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ON A SEMILINEAR SCHRÖDINGER EQUATION WITH CRITICAL SOBOLEV EXPONENT

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ABSTRACT. We consider the semilinear Schrödinger equation $-\Delta u+V(x)u = K(x)|u|^{2^*-2}u+g(x,u)$, $u\in W^{1,2}(\mathbf{R}^N)$, where $N\geq 4$, V,K,g are periodic in x_j for $1\leq j\leq N$, $K>0$, g is of subcritical growth and 0 is in a gap of the spectrum of $-\Delta+V$. We show that under suitable hypotheses this equation has a solution $u\neq 0$. In particular, such a solution exists if $K\equiv 1$ and $g\equiv 0$.

1. Introduction and statement of the main result

In this paper we shall be concerned with the semilinear Schrödinger equation

$$-\Delta u+V(x)u=K(x)|u|^{2^*-2}u+g(x,u),\qquad u\in W^{1,2}(\mathbf{R}^N), \tag{1.1}$$

where $N\geq 4$, $2^*:=2N/(N-2)$ is the critical Sobolev exponent and g is of subcritical growth. More precisely, we make the following assumptions:

- **(A1):** $V,K\in C(\mathbf{R}^N)$, $g\in C(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$, $K(x)>0$ in $\mathbf{R}^N$ and V,K,g are 1-periodic in x_j for $j=1,\dots,N$.
- **(A2):** $|g(x,u)|\leq c_0(1+|u|^{p-1})$ on $\mathbf{R}^N\times\mathbf{R}$ for some $c_0>0$ and $p\in(2,2^*)$.
- **(A3):** $g(x,u)/u\to 0$ uniformly in x as $u\to 0$.
- **(A4):** $0\leq 2G(x,u)\leq ug(x,u)$ on $\mathbf{R}^N\times\mathbf{R}$, where $G(x,u):=\int_0^u g(x,s)\,ds$.
- **(A5):** $0\notin\sigma(-\Delta+V)$ and $\sigma(-\Delta+V)\cap(-\infty,0)\neq\emptyset$, where σ denotes the spectrum in $L^2(\mathbf{R}^N)$.

Note that we do not exclude the case of $g\equiv 0$. It is well-known that under our hypotheses on V the spectrum of $-\Delta+V$ in $L^2(\mathbf{R}^N)$ is bounded below and is the union of disjoint closed intervals; see e.g. p. 161 and Theorem 4.5.9 in [12]. So (A5) is equivalent to 0 being in a spectral gap of $-\Delta+V$. According to (A3), $g(x,0)\equiv 0$. Hence $u=0$ is necessarily a solution of (1.1).

Our main result is the following

Theorem 1.1. *Suppose that conditions* (A1)–(A5) *are satisfied,* $N\geq 4$ *and* $K(x_0) = \max_{\mathbf{R}^N} K(x)$. *If* $K(x)-K(x_0)=o(|x-x_0|^2)$ *as* $x\to x_0$ *and* $V(x_0)<0$*, then equation* (1.1) *has a solution* $u\neq 0$.

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Remark 1.2. (i) If $N = 4$, then it suffices that $K(x) - K(x_0) = O(|x - x_0|^2)$ as $x \to x_0$ (see the comment at the end of Section 4). This condition is obviously satisfied if K is of class C^2.

(ii) The flatness condition $K(x) - K(x_0) = o(|x - x_0|^2)$ has been imposed by several authors; see e.g. [7].

As an immediate consequence of Theorem 1.1 we obtain the following:

Corollary 1.3. *If conditions* (A1)–(A5) *are satisfied,* $N \geq 4$ *and* $K(x) \equiv K$ *is a positive constant, then equation* (1.1) *has a solution* $u \neq 0$.

