrete topology and let $\beta\mathbb{N}$ be the Stone-Ćech compactification of \mathbb{N} . We will show that there exists an unbounded linear functional on $\mathcal{C}(\beta\mathbb{N})$ which is disjointness preserving. It is sufficient to define this functional on a Hamel basis. So, take an arbitrary element x_0 from $\beta\mathbb{N} \setminus \mathbb{N}$ and let $\{e_\lambda : \lambda \in \Lambda\}$ be a Hamel basis of the subspace of all functions $f \in \mathcal{C}(\beta\mathbb{N})$ having the property that $x_0 \notin \text{supp } f$. If the function h is given by $h(n) = \frac{1}{n}$, $n \in \mathbb{N}$, and if βh is its extension to the $\beta\mathbb{N}$, then $x_0 \in \text{supp } \beta h$, since \mathbb{N} is a dense subset in $\beta\mathbb{N}$. It follows that $\{\beta h\} \cup \{e_\lambda : \lambda \in \Lambda\}$ is a linearly independent subset of $\mathcal{C}(\beta\mathbb{N})$, so it can be completed to the Hamel basis $\{\beta h\} \cup \{e_\lambda : \lambda \in \Lambda\} \cup \{f_\pi : \pi \in \Pi\}$ of the vector space $\mathcal{C}(\beta\mathbb{N})$. On this basis we define the linear functional Φ to be $\Phi(\beta h) = 1$, $\Phi(e_\lambda) = 0$, $\lambda \in \Lambda$, and $\Phi(f_\pi) = 0$, $\pi \in \Pi$.

Let us show that Φ is unbounded. For every positive integer m we define a function h_m on \mathbb{N} by

$$h_m(n) = \begin{cases} \frac{1}{n} & \text{if } n \leq m, \\ 0 & \text{if } n > m, \end{cases}$$

and then extend it to a continuous function βh_m on $\beta\mathbb{N}$. Because $(\beta\mathbb{N} \setminus \mathbb{N}) \cap \text{supp } \beta h_m = \emptyset$ for every positive integer m and because $\beta h(x) = 0$ if $x \in \beta\mathbb{N} \setminus \mathbb{N}$, the limit of the sequence $\{\beta h_m\}$ equals βh . The functional Φ is therefore unbounded since $\Phi(\beta h_m) = 0$ for every positive integer m and $\Phi(\beta h) = 1$.

Let us prove that it is also disjointness preserving. Suppose $fg = 0$, where $f, g \in \mathcal{C}(\beta\mathbb{N})$. Then $N_1 = \{n \in \mathbb{N} : f(n) \neq 0\}$, $N_2 = \{n \in \mathbb{N} : g(n) \neq 0\}$ and $N_3 = \{n \in \mathbb{N} : f(n) = g(n) = 0\}$ are pairwise disjoint sets and $N_1 \cup N_2 \cup N_3 = \mathbb{N}$. Note that $f(x) = 0$ for every $x \in \beta\mathbb{N} \setminus \overline{N_1}$ and $g(x) = 0$ for every $x \in \beta\mathbb{N} \setminus \overline{N_2}$. We define a function k on \mathbb{N} by $k(n) = 0$ if $n \in N_1 \cup N_3$ and $k(n) = 1$ if $n \in N_2$, and then extend it to a continuous function βk on $\beta\mathbb{N}$. Since $\beta k^{-1}(0) \supset \overline{N_1}$ and $\beta k^{-1}(1) \supset \overline{N_2}$, at most one of the sets $\overline{N_1}$ or $\overline{N_2}$ contains x_0 , so at least one of $\Phi(f)$ and $\Phi(g)$ equals zero, meaning that Φ is disjointness preserving. \square

We will need the following lemma in the proof of the last theorem.

Lemma 4. *Suppose X and Y are compact Hausdorff spaces and $\Phi: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is a linear mapping. Then*

$$B = \{y \in Y : \Phi_y \text{ is a bounded linear functional}\}$$

and

$$D = \{y \in Y : \Phi_y \text{ is a disjointness preserving linear functional}\}$$

are closed subsets of Y .

