

ON THE L^p BOUNDEDNESS OF THE NON-CENTERED GAUSSIAN HARDY-LITTLEWOOD MAXIMAL FUNCTION

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ABSTRACT. The purpose of this paper is to prove the $L^p(\mathcal{R}^n, d\gamma)$ boundedness, for $p > 1$, of the non-centered Hardy-Littlewood maximal operator associated with the Gaussian measure $d\gamma = e^{-|x|^2} dx$.

Let $d\gamma = e^{-|x|^2} dx$ be a Gaussian measure in Euclidean space \mathcal{R}^n . We consider the non-centered maximal function defined by

$$\mathcal{M}f(x) = \sup_{x \in B} \frac{1}{\gamma(B)} \int_B |f| d\gamma,$$

where the supremum is taken over all balls B in \mathcal{R}^n containing x . P. Sjögren [2] proved that \mathcal{M} is not of weak type $(1,1)$ with respect to $d\gamma$ for $n > 1$. A more general result was obtained by A. Vargas [4], who characterized those radial and strictly positive measures for which the corresponding maximal operator is of weak type $(1,1)$. However, these papers leave open the question of the $L^p(d\gamma)$ boundedness of \mathcal{M} for $p > 1$ and $n > 1$.

The main result in this paper is

Theorem 1. \mathcal{M} is a bounded operator on $L^p(d\gamma)$ for $p > 1$, that is, there exists a constant $C = C(n, p)$ such that for $f \in L^p(d\gamma)$,

$$\|\mathcal{M}f\|_{L^p(d\gamma)} \leq C\|f\|_{L^p(d\gamma)}.$$

In a forthcoming paper [3], P. Sjögren and F. Soria prove estimates for the maximal operator associated with a more general radial measure with decreasing density.

We denote $S_r^{n-1} = \{x \in \mathcal{R}^n : |x| = r\}$ and $S^{n-1} = S_1^{n-1}$, and write $d\sigma$ for the area measure on S^{n-1} . The spherical maximal function

$$\mathcal{M}^e f(h) = \sup_{R>0} \frac{1}{\sigma(|z' - h| \leq R)} \int_{|z' - h| \leq R} |f(z')| d\sigma(z'), \quad h \in S^{n-1},$$

is bounded on $L^p(d\sigma)$. We extend \mathcal{M}^e to functions defined in \mathcal{R}^n by using polar coordinates $x = \rho x'$ with $x' \in S^{n-1}$ and applying \mathcal{M}^e in the x' variable. Then \mathcal{M}^e is bounded on $L^p(d\gamma)$.

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In order to prove Theorem 1, we need the following technical lemma, proved later.

Lemma 1. *Let B be a closed ball in \mathcal{R}^n of radius r . Denote by q the point of B whose distance to the origin is minimal. Assume that $|q| \geq 1$ and that $r \geq 1/|q|$. Then for all $x, y \in B$*

$$(1) \quad \gamma(B) \geq C \frac{e^{-|q|^2}}{|q|} \left(1 \wedge \frac{|y-x|^2}{|q|(|x| \vee |y| - |q|)} \right)^{\frac{n-1}{2}}.$$

Here and in the sequel, we write C for various positive finite constants and denote $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

Proof of Theorem 1. We assume that $n \geq 2$, since the case $n = 1$ is well known; see, e.g. [2]. Take $0 \leq f \in L^p(d\gamma)$ and $x \in \mathcal{R}^n$. For any ball B containing x , we must estimate the average $\mathcal{A}f(B) = \frac{1}{\gamma(B)} \int_B f d\gamma$. Let r and q be defined as in Lemma 1.