Equation (1.1) with $K \equiv 0$ and V, g satisfying (A1)–(A3), (A5) and a stronger version of (A4) (the subcritical case) has been considered by several authors; see e.g. [1, 3, 5, 9, 11, 13, 16, 17, 18] and the references there. Equation (1.1) under conditions similar to (A1)–(A5) was discussed in [6], and our Theorem 1.1 is an extension of the main result there. We also note that when $g \equiv 0$, (A5) cannot be replaced by the hypothesis that $0 \notin \sigma(-\Delta + V)$. Indeed, as was observed in [4], equation $-\Delta u + \lambda u = |u|^{2^*-2}u$, where $\lambda \neq 0$, has only the trivial solution $u = 0$ in $W^{1,2}(\mathbf{R}^N)$.

Recall [19] that there is a one-to-one correspondence between solutions of (1.1) and critical points of the functional

$$J(u) := \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + Vu^2)\,dx - \frac{1}{2^*}\int_{\mathbf{R}^N} K|u|^{2^*}\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx.$$

Moreover, $J \in C^1(E, \mathbf{R})$, where $E := W^{1,2}(\mathbf{R}^N)$. Later we shall see that the functional J has the so-called linking geometry.

In what follows we shall usually abbreviate $L^p(\mathbf{R}^N)$ by L^p and the Sobolev space $W^{m,p}(\mathbf{R}^N)$ by $W^{m,p}$. The norms will be respectively denoted by $\| \|_p$ and $\| \|_{m,p}$. The open ball centered at a and having radius r will be denoted by $B(a, r)$. The spaces L^p and $W^{m,p}$ are real except in Section 2 where they are complex.

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2. The linear operator

Let $\mathcal{L}_q : \mathcal{D}(\mathcal{L}_q) \subset L^q(\mathbf{R}^N) \to L^q(\mathbf{R}^N)$, $2 \leq q < \infty$, be the operator given by $\mathcal{L}_q u := -\Delta u + V(x)u$. If $q = 2$, we shall write $\mathcal{L}$ instead of $\mathcal{L}_2$. In this section we assume that $V \in L^\infty(\mathbf{R}^N)$, $N \geq 1$, and we do not require V to be periodic.

Lemma 2.1. *$\mathcal{L}_q$ is a closed operator with domain $\mathcal{D}(\mathcal{L}_q) = W^{2,q}(\mathbf{R}^N)$.*

Proof. The operator $u \mapsto (V(x) - 1)u$ is bounded in L^q. Therefore it suffices to prove the above statement for $-\Delta + 1$. However, this is an immediate consequence of the fact that $(-\Delta + 1)^{-1}$ is an isomorphism of L^q onto $W^{2,q}$ (a property of the Bessel potentials; see formula (41) and Theorem 3 of Chap. V in [14]). □

Recall that in this section the spaces L^p and $W^{m,p}$ are complex. By a result of Hempel and Voigt [8] (see also Arendt [2, Example 5.3]) $\sigma(\mathcal{L}_q) = \sigma(\mathcal{L})$ and $(\mathcal{L}_q - \lambda)^{-1}|_{L^q \cap L^2} = (\mathcal{L} - \lambda)^{-1}|_{L^q \cap L^2}$ for all complex $\lambda \notin \sigma(\mathcal{L})$.

Let $(E(\lambda))_{\lambda \in \mathbf{R}}$ be the spectral family of $\mathcal{L}$. Then for a fixed μ, $E(\mu)L^2$ is the subspace of L^2 corresponding to $\lambda \leq \mu$.

Proposition 2.2. *If $V \in L^\infty(\mathbf{R}^N)$ satisfies* (A5), *then $\|u\|_{1,\infty} \leq c_0\|u\|_2$ for some constant $c_0 > 0$ and all $u \in E(0)L^2$.*

Proof. Let Γ be a positively oriented smooth Jordan curve (in $\mathbf{C}$) containing $\sigma(\mathcal{L})\cap(-\infty,0)$ in its interior and the remaining part of $\sigma(\mathcal{L})$ in its exterior. Since $\mathcal{L}$ is a closed operator,

$$E(0) = -\frac{1}{2\pi i}\int_\Gamma (-\Delta+V-\lambda)^{-1}\,d\lambda \tag{2.1}$$

according to formula (III.6.19) in [10]. So

$$u = -\frac{1}{2\pi i}\int_\Gamma (-\Delta+V-\lambda)^{-1}u\,d\lambda \tag{2.2}$$