Proof of Lemma 4. Because $\{\Phi_y : y \in B\}$ is a collection of continuous linear functionals on the Banach space $\mathcal{C}(X)$ such that for every $f \in \mathcal{C}(X)$ the set $\{|\Phi_y(f)| : y \in B\}$ is bounded by $\|\Phi(f)\|$, it follows by the principle of uniform boundedness that there exists a constant M such that $\|\Phi_y\| \leq M$ for every $y \in B$. Also, since

$$\sup_{y \in \overline{B}} |\Phi(f)(y)| = \sup_{y \in B} |\Phi(f)(y)| \leq M\|f\|$$

for every $f \in \mathcal{C}(X)$, the set B is closed.

To deduce that D is closed, take $f, g \in \mathcal{C}(X)$ satisfying $fg = 0$. Then

$$\sup_{y \in \overline{D}} |\Phi(f)\Phi(g)(y)| = \sup_{y \in D} |\Phi(f)\Phi(g)(y)| = 0$$

and D is closed. \square

In the proof of Theorem 3 we will use a result of Jarosz [2, Theorem].

Proof of Theorem 3. Let Φ be a surjective ε -disjointness preserving linear mapping from $\mathcal{C}(X)$ onto $\mathcal{C}(Y)$ and let B and D be the closed subsets of Y defined in Lemma 4. Then B^C is a subset of D by Theorem 2. We will assume that $B^C \neq \emptyset$ and $D^C \neq \emptyset$ since these special cases follow easily from previous results. Let $\Phi_B: \mathcal{C}(X) \rightarrow \mathcal{C}(B)$ be given by

$$\Phi_B(f) = \Phi(f)|_B, \quad f \in \mathcal{C}(X).$$

Since $\{|\delta_y \circ \Phi_B(f)| : y \in B\}$ is bounded for every $f \in \mathcal{C}(X)$, it follows by the principle of uniform boundedness that Φ_B is bounded. It is clear that Φ_B is also ε -disjointness preserving, so we can define a continuous mapping $\phi_B: B_{6\sqrt{\varepsilon}} \rightarrow X$ by (7), where $B_K = \{y \in B : \|\delta_y \circ \Phi_B\| > K\}$.

In a similar way we define $\Phi_D: \mathcal{C}(X) \rightarrow \mathcal{C}(D)$ by $\Phi_D(f) = \Phi(f)|_D$, where $f \in \mathcal{C}(X)$. Then Φ_D is a disjointness preserving linear mapping. So, by [2, Theorem], the set D is a disjoint union of three sets D_1 , D_2 and D_3 , where $D_2 = \{y \in D : \delta_y \circ \Phi_D \text{ is unbounded}\}$, $D_3 = \{y \in D : \delta_y \circ \Phi_D = 0\}$ and $D_1 = D \setminus (D_2 \cup D_3)$, and there exists a continuous mapping $\phi_D: D_1 \cup D_2 \rightarrow X$ such that $\Phi_D(f)(y) = \Phi_D(1_X)(y)f(\phi_D(y))$ for every $y \in D_1$, $f \in \mathcal{C}(X)$, and $\phi_D(D_2)$ is a finite set $F = \{x_1, \dots, x_n\}$.

Because every function $f \in \mathcal{C}(D)$ can be extended to a function $f_1 \in \mathcal{C}(Y)$ and because the mapping Φ is surjective, it follows that Φ_D is also surjective. So, the set D_3 is empty and ϕ_D is defined on the whole set D . Then $\phi_D^{-1}(x_1), \dots, \phi_D^{-1}(x_n)$ are pairwise disjoint closed sets in Y since ϕ_D is continuous and D is closed in Y .

