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THE ROOT LATTICE A_n^* AND RAMANUJAN'S CIRCULAR SUMMATION OF THETA FUNCTIONS

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Abstract. We relate a formula of Ramanujan on the circular summation of the nth power of theta functions, $F_n(q)$, to the theta series of the root lattice A_n^*. We then use properties of the lattice to show that F_n includes an $\mathrm{SL}_2(\mathbf{Z})$ modular form when n is an odd perfect square as well as to derive a very simple expression for $F_9(q)$.

1. Introduction

Let $\theta(z,\tau) = \sum_{m=-\infty}^{\infty} q^{m^2} e^{2\pi imz}$ be the Jacobi theta function. We assume throughout this paper that $q = e^{\pi i\tau}$, where τ lies in the upper half plane. On page 54 of his "Lost Notebook" [7], Ramanujan claimed (in his own notation) the following result:

Ramanujan's Claim. *If n is a positive integer, then*

$$\sum_{k=0}^{n-1} q^{k^2} e^{2\pi ikz}\theta(z+k\tau, n\tau)^n = \theta(z,\tau)F_n(q^2) \tag{1.1}$$

where $F_n(q^2) = 1 + 2nq^{n-1} + \cdots$.

Ramanujan's claim was proven by Rangachari [8] in 1994 and also more recently by Son [9] using elementary methods. Ramanujan also stated very simple and elegant formulae for $F_n(q)$ for $n = 2,3,4,5,7$. In order to prove Ramanujan's formulae for $F_n(q)$, Rangachari noticed an interesting relation with the theta series of the root lattice A_{n-1}^* (the dual of A_{n-1}). However, because he used the theory of linear codes over a finite field, Rangachari was only able to prove his relation for the case $n = p$ a prime. Our first result is a generalization of Rangachari's observation to all n where we use essentially only the Jacobi Inversion Formula (the reader should refer to Section 2 for the scaling of our lattice):

Theorem 1. *For all $n \geq 2$, $F_n(q^2) = \theta_{A_{n-1}^*}(n\tau)$.*

Theorem 1 has several immediate consequences (see Corollaries 2.1 and 2.2 in Section 2) and a surprising application for efficient computation for the theta series of A_{n-1}^* (see Lemma 2.2) and even of A_{n-1} when n is square free (see Lemma 3.4). The problem of exhibiting 'simple' explicit identities for $F_n(q^2)$ has been the

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subject of a number of recent works (see for example [1, 6, 9]). Using Theorem 1 and some properties of A^*_{n-1}, we will derive further results in this direction. One of our key observations is that if $n = r^2$ is an odd perfect square (note this implies $n - 1 \equiv 0 \bmod 8$), there is an even unimodular lattice midway between A_{n-1} and A^*_{n-1} (see Lemma 3.1 below). This allows us to determine a part of $F_n(q^2)$ in this case. More precisely, let

$$F_n(q^2) = \sum_{m=0}^{\infty} \alpha_n(m) q^m, \tag{1.2}$$

and set

$$G_n(\tau) = \sum_{m=0}^{\infty} \alpha_n(2nm) q^{2m}. \tag{1.3}$$

Then we have the following:

Theorem 2. *If $n = r^2$ is an odd perfect square, then $G_n(\tau)$ is a holomorphic modular form for $\mathrm{SL}_2(\mathbf{Z})$ of weight $(n-1)/2$ and is given by the theta series of E_{n-1}, the unique even unimodular lattice between A_{n-1} and A^*_{n-1}.*

The $G_n(\tau)$ above are effectively computable as a polynomial in the Eisenstein series and $\Delta(\tau)$ the unique cusp form of weight 12, for any given n, and we will give explicit examples in Section 3. Theorem 2 however, only determines a part of $F_n(q^2)$. We can sometimes determine the other part (which corresponds to the theta series of the translates of the intermediate even unimodular lattice). We give as one example an explicit evaluation of $F_9(q^2)$ where a different expression has been given by S. Ahlgren in ([1], Theorem 1, (1.5)). We will also indicate in Section 3 how we are led to our formula. As has been remarked several times earlier [1, 6], the proof of such formula, once it is found is a trivial verification by computing enough terms of the expansion since they belong to a space of some finite-dimensional modular forms. It is therefore the process of discovering the formula, via lattices in our case, which is more interesting. The problem of evaluating the next interesting case of $F_{25}(q^2)$ is apparently still open. We have already determined a part of it, namely $G_{25}(\tau)$.