whenever $u\in E(0)L^2$. Since Γ is compact and $-\Delta+V-\lambda$ is invertible for each $\lambda\in\Gamma$ (as an operator from $\mathcal{D}(\mathcal{L})$ into L^2), it is easy to see from (2.2) and the Sobolev embedding theorem that $\|u\|_{q_1}\le c_1\|u\|_{2,2}\le c_2\|u\|_2$, where $q_1=2N/(N-4)$ if $N>4$ and q_1 may be chosen arbitrarily large if $N\le 4$ (here and in what follows c_1, c_2, etc. denote positive constants whose numerical values are immaterial). Keeping in mind that $\mathcal{L}_q$ is closed and $\mathcal{L}_q-\lambda$ is invertible on Γ for all q, we may employ the usual bootstrap argument: we get $\|u\|_{q_2}\le c_3\|u\|_{2,q_1}\le c_4\|u\|_{q_1}\le c_5\|u\|_2$, where $q_2=2N/(N-8)$; after a finite number of iterations $q_k>N$ and by (2.2) again, $\|u\|_{2,q_k}\le\tilde{c}\|u\|_2$. Now the conclusion follows by the Sobolev embedding $W^{2,q_k}\hookrightarrow W^{1,\infty}$. □

Proposition 2.3 (Troestler [17])**.** *If $V\in L^\infty(\mathbf{R}^N)$ satisfies* (A5) *and $q\in(2,\infty)$, then $E(0)|_{L^2\cap L^q}$ is L^q-continuous. In particular, $E(0)$ and $I-E(0)$ extend to continuous projections of L^q onto the complementary subspaces $\mathrm{cl}_{L^q}\big(E(0)L^2\big)$ and $\mathrm{cl}_{L^q}\big((I-E(0))L^2\big)$ (*cl *denotes the closure).*

Proof. By (2.1), $\|E(0)u\|_q\le\|E(0)u\|_{2,q}\le c_0\|u\|_q$ for all $u\in L^2\cap L^q$ and some $c_0>0$. Hence $E(0)$ and $I-E(0)$ may be extended to continuous projections of L^q onto the complementary subspaces as required. □

Proposition 2.4. *If $V\in L^\infty(\mathbf{R}^N)$, then for each $\mu\in\mathbf{R}$ there exist constants c_1 and $c_2=c_2(\mu)$ such that $\|u\|_q\le c_1\|u\|_{2,2}\le c_2\|u\|_2$ whenever $u\in E(\mu)L^2$. Here $q=2N/(N-4)$ if $N>4$, q may be taken arbitrarily large if $N=4$ and $q=\infty$ if $N<4$.*

Proof. The operator $\mathcal{L}^\mu:=\mathcal{L}|_{E(\mu)L^2}:E(\mu)L^2\to E(\mu)L^2$ is bounded. Let Γ be a positively oriented smooth Jordan curve enclosing the spectrum of $\mathcal{L}^\mu$. Then (2.2) still holds for all $u\in E(\mu)L^2$ (with $(-\Delta+V-\lambda)^{-1}$ replaced by $(\mathcal{L}^\mu-\lambda)^{-1}$). Therefore $\|u\|_q\le c_1\|u\|_{2,2}\le c_2\|u\|_2$. □

3. Existence of a Palais-Smale sequence

In this section we assume that the hypotheses (A1)–(A5) are satisfied. Recall $E=W^{1,2}(\mathbf{R}^N)$ and let $E^-:=E(0)L^2\cap E$ and $E^+:=(I-E(0))L^2\cap E$ ($E(\lambda)$ is as in the preceding section). Then the quadratic form $\int_{\mathbf{R}^N}(|\nabla u|^2+Vu^2)\,dx$ is positive definite on E^+ and negative definite on E^- [15, Sections 8 and 9]. Hence we may introduce a new inner product $\langle\ ,\ \rangle$ in E such that the corresponding norm $\|\ \|$ is equivalent to $\|\ \|_{1,2}$ and $\int_{\mathbf{R}^N}(|\nabla u|^2+Vu^2)\,dx=\|u^+\|^2-\|u^-\|^2$, where $u^\pm\in E^\pm$.