We first consider small balls B , and denote by $\mathcal{M}_0f(x)$ the supremum of $\mathcal{A}f(B)$ taken only over balls B containing x and verifying $r < 1 \wedge |q|^{-1}$. Split \mathcal{R}^n into rings $R_k = \{x : \sqrt{k-1} \leq |x| < \sqrt{k}\}$, $k = 1, 2, \dots$. The width of R_k is no larger than $1/\sqrt{k}$, and so the Gaussian density is of constant order of magnitude in each R_k . Using Lebesgue measure arguments, one can easily estimate the $L^p(d\gamma)$ norm of \mathcal{M}_0f in R_k in terms of the $L^p(d\gamma)$ norm of f in $\bigcup\{R_{k'} : |k' - k| \leq C\}$. This takes care of small balls.

Consider now balls B with $r \geq 1 \wedge |q|^{-1}$. To begin with observe that the case $|q| < 2$ is simple, since then $\gamma(B) \geq C$ and thus

$$\mathcal{A}f(B) \leq C \int f d\gamma \leq C \|f\|_{L^p(d\gamma)}.$$

The corresponding part of $\mathcal{M}f$ thus satisfies the $L^p(d\gamma)$ estimate.

It remains to consider $\tilde{\mathcal{M}}f(x) = \sup \mathcal{A}f(B)$, the supremum taken over balls B containing x and with the property that $r \geq |q|^{-1}$ and $|q| \geq 2$. Let B be such a ball, and observe that it satisfies the hypotheses of Lemma 1.

For each $\rho \geq 1$ such that S_ρ^{n-1} intersects B , let $y_\rho \in S_\rho^{n-1} \cap \partial B$ be such that $|y_\rho - x| = \sup_{z \in B \cap S_\rho^{n-1}} |z - x|$. Write $x' = x/|x|$.

For each $z' \in S^{n-1}$ such that $\rho z' \in B$ we have

$$(2) \quad \begin{aligned} |x' - z'| &= \frac{1}{\rho} |\rho x' - \rho z'| \\ &\leq \frac{1}{\rho} [|x - \rho z'| + |\rho - |x||] \\ &\leq \frac{2}{\rho} |y_\rho - x|, \end{aligned}$$

and trivially $|x' - z'| \leq 2$.

Because of (2) and the definition of \mathcal{M}^e ,

$$\begin{aligned}
(3) \quad \mathcal{A}f(B) &= \int_{|q|}^{|q|+2r} \frac{1}{\gamma(B)} \int_{S^{n-1}} \chi_B(\rho z') f(\rho z') d\sigma(z') \rho^{n-1} e^{-\rho^2} d\rho \\
&\leq \int_{|q|}^{|q|+2r} \frac{1}{\gamma(B)} \int_{|z'-x'| \leq 2 \left(1 \wedge \frac{|y_\rho - x|}{\rho}\right)} f(\rho z') d\sigma(z') \rho^{n-1} e^{-\rho^2} d\rho \\
&\leq C \int_{|q|}^{|q|+2r} \frac{\left\{1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{n-1}\right\}}{\gamma(B)} \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho \\
&\leq C \int_{|q|}^{|q|+2r} |q| e^{|q|^2} \left\{1 \vee \left(\frac{|q|(\rho \vee |x| - |q|)}{|x - y_\rho|^2}\right)^{\frac{n-1}{2}}\right\} \left\{1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{n-1}\right\} \\
&\quad \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho,
\end{aligned}$$

where we applied Lemma 1 with $y = y_\rho$ to get the last inequality.

Write $M = \rho \vee |x|$ and $m = \rho \wedge |x|$, so that $|q| \leq m \leq M$.