2. The root lattice A^*_n

It follows from Ramanujan's claim (1.1) that

$$F_n(q^2) = \sum_{k=0}^{n-1} q^{k^2} \theta(k\tau, n\tau)^n / \theta_3(\tau) = \sum_{k=0}^{n-1} \left(\sum_{m=-\infty}^{\infty} q^{(nm+k)^2/n} \right)^n \Bigg/ \sum_{m=-\infty}^{\infty} q^{m^2}. \tag{2.1}$$

We will now state precisely some relevant facts on lattices following the standard reference [5]. The root lattice A_n is defined by

$$A_n = \{(x_0, \ldots, x_n) \in Z^{n+1} : \sum x_i = 0\} \tag{2.2}$$

and its dual is given by

$$A^*_n = \left\{ y = (y_0, \ldots, y_n) \in R^{n+1} : \sum y_i = 0, y^T x \in Z, x \in A_n \right\}. \tag{2.3}$$

We note that A_n has determinant $n+1$ and so A_n^* has determinant $1/(n+1)$. For an n-dimensional lattice Λ, its theta series is defined by

$$\theta_\Lambda(\tau) = \sum_{x\in\Lambda} e^{\pi i\tau\langle x.x\rangle}. \tag{2.4}$$

The theta series of a lattice and its dual is related by the following well-known Jacobi Transformation formula (see [5])

$$\theta_{\Lambda^*}(\tau) = \sqrt{\det(\Lambda)}(i/\tau)^{n/2}\theta_\Lambda(-1/\tau). \tag{2.5}$$

We also recall the following well-known transformation rule for the Jacobi theta function $\theta(z,\tau)$ (see [4]):

$$\theta(z+\mu+\lambda\tau,\tau) = e^{-\pi i(\lambda^2\tau+2\lambda z)}\theta(z,\tau), \qquad (\mu,\lambda)\in Z^2, \tag{2.6a}$$

$$\theta\left(\frac{z}{\tau},-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\frac{\pi i z^2}{\tau}}\theta(z,\tau), \tag{2.6b}$$

$$\theta(z,\tau+2) = \theta(z,\tau). \tag{2.6c}$$

These show incidentally that $\theta(z,\tau)$ ought to be a singular Jacobi form of weight $1/2$ and index $1/2$ for the subgroup $G(2)$. We note that Rangachari's proof of Ramanujan's claim essentially consists of verifying that the left-hand side of (1.1) also satisfies (2.6a) and hence its quotient by $\theta(z,\tau)$ is an elliptic function of order ≤ 1 and hence must be independent of z. We also need the following explicit formula for the theta series of A_{n-1} (see [5], pg. 110, formula (56)).

Lemma 2.1. *$\theta_{A_{n-1}}(\tau) = \sum_{k=1}^{n-1}(\sum_{m=-\infty}^{\infty} q^{m^2}\zeta^{km})^n/n\theta_3(n\tau)$, where $\zeta = e^{2\pi i/n}$ and $\theta_3(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2}$.*

Proof. We have $(\sum_{m=-\infty}^{\infty} q^{m^2}\zeta^{km})^n = \sum_{x\in Z^n}(q^{\sum x_i^2}\zeta^{k\sum x_i})$, so

$$\sum_{k=0}^{n-1}\left(\sum_m q^{m^2}\zeta^{km}\right)^n = \sum_{x\in Z^n}\left(\sum_{k=0}^{n-1}\zeta^{k\sum x_i}\right)q^{\sum x_i^2} = n\sum_m \sum_{x\in Z^n,\sum x_i=mn} q^{\sum x_i^2}, \tag{2.7}$$

where we have used the well-known orthogonality relation

$$\sum_{k=0}^{n-1}\zeta^{ka} = \begin{cases} n & a=mn,\\ 0 & \text{otherwise}\end{cases}$$

in the last equality. Setting $y_i = x_i - m$ in the inner sum on the right-hand side of (2.7) gives $\sum_{y\in Z^n,\sum y_i=0}(q^{\sum y_i^2+2m(\sum y_i)+nm^2}) = q^{nm^2}\theta_{A_{n-1}}(\tau)$ and substituting back into (2.7) we are done. □

We can now prove Theorem 1.