Set $\psi(u) := (2^*)^{-1} \int_{\mathbf{R}^N} K|u|^{2^*}\,dx + \int_{\mathbf{R}^N} G(x,u)\,dx$; then

(3.1)

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + Vu^2)\,dx - \frac{1}{2^*}\int_{\mathbf{R}^N} K|u|^{2^*}\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx \\ &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \psi(u). \end{aligned}$$

Let $z_0 \in E^+ \setminus \{0\}$,

$$M := \{u = u^- + sz_0 : u^- \in E^-,\ s \geq 0 \text{ and } \|u\| \leq R\}$$

and denote the boundary of M in $E^- \oplus \mathbf{R}z_0$ by ∂M. We summarize the properties of J in the following:

Proposition 3.1. (i) *There exist $\alpha, \rho > 0$ and $R > \rho$ (R depending on z_0) such that $J(u) \geq \alpha$ for all $u \in E^+ \cap \partial B(0,\rho)$ and $J(u) \leq 0$ for all $u \in \partial M$.*

(ii) *$\psi \geq 0$, ψ is weakly sequentially lower semicontinuous and ψ' is weakly sequentially continuous.*

Functionals satisfying (i) above are said to have the linking geometry.

Proof. (i) See e.g. [11, 18, 19]. The proofs given there are for nonlinearities of subcritical growth but the argument remains unchanged in our case (the part showing $J|_{\partial M} \leq 0$ is in fact somewhat simpler here; observe only that $(2^*)^{-1}K(x)|u|^{2^*} + G(x,u) \geq c_0|u|^{2^*}$ for some $c_0 > 0$).

(ii) It is obvious that $\psi \geq 0$. Let $u_n \rightharpoonup u$. Then $u_n \to u$ a.e. in $\mathbf{R}^N$, possibly after passing to a subsequence. Hence it follows from the Fatou lemma that ψ is weakly sequentially lower semicontinuous. Moreover, since $u_n \to u$ in L^p_{loc}, it is easy to see from (A2) and (A3) that

$$\int_{\mathbf{R}^N} g(x,u_n)v\,dx \to \int_{\mathbf{R}^N} g(x,u)v\,dx \text{ for each } v \in E.$$

Finally, $u_n \to u$ in $L^{(N+2)/(N-2)}_{loc}$; therefore $K|u_n|^{2^*-2}u_n \to K|u|^{2^*-2}u$ in L^1_{loc} and

$$\int_{\mathbf{R}^N} K|u_n|^{2^*-2}u_n\varphi\,dx \to \int_{\mathbf{R}^N} K|u|^{2^*-2}u\varphi\,dx \text{ whenever } \varphi \in C_0^\infty.$$

Taking into account that the sequence $(K|u_n|^{2^*-1})$ is bounded in $L^{2N/(N+2)}$, we may replace φ by $v \in E$. This completes the proof. □

Proposition 3.2. *If J is a functional of the form appearing in the second line of* (3.1) *and if* (i), (ii) *of Proposition 3.1 are satisfied, then there exists a Palais-Smale sequence (u_n) for J such that $J(u_n) \to c \in [\alpha, \sup_M J]$.*

This is a special case of Theorem 3.4 in [11]; see also Theorem 6.10 in [19].

We have thus shown that the functional J associated with (1.1) possesses a Palais-Smale sequence (u_n) with $J(u_n) \to c$.

Proposition 3.3. *The Palais-Smale sequence above is bounded.*

Proof. It follows from (A2)–(A3) that for each $\varepsilon > 0$ there exists $c_1(\varepsilon)$ such that $|g(x,u)| \leq \varepsilon|u| + c_1(\varepsilon)|u|^{2^*-1}$. By (A4),

$$c + 1 + \|u_n\| \geq J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle \geq \frac{1}{N}\int_{\mathbf{R}^N} K|u_n|^{2^*}\,dx$$