Lemma 2. For $|q| < \rho < |q| + 2r$ and some C ,

$$\begin{aligned}
&e^{|q|^2} \left\{1 \vee \left(\frac{|q|(M - |q|)}{|x - y_\rho|^2}\right)^{\frac{n-1}{2}}\right\} \left\{1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{n-1}\right\} \\
&\leq C e^{m^2} \left(\frac{1}{m^2} \vee \frac{M - m}{m}\right)^{\frac{n-1}{2}}.
\end{aligned}$$

Assuming this lemma for the moment, we conclude from (3) that

$$\mathcal{A}f(B) \leq C \int_1^\infty m e^{m^2} \left(\frac{1}{m^2} \vee \frac{M - m}{m}\right)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho.$$

We split this integral into five integrals taken over the following intervals:

$$\begin{aligned}
I_1 &= \left[1, \frac{|x|}{2}\right], \quad I_2 = \left(\frac{|x|}{2}, |x| - \frac{1}{|x|}\right], \quad I_3 = \left(|x| - \frac{1}{|x|}, |x| + \frac{1}{|x|}\right], \\
I_4 &= \left(|x| + \frac{1}{|x|}, \frac{5}{4}|x|\right], \quad I_5 = \left(\frac{5}{4}|x|, +\infty\right).
\end{aligned}$$

Let for $i = 1, \dots, 5$

$$\mathcal{M}_i f(x) = \int_{I_i} m e^{m^2} \left(\frac{1}{m^2} \vee \frac{M - m}{m}\right)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho.$$

Then $\tilde{\mathcal{M}}f \leq C \sum_1^5 \mathcal{M}_i f$.

Bound for $\mathcal{M}_1 f(x)$. One finds that

$$\mathcal{M}_1 f(x) \leq |x|^n \int_1^{\frac{|x|}{2}} \mathcal{M}^e f(\rho x') d\rho.$$

Hölder's inequality and the $L^p(d\sigma)$ boundedness of \mathcal{M}^e imply

$$\begin{aligned} \|\mathcal{M}_1 f\|_{L^p(d\gamma)}^p &\leq \int_1^{+\infty} \int_{S^{n-1}} \left(s^n \int_1^{\frac{s}{2}} \mathcal{M}^e f(\rho x') d\rho \right)^p d\sigma(x') s^{n-1} e^{-s^2} ds \\ &\leq \int_1^{+\infty} \int_{S^{n-1}} \int_1^{\frac{s}{2}} |\mathcal{M}^e f(\rho x')|^p \rho^{n-1} e^{-\rho^2} d\rho \left(\int_1^{\frac{s}{2}} \rho^{-(n-1)\frac{p'}{p}} e^{\frac{p'}{p}\rho^2} d\rho \right)^{\frac{p}{p'}} d\sigma(x') s^{n-1} e^{-s^2} ds \\ &\leq \left(\int_1^{+\infty} s^C e^{-\frac{3}{4}s^2} ds \right) \|f\|_{L^p(d\gamma)}^p \leq C \|f\|_{L^p(d\gamma)}^p. \end{aligned}$$

Bound for $\mathcal{M}_2 f(x)$. Making the change of variable $\rho = |x| - \frac{t}{|x|}$, we get

$$\begin{aligned} \mathcal{M}_2 f(x) &\leq |x|^{\frac{n+1}{2}} \int_{|x|/2}^{|x| - \frac{1}{|x|}} (|x| - \rho)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x') d\rho \\ &\leq \int_1^{|x|^2/2} t^{\frac{n-1}{2}} \mathcal{M}^e f\left(\left(|x| - \frac{t}{|x|}\right)x'\right) dt. \end{aligned}$$

From Minkowski's integral inequality and the $L^p(d\sigma)$ boundedness of \mathcal{M}^e , we obtain

$$\begin{aligned} \|\mathcal{M}_2 f\|_{L^p(d\gamma)} &\leq \int_1^{+\infty} t^{\frac{n-1}{2}} \left\| \mathcal{M}^e f\left(\left(|x| - \frac{t}{|x|}\right)x'\right) \chi_{\{1 \leq t \leq \frac{|x|^2}{2}\}} \right\|_{L^p(d\gamma)} dt \\ &= \int_1^{+\infty} t^{\frac{n-1}{2}} \left[\int_{S^{n-1}} \int_{\sqrt{2t}}^{+\infty} f\left(\left(s - \frac{t}{s}\right)x'\right)^p s^{n-1} e^{-s^2} ds d\sigma(x') \right]^{\frac{1}{p}} dt. \end{aligned}$$