Proof of Theorem 1. By Lemma 2.1, we can rewrite

$$\theta_{A_{n-1}}(\tau) = \frac{1}{n\theta_3(n\tau)}\sum_{k=0}^{n-1}\theta^n\left(\frac{k}{n},\tau\right).$$

By the Jacobi Inversion Formula (2.5), we get

$$\theta_{A^*_{n-1}}(\tau) = \sqrt{n}\left(\frac{i}{\tau}\right)^{(n-1)/2}\sum_{k=0}^{n-1}\theta^n\left(\frac{k}{n},-\frac{1}{\tau}\right)\Big/ n\theta_3\left(-\frac{n}{\tau}\right). \tag{2.8}$$

From (2.6b), we get by setting $z = 0, \tau' = \tau/n$ and $z = k\tau/n$ respectively that

$$\theta_3(-n/\tau) = \sqrt{\tau/in}\theta_3(\tau/n) \tag{2.9a}$$

and

$$\theta\left(\frac{k}{n}, -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} e^{\frac{\pi i k^2 \tau}{n^2}} \theta\left(\frac{k\tau}{n}, \tau\right). \tag{2.9b}$$

Substituing (2.9a), (2.9b) into (2.8) gives

$$\theta_{A^*_{n-1}}(\tau) = \sum_{k=0}^{n-1} q^{k^2/n} \theta^n(k\tau/n, \tau)/\theta_3(\tau/n) = F_n(q^{2/n}) \quad \text{by (2.1).} \quad \square$$

Theorem 1 has many immediate corollaries. First, as Rangachari noted, by utilising the explicit theta series for the translate of A_{n-1} on (pg. 110 of [5], formula (57)), we have immediately

Corollary 2.1. *For every positive integer n and $\zeta = e^{2\pi i/n}$, we have*

$$\theta_{A^*_{n-1}}(n\tau) = \sum_{k=0}^{n-1} q^{k^2} \theta^n(k\tau, n\tau)/\theta_3(\tau) = \sum_{l=0}^{n-1} \frac{\sum_{k=0}^{n-1} \zeta^{-kl} \theta^n(k/n, n\tau)}{nq^{l^2} \theta(nl\tau, n^2\tau)}, \tag{2.10}$$

where as usual $\theta_3(\tau) = \theta(0, \tau)$.

Remark. Formula (2.10) is easily verified, for small $n = 2, 3, \ldots$ using the standard theta identities in [5], pg. 104. It seems to be nontrivial in general.

One can also view Theorem 1 as stating that $F_n(q^2)$ is the theta series of the scaled lattice $\sqrt{n}A^*_{n-1}$ (with determinant n^{n-2}). Using the known quadratic form (eq. (77) in pg. 115 of [5]) we have the following multi-dimensional sum:

Corollary 2.2. $F_n(q^2) = \sum_{x \in Z^{n-1}} q^{(n-1)\sum_i x_i^2 - \sum_{i \neq j} x_i x_j}$.

The multi-dimensional sum in Corollary 2.2 is useless as a formula for computing the theta series for A^*_{n-1} as it runs in exponential time. It seems the only known way previously for actually computing this series is via the formula (57) on p. 110 of [5] where one actually has to compute over the ring $Z[\zeta]$ (see the analogous algorithm of Robert Harley for computing A_{11} in sequence A023923 of [10]). A surprising consequence of Ramanujan's formula and Theorem 1 is that it gives an efficient way to compute theta series of A^*_{n-1} via the formula on the RHS of (2.1). We state it as a lemma.

Lemma 2.2. *The theta series of $\sqrt{n}A^*_{n-1}$ $(= F_n(q^2))$ can be computed efficiently (involving only polynomial operations over the integers) using the following expressions:*

(2.11)

$$\theta_3(\tau)F_n(q^2) = \begin{cases} \left(\sum_m q^{nm^2}\right)^n + 2\sum_{k=0}^{(n-1)/2} q^{k^2} \left(\sum_m q^{nm^2+2km}\right)^n & n \text{ odd}, \\ \left(\sum_m q^{nm^2}\right)^n + q^{n^2/4} \left(\sum_m q^{nm^2+nm}\right)^n & \\ \quad +2\sum_{k=1}^{(n-2)/2} q^{k^2} \left(\sum_m q^{nm^2+2km}\right)^n & n \text{ even}, \end{cases}$$

where the summation of m is over all integers.

Proof. Equation (2.11) follows easily from the right-hand side of (2.1) and the fact that $\sum_m q^{(nm+k)^2/n} = \sum_m q^{(nm+n-k)^2/n}$. We also note that in our expression (2.11), the summation in m is only over nonnegative powers of q so the right-hand side is easily computed to any accuracy using only polynomial series expansion and the same holds for the division by θ_3. □