for almost all n, and since $K(x)$ is bounded below by a positive constant,

$$\|u_n\|_{2^*}^{2^*} \le c_2 + c_3\|u_n\|. \tag{3.2}$$

Using the Hölder and Sobolev inequalities we obtain, for large n,

$$\begin{aligned}\|u_n^+\|^2 &= \langle J'(u_n), u_n^+\rangle + \int_{\mathbf{R}^N} K|u_n|^{2^*-2}u_n u_n^+\,dx + \int_{\mathbf{R}^N} g(x,u_n)u_n^+\,dx \\ &\le \|u_n^+\| + c_4\|u_n\|_{2^*}^{2^*-1}\|u_n^+\| + c_5(\varepsilon\|u_n\| + c_1(\varepsilon)\|u_n\|_{2^*}^{2^*-1})\|u_n^+\|.\end{aligned}$$

Hence by (3.2),

$$\|u_n^+\| \le c_6(\varepsilon) + c_7(\varepsilon)\|u_n\|^{(2^*-1)/2^*} + c_5\varepsilon\|u_n\|,$$

and a similar inequality holds for $\|u_n^-\|$. Choosing ε sufficiently small, we see that (u_n) must be bounded. □

4. Proof of Theorem 1.1

In the preceding section we have shown that there exists a bounded Palais-Smale sequence (u_n) such that $J(u_n) \to c \in [\alpha, \sup_M J]$. Clearly, (u_n) is either

(i) *Vanishing:* For each $r > 0$, $\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,r)} u_n^2\,dx = 0$, or

(ii) *Non-vanishing:* There exist $r, \eta > 0$ and a sequence $(y_n) \subset \mathbf{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B(y_n,r)} u_n^2\,dx \ge \eta.$$

In (ii) we may assume $y_n \in \mathbf{Z}^N$ by taking a larger r if necessary. Suppose (ii) holds and let $\tilde u_n(x) := u_n(x + y_n)$. Since J is invariant with respect to the translation of x by elements of $\mathbf{Z}^N$ (i.e. $J(u(.)) = J(u(.+y))$ whenever $y \in \mathbf{Z}^N$), $\|\tilde u_n\| = \|u_n\|$ and $\|J'(\tilde u_n)\| = \|J'(u_n)\|$. Hence $\tilde u_n \rightharpoonup \tilde u$ after passing to a subsequence, $J'(\tilde u) = 0$ and since $\limsup_{n\to\infty} \int_{B(0,r)} \tilde u_n^2\,dx \ge \eta$, $\tilde u \ne 0$. So $\tilde u$ is a nontrivial solution of (1.1). To complete the proof of Theorem 1.1 it remains therefore to show that vanishing cannot occur. This will be done in the following two propositions. Let

$$S := \inf_{u\in E\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}. \tag{4.1}$$

Proposition 4.1. *If* $0 < c < c^* := \dfrac{S^{N/2}}{N\|K\|_\infty^{(N-2)/2}}$, *then* (u_n) *cannot be vanishing.*

Proof. If (u_n) is vanishing, then it follows from P.L. Lions' lemma [19, Lemma 1.21] that $u_n \to 0$ in L^r whenever $2 < r < 2^*$. Let (z_n) be a bounded sequence in E. Since for each $\varepsilon > 0$ there is $c_1(\varepsilon)$ such that $|g(x,u)| \le \varepsilon|u| + c_1(\varepsilon)|u|^{p-1}$,

$$\int_{\mathbf{R}^N} |g(x,u_n)|\,|z_n|\,dx \le c_2\varepsilon\|u_n\|\,\|z_n\| + c_3(\varepsilon)\|u_n\|_p^{p-1}\|z_n\|.$$

Using this and a similar argument for G we see that

$$\int_{\mathbf{R}^N} g(x,u_n)z_n\,dx \to 0 \qquad \text{and} \qquad \int_{\mathbf{R}^N} G(x,u_n)\,dx \to 0. \tag{4.2}$$

Hence

$$J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle = \frac{1}{N}\int_{\mathbf{R}^N} K|u_n|^{2^*}\,dx + o(1) \to c. \tag{4.3}$$