Let us show that $\phi_D(y) = \phi_B(y)$ for every $y \in B_{6\sqrt{\varepsilon}} \cap D$. Otherwise there would exist a $y \in B_{6\sqrt{\varepsilon}} \cap D$ and a $g \in \mathcal{C}(X)$ with $g(\phi_D(y)) = 1$ and $g(\phi_B(y)) = 0$. Then $\Phi(1_X)(y) = \Phi(1_X)(y)g(\phi_D(y)) = \Phi(g)(y) = w_y(\phi_B(y))\|\Phi_y\|g(\phi_B(y)) = 0$ by (5) and (7), which means that $y \in D_3$, a contradiction.

Next, we claim that ϕ_D is injective on D_1 . Assume, on the contrary, that there exist different $y_1, y_2 \in D_1$ such that $\phi_D(y_1) = \phi_D(y_2)$. But then any function $g \in \mathcal{C}(D)$, for which $g(y_1) = 0$ and $g(y_2) = 1$, would not be in the image of Φ_D , a contradiction.

If we define $N_i = \phi_D^{-1}(x_i)$, then there is at most one element from D_1 in each of the closed sets N_i , $i = 1, \dots, n$. Let $\{N_i : i \in I \subset \{1, \dots, n\}\}$ be the collection of sets for which there exists a $y_i \in N_i$ such that $\|\Phi_{y_i}\| \leq 6\sqrt{\varepsilon}$. Let us in addition denote $N_0 = \overline{B_{10\sqrt{\varepsilon}}} \cup (\bigcup_{i \in \{1, \dots, n\} \setminus I} N_i)$ and $I_0 = \{0\} \cup I$. By (8), $\{N_i : i \in I_0\}$ is a family of pairwise disjoint closed subsets of the compact Hausdorff space Y , so there exist pairwise disjoint open sets $V_0 \supset N_0$ and $U_i \supset N_i$, $i \in I$. Since N_0 is a subset of the open set $(B \setminus B_{6\sqrt{\varepsilon}})^C$, also $U_0 = V_0 \cap (B \setminus B_{6\sqrt{\varepsilon}})^C$ is an open set containing N_0 . By Urysohn's lemma there exist $h_i \in \mathcal{C}(Y)$ such that $0 \leq h_i \leq 1_Y$, $\text{supp } h_i \subset U_i$ and $h_i(y) = 1$ for $y \in N_i$, $i \in I_0$. Define a mapping $\phi: Y \rightarrow X$ by

$$\phi(y) = \begin{cases} \phi_B(y) & \text{if } y \in U_0 \cap B, \\ x_i & \text{if } y \in U_i, i \in I, \\ x_1 & \text{otherwise,} \end{cases}$$

and a linear mapping Ψ by

$$\Psi(f)(y) = \begin{cases} \Phi(f)(y) & \text{if } y \in D_2, \\ \Phi(1_X)(y)f(\phi(y))(\sum_{i \in I_0} h_i(y)) & \text{if } y \in B, \end{cases}$$

for every $f \in \mathcal{C}(X)$. Because ϕ is continuous on $B \cap (\bigcup_{i \in I_0} U_i)$ and $\text{supp } \sum_{i \in I_0} h_i \subset \bigcup_{i \in I_0} U_i$, we see that $\Psi(f)$ is continuous on B . And because $\phi_D(y) = \phi(y)$ and $\sum_{i \in I_0} h_i(y) = 1$ for every $y \in \overline{D_2} \cap B$, it follows that $\Phi(f)(y) = \Psi(f)(y)$, $y \in \overline{D_2} \cap B$. So, $\Psi(f) \in \mathcal{C}(Y)$ for every $f \in \mathcal{C}(X)$. Since Ψ_y is a disjointness preserving functional for every $y \in B$ and also for every $y \in D_2$, Ψ is also a disjointness preserving map. Finally, by (9), (10) and (11)

$$\|\Phi(f) - \Psi(f)\| \leq 20\sqrt{\varepsilon}\|f\| \quad \text{for all } f \in \mathcal{C}(X).$$

□

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