As was noted by Rangachari, one can also use Theorem 1 to evaluate $F_n(q^2)$. For example if $n = 4$, since we know $A_3^* \approx D_3^*$ and the theta series of D_3^* has a simple expression ((96) of [5]), one gets immediately $F_4(q^2) = \theta_3^3(q^4)+\theta_2^3(q^4)$, as was noted by Ramanujan. We mention also that for $n = 3$, since $A_2^* \approx A_2$ is the hexagonal lattice and determinant $(\sqrt{3}A_2^*) = 1$, we must have $F_3(q) = \sum_{m,n} q^{m^2+mn+n^2} = a(q)$ and the Ramanujan expression for $n = 3$ as given in [6]:

$$F_3^3(q) = \left(\frac{f^3(-q)}{f(-q^3)}\right)^3 + \left(3q\frac{f^3(-q^3)}{f(-q)}\right)^3$$

is exactly the Cubic Identity of [3] $a(q)^3 = b(q)^3 + c(q)^3$. The fact that $b(q) = f^3(-q)/f(-q^3)$ and $c(q) = 3q^{1/3}f^3(-q^3)/f(-q)$ has been noted by Berndt [2]. It is however possible to derive further properties of $F_n(q^2)$ using Theorem 1 and more detail structure of A_{n-1}^* as we will see in Section 3.

3. Explicit evaluation of $F_n(q^2)$

Since determinant$(A_{n-1}) = n$, A_{n-1} is a subgroup of A_{n-1}^* of index n and the glue group A_{n-1}^*/A_{n-1} is cyclic of order n (see [5]). By (2.2) and (2.3), the vector

$$w = (1, \dots, 1, -(n-1))/n \in R^n \tag{3.1}$$

is clearly in A_{n-1}^*/A_{n-1} and it is indeed a generator. For $0 \le j \le n-1$, let C_j be the coset $A_{n+1} + jw$, so that

$$A_{n-1}^* = \bigcup_{j=0}^{n-1} C_j. \tag{3.2}$$

Note that we can rewrite Theorem 1 as $F_n(q^2) = \theta_{\sqrt{n}A_{n-1}^*}(\tau)$, where $\sqrt{n}A_{n-1}^*$ is the lattice A_{n-1}^* scaled by a factor of $\sqrt{n}$. It follows from (3.1) and (3.2) that $\sqrt{n}A_{n-1}^*$ is an integral lattice and is even integral if n is odd. One of our key observations is the following:

Lemma 3.1. *If $n = r^2$ is an odd perfect square (note this means $n-1 \equiv 0 \bmod 8$), there is an even unimodular $(n-1)$-dimensional lattice E_{n-1} midway between A_{n-1} and its dual, i.e. $A_{n-1} \subset E_{n-1} \subset A_{n-1}^*$. We also have*

$$E_{n-1} = \bigcup_{j=0}^{r-1} C_{jr} \tag{3.3a}$$

and

$$A_{n-1}^* = \bigcup_{j=0}^{r-1} E_{n-1} + jw. \tag{3.3b}$$

Proof. We define E_{n-1} by (3.3a) which is clearly the unique subgroup of index r in A_{n-1}^* since A_{n-1} is of index $n = r^2$. It follows that determinant$(E_{n-1}) = 1$. The fact that E_{n-1} is integral and even follows from Lemma 3.2 below. □

The lattice E_{n-1} is easily identified for small n. For $n = 9$, it must be the lattice E_8 since there is only one such lattice. For $n = 25$, E_{24} is clearly the 24-dimensional Niemeier lattice of type A_{24} with Coxeter number 25 (see [5], table 16.1, pg. 407). Now a standard observation of Hecke states that the theta series of an even unimodular m-dimensional lattice is a modular form of weight $m/2$ for $\mathrm{SL}_2(\mathbf{Z})$ and it follows from (3.3b) that we may hope to see an $\mathrm{SL}_2(\mathbf{Z})$ modular form to appear in the theta series of A^*_{n-1}. That this is indeed the case follows from the next two lemmas.

Lemma 3.2. *Let $y = \sqrt{n}(x+jw)$, where $x \in A_{n-1}$, be a vector in the coset $\sqrt{n}C_j$. Then we have*

$$\|y\|^2 \equiv j^2(n-1) \bmod 2n. \tag{3.4}$$

Proof. By (2.2) and (3.1) we have clearly $\langle x, w\rangle = \sum_{j=1}^{n-1} x_j$ is integral for any $x \in A_{n-1}$ and $\|w\|^2 = (n-1)/n$. So we have $\|y\|^2 = n\|x\|^2 + 2jn\langle x, w\rangle + j^2(n-1)$. (3.4) follows from this and the fact that A_{n-1} is even integral ($\sum x_i^2 \equiv \sum x_i = 0 \bmod 2$). □

It follows from Lemma 3.2 and (3.2) that $\alpha_n(m)$ is only nonzero for certain arithmetic progressions of $m \bmod 2n$. In particular, we have the following

Lemma 3.3. *If $n = r^2$ is odd and $y \in \sqrt{n}A^*_{n-1}$, then $\|y\|^2 \equiv 0 \bmod 2n$ iff $y \in \sqrt{n}E_{n-1}$.*

Proof. The "if" part follows from equations (3.4) and (3.3a). Conversely, since $\gcd(n-1, 2n) = 2$, (3.4) implies that $\|y\|^2 \equiv 0 \bmod 2n$ can only occur for $y \in$ a coset C_j where r divides j and the result follows again from (3.3a). □

We can now prove Theorem 2.