Recall $(E(\lambda))_{\lambda\in\mathbf{R}}$ is the spectral family of $-\Delta+V$ in L^2. Let $u=u^++u^-\in E^+\oplus E^-$ and write $u^+=w+z$, where $w\in E(\mu)L^2$, $z\in(I-E(\mu))L^2$, $\mu>0$ large (to be determined). By Proposition 2.4, $w\in E$, hence also $z\in E$; moreover, $\|u_n^-\|_q\le c_4\|u_n^-\|_2\le c_5\|u_n\|$ and $\|w_n\|_q\le c_4\|w_n\|_2\le c_5\|u_n\|$, where $q=2N/(N-4)$ if $N>4$ and q may be taken arbitrarily large if $N=4$. Let r be such that $(2^*-1)/r+1/q=1$. Then $2<r<2^*$ (for $N=4$, q needs to be larger than 4). Since $\|u_n^-\|_q$ is bounded and $u_n\to 0$ in L^r, we obtain using (4.2) and the Hölder inequality that

$$\begin{aligned}\|u_n^-\|^2 &= -\langle J'(u_n),u_n^-\rangle-\int_{\mathbf{R}^N}K|u_n|^{2^*-2}u_nu_n^-\,dx-\int_{\mathbf{R}^N}g(x,u_n)u_n^-\,dx\\ &\le \|K\|_\infty\|u_n\|_r^{2^*-1}\|u_n^-\|_q+o(1)\to 0.\end{aligned}$$

Similarly,

$$\|w_n\|^2=\int_{\mathbf{R}^N}K|u_n|^{2^*-2}u_nw_n\,dx+o(1)\to 0.$$

Hence

$$u_n-z_n=w_n+u_n^-\to 0, \tag{4.4}$$

and therefore

$$\begin{aligned}\|z_n\|^2 &= \int_{\mathbf{R}^N}(|\nabla z_n|^2+Vz_n^2)\,dx = \int_{\mathbf{R}^N}K|u_n|^{2^*-2}u_nz_n\,dx+o(1)\\ &= \int_{\mathbf{R}^N}K|u_n|^{2^*}\,dx+o(1).\end{aligned}\tag{4.5}$$

Furthermore, for each $\delta>0$ we may find $\mu>0$ such that

$$(1-\delta)\int_{\mathbf{R}^N}|\nabla z_n|^2\,dx\le\int_{\mathbf{R}^N}(|\nabla z_n|^2+Vz_n^2)\,dx. \tag{4.6}$$

Indeed, since $z_n\in(I-E(\mu))L^2\cap E$, we have $\int_{\mathbf{R}^N}(|\nabla z_n|^2+Vz_n^2)\,dx\ge\mu\|z_n\|_2^2$ and

$$\delta\int_{\mathbf{R}^N}|\nabla z_n|^2\,dx\ge\delta(\mu-\|V\|_\infty)\|z_n\|_2^2\ge-\int_{\mathbf{R}^N}Vz_n^2\,dx$$

whenever μ is large enough. Combining (4.4), (4.1), (4.6) and (4.5) gives

$$\begin{aligned}(1-\delta)S\|K\|_\infty^{-2/2^*}\left(\int_{\mathbf{R}^N}K|u_n|^{2^*}\,dx\right)^{2/2^*} &\le (1-\delta)S\|u_n\|_{2^*}^2\\ = (1-\delta)S\|z_n\|_{2^*}^2+o(1)\le (1-\delta)\int_{\mathbf{R}^N}|\nabla z_n|^2\,dx+o(1)&\\ \le \int_{\mathbf{R}^N}K|u_n|^{2^*}\,dx+o(1).&\end{aligned}$$

Passing to the limit and using (4.3) we obtain

$$(1-\delta)S\|K\|_\infty^{-2/2^*}(cN)^{2/2^*}\le cN;$$

hence either $c=0$ which is impossible or $(1-\delta)^{N/2}c^*\le c<c^*$ which is also impossible because δ may be chosen arbitrarily small. □