We now make the change of variables $s \rightarrow \rho = s - t/s$, observing that $s \leq 2\rho$ and $-s^2 = -\rho^2 - 2t + t^2/s^2 \leq -\rho^2 - 3t/2$ and $d\rho/ds \geq 1$. Thus

$$\begin{aligned} \|\mathcal{M}_2 f\|_{L^p(d\gamma)} &\leq C \int_1^{+\infty} t^{\frac{n-1}{2}} \left[\int_{S^{n-1}} \int_{\sqrt{t/2}}^{+\infty} |f(\rho x')|^p \rho^{n-1} e^{-\rho^2} e^{-3t/2} d\rho d\sigma(x') \right]^{\frac{1}{p}} dt \\ &\leq C \|f\|_{L^p(d\gamma)} \left(\int_1^{+\infty} t^{\frac{n-1}{2}} e^{-\frac{3t}{2p}} dt \right) \leq C \|f\|_{L^p(d\gamma)}. \end{aligned}$$

Bound for $\mathcal{M}_3 f(x)$. Let $d\mu = \rho^{n-1} e^{-\rho^2} d\rho$ in \mathcal{R}_+ . We have

$$\begin{aligned} \mathcal{M}_3 f(x) &\leq C|x| \int_{|x|-1/|x|}^{|x|+1/|x|} \mathcal{M}^e f(\rho x') d\rho \\ &\leq C(\mu(|x| - 1/|x|, |x| + 1/|x|))^{-1} \int_{|x|-1/|x|}^{|x|+1/|x|} \mathcal{M}^e f(\rho x') d\mu(\rho). \end{aligned}$$

Let \mathcal{M}^μ denote the one-dimensional centered maximal operator defined in terms of μ , acting in the ρ variable. Then

$$\mathcal{M}_3 f(x) \leq C \mathcal{M}^\mu \mathcal{M}^e f(|x|x').$$

But \mathcal{M}^μ is known to be bounded on $L^p(d\mu)$; see [1] or [2]. The $L^p(d\gamma)$ boundedness of \mathcal{M}_3 follows.

Bound for $\mathcal{M}_4f(x)$. Making the change of variable $\rho = |x| + \frac{t}{|x|}$, we have

$$\begin{aligned} \mathcal{M}_4f(x) &\leq C|x|^{\frac{n+1}{2}} e^{|x|^2} \int_{|x|+\frac{1}{|x|}}^{\frac{5}{4}|x|} (\rho - |x|)^{\frac{n-1}{2}} \mathcal{M}^e f(\rho x') e^{-\rho^2} d\rho \\ &\leq C \int_1^{\frac{|x|^2}{4}} t^{\frac{n-1}{2}} \mathcal{M}^e f\left(\left(|x| + \frac{t}{|x|}\right)x'\right) e^{-2t} e^{-\frac{t^2}{|x|^2}} dt. \end{aligned}$$

Minkowski's integral inequality implies

$$\|\mathcal{M}_4f\|_{L^p(d\gamma)} \leq C \int_1^{+\infty} t^{\frac{n-1}{2}} \left\| \mathcal{M}^e f\left(\left(|x| + \frac{t}{|x|}\right)x'\right) e^{-\frac{t^2}{|x|^2}} \chi_{\{1 \leq t \leq \frac{|x|^2}{4}\}} \right\|_{L^p(d\gamma)} e^{-2t} dt.$$

But \mathcal{M}^e is bounded on $L^p(d\sigma)$, so that

$$\begin{aligned} &\left\| \mathcal{M}^e f\left(\left(|x| + \frac{t}{|x|}\right)x'\right) e^{-\frac{t^2}{|x|^2}} \chi_{\{1 \leq t \leq \frac{|x|^2}{4}\}} \right\|_{L^p(d\gamma)}^p \\ &\leq C \int_{2\sqrt{t}}^{\infty} \int_{S^{n-1}} |f\left(\left(s + \frac{t}{s}\right)x'\right) e^{-\frac{t^2}{s^2}}|^p d\sigma(x') s^{n-1} e^{-s^2} ds. \end{aligned}$$