Proof of Theorem 2. By Lemma 3.3, a vector in $\sqrt{n}A^*_{n-1}$ with norm $2nm$ must lie in the scaled even unimodular sublattice $\sqrt{n}E_{n-1}$ and only such vectors can have norm dividing $2n$. It follows that $\sum_{m=0}^{\infty} \alpha_n(2nm)q^{2nm} = \theta_{\sqrt{n}E_{n-1}}(\tau) = \theta_{E_{n-1}}(n\tau)$. By (1.3) we must have $G_n(\tau) = \theta_{E_{n-1}}(\tau)$, and so it must be a modular form of weight $(n-1)/2$ for $\mathrm{SL}_2(\mathbf{Z})$. □

Remark 1. Let $m = (n-1)/2$ and $k = \lfloor m/12 \rfloor$, the space of modular form of weight m is spanned by $\{\Delta^j(\tau)E_{m-12j}(\tau) : 0 \le j \le k\}$ and $\theta_{E_{n-1}}$ can be computed as a linear combination of these by computing the coefficients $\alpha_n(2nj)$ for $0 \le j \le k$ using Lemma 2.2. For $n = 9$, the weight 4 modular form is unique and $G_9(\tau) = E_4(\tau) = 1 + 240\sum_{j=0}^{\infty} \sigma_3(j)q^{2j}$. For $n = 25$, one need only two coefficients $\alpha_{25}(0) = 1$ and $\alpha_{25}(50) = 600$ to get $G_{25}(\tau) = E_{12}(\tau) + (600 - \frac{65520}{691})\Delta(\tau)$.

Remark 2. Theorem 2 determines only a part of $F_n(q^2)$, one still needs the theta series for the $r-1$ tranlsates of E_{n-1} in A^*_{n-1}. For $n = 9$, computing $F_9(q^2) - E_4(9\tau)$ by Lemma 2.2 gives us the series

$$18q^8 + 72q^{14} + 252q^{20} + 504q^{26} + 1026q^{32} + 1512q^{38} + 2664q^{44} + 3528q^{50} + \cdots.$$

Dividing out by 18, we search for the sequence $1, 4, 14, 28, 57, 84, 148, \ldots$ on Neil Sloane's on line Encyclopedia of Integer Sequence [10] and discover it as the theta series of the direct sum of 4 copies of the translate of the Hexangonal lattice (sequence A033690), more exactly, if we set $f(q) = 1+4q+14q+28q^3+57q^4+84q^5+\cdots$, we have, in the notation of [5], pg. 111,

$$f(q) = (1 + q + 2q^2 + 2q^4 + \cdots)^4 = \left(\frac{\theta_{\mathrm{hex}+[1]}(\tau)}{3q^{1/3}}\right)^4. \tag{3.5}$$

The lattice Hex +[1] above is just the translate of the Hexagonal lattice $= A_2$ and it is easy to see that the series of this translate is given by

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m-1/3)^2+(m-1/3)(n-1/3)+(n-1/3)^2}$$

which appears in the cubic identity [3]. We have arrived at the rather simple formula $F_9(q^2) = E_4(9\tau) + \frac{2}{9}[c(q^6)]^4$ which can be easily verified.

Finally, we can also determine $G_n(\tau)$ in some cases when n is not a square. One observes that A_{n-1} is a coset of A^*_{n-1}, and try to recover the theta series of A_{n-1} from that of A^*_{n-1}.

Lemma 3.4. *Let* $\alpha_n(m)$ *and* $G_n(\tau)$ *be as defined in* (1.2) *and* (1.3). *If* n *is square free,* $G_n(\tau) = \theta_{A_{n-1}}(\tau)$.

Proof. The coset C_0 gives A_{n-1} but conversely (3.4) of Lemma 3.2 implies that a vector y in any other coset cannot have norm which is a multiple of $2n$ since $j^2(n-1) \neq 0 \bmod 2n$ for any $j \neq 0$ for square free n. So the vectors in $\sqrt{n}A^*_{n-1}$ with norm a multiple of $2n$ are exactly those in the lattice $\sqrt{n}A_{n-1}$.

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