Let

$$\varphi_\varepsilon(x):=\frac{c_N\psi(x)\varepsilon^{(N-2)/2}}{(\varepsilon^2+|x|^2)^{(N-2)/2}},$$

where $c_N = (N(N-2))^{(N-2)/4}$, $\varepsilon > 0$ and $\psi \in C_0^\infty(\mathbf{R}^N, [0,1])$ is such that $\psi(x) = 1$ for $|x| \le r/2$ and $\psi(x) = 0$ for $|x| \ge r$ (r to be determined). We shall need the following asymptotic estimates as $\varepsilon \to 0^+$ (see e.g. pp. 35 and 52 in [19]):

(4.7)
$$\|\nabla\varphi_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \quad \|\nabla\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}),$$
$$\|\varphi_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N), \quad \|\varphi_\varepsilon\|_{2^*-1}^{2^*-1} = O(\varepsilon^{(N-2)/2}), \quad \|\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2})$$

and

$$\|\varphi_\varepsilon\|_2^2 = \begin{cases} b\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ b\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \ge 5, \end{cases} \tag{4.8}$$

where b is a positive constant. Finally, let

$$Z_\varepsilon := E^- \oplus \mathbf{R}\varphi_\varepsilon \equiv E^- \oplus \mathbf{R}\varphi_\varepsilon^+.$$

We may assume without loss of generality that $K(0) = \|K\|_\infty$ and $V(0) < 0$. Moreover, r in the definition of φ_ε may be chosen so that $V(x) \le -\beta$ for some $\beta > 0$ and all x with $|x| \le r$.

Proposition 4.2. *If $\varepsilon > 0$ is small enough, then $\sup_{Z_\varepsilon} J < c^*$. So in particular, if $z_0 = \varphi_\varepsilon^+$ with ε small enough, then $c \le \sup_M J < c^*$.*

Proof. Let

$$I(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2^*}\int_{\mathbf{R}^N} K|u|^{2^*}\,dx.$$

Since $I(u) \ge J(u)$ for all u, it suffices to show that $\sup_{Z_\varepsilon} I < c^*$.

In what follows we adapt the argument on pp. 52-53 in [19]. If $u \ne 0$, then

$$\max_{t\ge 0} I(tu) = \frac{1}{N}\frac{\left(\int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)\,dx\right)^{N/2}}{\left(\int_{\mathbf{R}^N} K|u|^{2^*}\,dx\right)^{(N-2)/2}} \tag{4.9}$$

whenever the integral in the numerator above is positive, and the maximum is 0 otherwise. Let $\|u\|_{2^*,K}^{2^*} := \int_{\mathbf{R}^N} K|u|^{2^*}\,dx$. It is easy to see from (4.9) that if

$$m_\varepsilon := \sup_{\substack{u\in Z_\varepsilon \\ \|u\|_{2^*,K}=1}} \int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)\,dx < \frac{S}{\|K\|_\infty^{(N-2)/N}}, \tag{4.10}$$

then $\sup_{Z_\varepsilon} J \le \sup_{Z_\varepsilon} I < c^*$. So it remains to show (4.10) is satisfied for all small $\varepsilon > 0$.

Below we shall repeatedly use (4.7) and (4.8). Since $\int_{\mathbf{R}^N}(|\nabla\varphi_\varepsilon^-|^2 + V(\varphi_\varepsilon^-)^2)\,dx \le 0$, $\int_{\mathbf{R}^N}|\nabla\varphi_\varepsilon^-|^2\,dx \le c_1\|\varphi_\varepsilon^-\|_2^2 \le c_1\|\varphi_\varepsilon\|_2^2 \to 0$ as $\varepsilon \to 0$; therefore $\|\varphi_\varepsilon^-\|_{2^*} \le c_2\|\varphi_\varepsilon^-\| \to 0$ and $\|\varphi_\varepsilon^+\|_{2^*}^{2^*} \to S^{N/2}$. Suppose $\|u\|_{2^*,K} = 1$ and write $u = u^- + s\varphi_\varepsilon = (u^- + s\varphi_\varepsilon^-) + s\varphi_\varepsilon^+$. It follows from Proposition 2.3 and the argument above that $\|u^-\|_{2^*} \le c_3$ and $|s| \le c_3$ for some constant c_3 independent of ε. By Proposition 2.2 and convexity of $\|\ \|_{2^*,K}$ we obtain

$$\begin{aligned} 1 = \|u\|_{2^*,K}^{2^*} &\ge \|s\varphi_\varepsilon\|_{2^*,K}^{2^*} + 2^*\int_{\mathbf{R}^N}(s\varphi_\varepsilon)^{2^*-1}u^-\,dx \\ &\ge \|s\varphi_\varepsilon\|_{2^*,K}^{2^*} - c_4\|\varphi_\varepsilon\|_{2^*-1}^{2^*-1}\|u^-\|_2. \end{aligned} \tag{4.11}$$