Almost as in the case of \mathcal{M}_2 , we make the change of variable $\rho = s + t/s$ and observe that $s \leq \rho$ and $-s^2 = -\rho^2 + 2t + t^2/s^2$ and $d\rho/ds \geq 1/2$. Since $e^{-\rho t^2/s^2} e^{t^2/s^2} < 1$, it follows that the above double integral is at most

$$C \int_{S^{n-1}} \int_1^{+\infty} |f(\rho x')|^p \rho^{n-1} e^{-\rho^2} d\rho d\sigma(x') e^{2t} \leq C \|f\|_{L^p(d\gamma)}^p e^{2t}.$$

Thus

$$\|\mathcal{M}_4f\|_{L^p(d\gamma)} \leq C \int_1^{+\infty} t^{\frac{n-1}{2}} \|f\|_{L^p(d\gamma)} e^{\frac{2t}{p}} e^{-2t} dt \leq C \|f\|_{L^p(d\gamma)}.$$

Bound for $\mathcal{M}_5f(x)$. Observe that

$$\mathcal{M}_5f(x) \leq |x|^{\frac{3-n}{2}} e^{|x|^2} \int_{\frac{5}{4}|x|}^{+\infty} \mathcal{M}^e f(\rho x') \rho^{\frac{n-1}{2}} \rho^{n-1} e^{-\rho^2} d\rho.$$

We take the L^p norm and then apply Hölder's inequality, getting

$$\begin{aligned} \|\mathcal{M}_5f\|_{L^p(d\gamma)}^p &\leq \int_1^{+\infty} \int_{S^{n-1}} \frac{e^{ps^2}}{s^{p\frac{n-3}{2}}} \left(\int_{\frac{5s}{4}}^{+\infty} \mathcal{M}^e f(\rho x') \rho^{\frac{3(n-1)}{2}} e^{-\rho^2} d\rho \right)^p d\sigma(x') s^{n-1} e^{-s^2} ds \\ &\leq \int_1^{+\infty} \int_{S^{n-1}} \frac{e^{ps^2}}{s^{p\frac{n-3}{2}}} \int_0^{+\infty} |\mathcal{M}^e f(\rho x')|^p \rho^{n-1} e^{-\rho^2} d\rho \left(\int_{\frac{5s}{4}}^{+\infty} \rho^{(\frac{p}{2}+1)(n-1)} e^{-\rho^2} d\rho \right)^{\frac{p}{p'}} d\sigma(x') s^{n-1} e^{-s^2} ds \\ &\leq \|f\|_{L^p(d\gamma)}^p \left(\int_1^{+\infty} s^C e^{(p-1)s^2} e^{-(p-1)(\frac{5}{4}s)^2} ds \right) \\ &\leq C \|f\|_{L^p(d\gamma)}^p. \end{aligned}$$

To finish the proof of Theorem 1, it now only remains to prove the two lemmas.

Proof of Lemma 1. Consider the hyperplane orthogonal to q whose distance from the origin is $|q| + t$, with $1/(2|q|) < t < 1/|q|$. Its intersection with B is an $(n - 1)$ -dimensional ball whose radius is at least $C\sqrt{rt} \geq C\sqrt{r/|q|}$. Integrating the Gaussian density first along this $(n - 1)$ -dimensional ball and then in t , we get

$$\gamma(B) \geq \int_{1/(2|q|)}^{1/|q|} e^{-(|q|+t)^2} dt \int_{|v| < C\sqrt{r/|q|}} e^{-|v|^2} dv,$$

where v is an $(n - 1)$ -dimensional variable. The inner integral here is at least $C \min(1, (r/|q|)^{(n-1)/2})$, and $e^{-(|q|+t)^2} \geq Ce^{-|q|^2}$ for these t ; therefore

$$(4) \quad \gamma(B) \geq C \frac{e^{-|q|^2}}{|q|} \left(1 \wedge \left(\frac{r}{|q|} \right)^{\frac{n-1}{2}} \right).$$