Moreover, by Proposition 2.2 again,

$$\int_{\mathbf{R}^N} (\nabla\varphi_\varepsilon \cdot \nabla u^- + V\varphi_\varepsilon u^-)\, dx \leq c_5(\|\nabla\varphi_\varepsilon\|_1 + \|\varphi_\varepsilon\|_1)\|u^-\|_2 = O(\varepsilon^{(N-2)/2})\|u^-\|_2. \tag{4.12}$$

Since $V(x) \leq -\beta < 0$ for $x \in \operatorname{supp}\varphi_\varepsilon$ and $K(x) - K(0) = o(|x|^2)$ as $x \to 0$,

$$\int_{\mathbf{R}^N} V\varphi_\varepsilon^2\, dx \leq \begin{cases} -d\varepsilon^2 & \text{if } N \geq 5, \\ -d\varepsilon^2|\log\varepsilon| & \text{if } N = 4, \end{cases} \tag{4.13}$$

for some $d > 0$ and

$$\|\varphi_\varepsilon\|_{2^*,K}^{2^*} = \|K\|_\infty \int_{\mathbf{R}^N} \varphi_\varepsilon^{2^*}\, dx + \int_{\mathbf{R}^N} (K(x) - K(0))\varphi_\varepsilon^{2^*}\, dx = \|K\|_\infty S^{N/2} + o(\varepsilon^2). \tag{4.14}$$

Let $N \geq 5$. Using (4.12), (4.14), (4.11), (4.13) and the fact that

$$-\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \leq O(\varepsilon^{N-2}),$$

we obtain

$$\begin{aligned} m_\varepsilon &\leq -\|u^-\|^2 + \frac{\int_{\mathbf{R}^N}(|\nabla\varphi_\varepsilon|^2 + V\varphi_\varepsilon^2)\, dx}{\|\varphi_\varepsilon\|_{2^*,K}^2}\|s\varphi_\varepsilon\|_{2^*,K}^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\leq -c_6\|u^-\|_2^2 + \frac{\int_{\mathbf{R}^N}(|\nabla\varphi_\varepsilon|^2 + V\varphi_\varepsilon^2)\, dx}{\|K\|_\infty^{(N-2)/N} S^{(N-2)/2} + o(\varepsilon^2)}(1 + c_4\|\varphi_\varepsilon\|_{2^*-1}^{2^*-1}\|u^-\|_2)^{2/2^*} \\ &\quad + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &= -c_6\|u^-\|_2^2 + \frac{S^{N/2} - d\varepsilon^2 + O(\varepsilon^{N-2})}{\|K\|_\infty^{(N-2)/N} S^{(N-2)/2} + o(\varepsilon^2)} + O(\varepsilon^{(N-2)/2)})\|u^-\|_2 \\ &\leq \frac{S}{\|K\|_\infty^{(N-2)/N}} - d_0\varepsilon^2 + o(\varepsilon^2), \end{aligned}$$

where $d_0 > 0$. If $N = 4$, then in a similar way,

$$m_\varepsilon \leq \frac{S}{\|K\|_\infty^{(N-2)/N}} - d_0\varepsilon^2|\log\varepsilon| + o(\varepsilon^2).$$

Hence (4.10) holds provided ε is sufficiently small. □

Note that if $K(x) - K(0) = O(|x|^2)$ as $x \to 0$, then (4.14) holds with $O(\varepsilon^2)$ replacing $o(\varepsilon^2)$. This does not affect the estimate of m_ε if $N = 4$. Hence for such N the conclusion of Theorem 1.1 remains valid under the weaker hypothesis on K as in Remark 1.2(i).

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