To estimate r from below, we let z be the center of B and w the projection of x onto the line passing through 0 , q and z . Write $h = |x - w|$ and $a = |w - q|$. Applying the Pythagorean Theorem twice, we get

$$|x - z|^2 - (r - a)^2 = h^2 = |x - q|^2 - a^2.$$

Since $|x - z| \leq r$, we conclude that $2ar \geq |x - q|^2$. Clearly $a \leq |x| - |q|$ so that

$$r \geq \frac{|x - q|^2}{2(|x| - |q|)} \geq \frac{|x - q|^2}{2(|x| \vee |y| - |q|)}.$$

Since x and y are arbitrary points of B , the same argument also implies

$$r \geq \frac{|y - q|^2}{2(|x| \vee |y| - |q|)}.$$

From the triangle inequality we conclude that $2|x - q| \vee |y - q| \geq |x - y|$, and so

$$r \geq \frac{|x - y|^2}{8(|x| \vee |y| - |q|)}.$$

Combining this with (4), we obtain the inequality of Lemma 1.

Proof of Lemma 2. We write LHS for the left-hand side of the inequality to be proved. Assume first that

$$(5) \quad \left(\frac{|q|(M - |q|)}{|x - y_\rho|^2} \right)^{\frac{n-1}{2}} \leq 1.$$

Then $\text{LHS} \leq e^{|q|^2} (|x - y_\rho|/\rho)^{n-1}$. The angles at q of the triangles $0qx$ and $0qy_\rho$ are obtuse, so that $|x|^2 \geq |q|^2 + |x - q|^2$ and $|y_\rho|^2 \geq |q|^2 + |y_\rho - q|^2$. But $|x - y_\rho| \leq |x - q| + |y_\rho - q|$, and this implies

$$\begin{aligned} |x - y_\rho|^2 &\leq 4 \max(|x - q|^2, |y_\rho - q|^2) \\ &\leq 4 \max(|x|^2 - |q|^2, |y_\rho|^2 - |q|^2) = 4(M^2 - |q|^2). \end{aligned}$$

If $|x| \leq 2\rho$, this last quantity is at most $16\rho(M - |q|)$, and then

$$(6) \quad \text{LHS} \leq Ce^{|q|^2} \left(\frac{M - |q|}{\rho} \right)^{\frac{n-1}{2}}.$$

In the contrary case $|x| > 2\rho$, we simply observe that $\text{LHS} \leq Ce^{|q|^2}$ whereas the right-hand side is at least Ce^{m^2} . This case of the lemma is thus trivial.

Assume now that (5) is false. Then

$$\text{LHS} \leq e^{|q|^2} \frac{(|q|(M - |q|))^{\frac{n-1}{2}}}{\rho^{n-1}}$$

and we arrive again at (6).

It thus only remains to see that (6) implies Lemma 2. This would follow from the estimate

$$(7) \quad e^{|q|^2 - m^2} (M - |q|)^{\frac{n-1}{2}} \leq C((1/m) \vee (M - m))^{\frac{n-1}{2}}.$$

To prove (7), we use the fact that

$$(M - |q|)^{\frac{n-1}{2}} \leq C \left((M - m)^{\frac{n-1}{2}} + (m - |q|)^{\frac{n-1}{2}} \right)$$

and when $m - |q| > 1/m$ also

$$e^{|q|^2 - m^2} = e^{-(m-|q|)(m+|q|)} \leq \frac{C}{(m - |q|)^{\frac{n-1}{2}} m^{\frac{n-1}{2}}}.$$

Now (7) and Lemma 2 follow.

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