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ON SOME INEQUALITIES INVOLVING THE ZEROS AND WEIGHTED L^p NORMS OF POLYNOMIALS

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Abstract. Using Parseval's identity and the Hardy-Littlewood-Pólya inequality on the maximal decreasing rearrangement, we establish some sharp inequalities involving the weighted L^p norm and the zeros of polynomials.

1. Introduction

Let $\{z_j\}$ be the zeros (counting the multiplicity) of the polynomial

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$$

and let

$$\|p\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 dt \right)^{\frac{1}{2}}.$$

Recently, Kroo and Pritsker [1, Theorem 2.4] proved

Theorem A.

$$\prod_{j=1}^{n} (1 + |z_j|^2) \le 2^{n-1} \|p\|_2^2. \tag{1.1}$$

The inequality is best possible and the equality is achieved if and only if

$$p(z) = z^n + a, \text{ where } |a| = 1.$$

Moreover

$$\|p\|_2^2 \le 2^{-n} \binom{2n}{n} \prod_{j=1}^{n} (1 + |z_j|^2). \tag{1.2}$$

The inequality is best possible and the equality is achieved if and only if

$$p(z) = (z + a)^n, \text{ where } |a| = 1.$$

The main goal of this note is to establish the following generalization of (1.2).

Theorem 1. *Let $w(t)$ be a non-negative weight function defined on $[-\pi, \pi]$ with the following properties:*

(1) *$w(-t) = w(t)$,*

(2) *$w(t)$ is monotonically decreasing on $[0, \pi]$, and*

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(3) *$w(t)$ is integrable on $[-\pi, \pi]$.*

Then, for $c \geq 2$,

$$\|p\|_c \leq 2^{-\frac{n}{2}} \prod_{j=1}^{n} \left(1 + |z_j|^2\right)^{\frac{1}{2}} \|P^*\|_c,$$

where $P^*(z) = (1+z)^n$ and $\|p\|_c = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |p(e^{it})|^c w(t) dt\right)^{\frac{1}{c}}$.

Since the letter p is reserved for polynomials, to avoid unnecessary confusion, the letter c takes the place of the standard letter p for the notation of the L^p norm.

The method used in [1] does not seem applicable in dealing with the weighted L^p norm, whence a completely different approach is employed in this work. In Section 3, we will give a very different proof of (1.1) based on a simple property of the Blaschke product and the Parseval identity.

2. Proof of Theorem 1

One key ingredient of the proof is based on the following extremal property of the decreasing symmetric rearrangements of non-negative functions (see [2, p. 278] for the definition of the symmetric rearrangement of a function).

Lemma 2.1. *Let g_j be continuous and non-negative on the interval $[-a, a]$ and let g_j^* be the symmetrical rearrangement of g_j. Then, for any measurable set $E \subset [-a, a]$ of measure $m(E) = 2c$ with $c \leq a$,*

$$\int_E \prod g_j(x) dx \leq \int_{-c}^{c} \prod g_j^*(x) dx.$$

The result is a slight generalization of Theorem 378 in [2, p. 278]. For the reader's convenience we briefly outline the proof.

Proof. Let $E_j \subset [-a, a]$, $j = 1, 2, \ldots, n$, be measurable and let $K_{E_j}(x)$ be the characteristic functions of E_j and let $E^* = [-c, c]$ and $E_j^* = [-c_j, c_j]$, where $2c_j = m(E_j)$. Then, the decreasing symmetric rearragement of $K_{E_j}(x)$ is $K_{E_j^*}(x)$. Clearly, we have

$$\begin{aligned} \int_E \prod K_{E_j}(x) dx &= m\left(\bigcap_{j=1}^{n} E_j \cap E\right) \\ &\leq \min_j m(E \cap E_j) \\ &\leq \min_j m(E^* \cap E_j^*) \\ &= \int_{E^*} \prod K_{E_j^*}(x) dx. \end{aligned} \tag{2.1}$$

Let $s(x) > 0$ be a simple function. Then (see Section 10.13 of [2]) we can represent $s(x)$ in the form

$$s(x) = a_1 K_{E_1}(x) + a_2 K_{E_2}(x) + \cdots + a_m K_{E_m}(x)$$

with $E_i + 1 \subset E_i$, so that one has

$$s^*(x) = a_1 K_{E_1^*}(x) + a_2 K_{E_2^*}(x) + \cdots + a_m K_{E_m^*}(x),$$

where $a_j > 0$ and $j = 1, 2, \dots, m$. For the simple functions $s_j(x)$, the inequality

$$\int_E \prod s_j(x)dx \le \int_{E^*} \prod s_j^*(x)dx$$

follows from a linear combination of (2.1).

Finally, we establish the lemma in the general case by approximating g in terms of simple functions. $\square$

Corollary 1. *Let* $p(z) = \prod_{j=1}^n (z - z_j)$ *and* $p^*(z) = \prod_{j=1}^n (z + x_j)$, *where* $x_j = |z_j|$. *Then*

$$\int_{-\pi}^{\pi} |p(e^{it})|^c w(t)dt \le \int_{-\pi}^{\pi} |p^*(e^{it})|^c w(t)dt$$

for all $c \ge 0$.

Lemma 2.2. *Let* $f \in L^1[0, \pi]$.

(a) *Suppose f is non-negative and monotonically decreasing on* $[0, \pi]$. *Then*

$$\int_0^{\pi} f(t) \cos t \, dt > 0 \, .$$

(b) *Let $c \ge 2$ and let f be as in* (a). *Suppose, for* $0 \le x < \infty$,

$$F(x) = \left(1 + x^2\right)^{-\frac{c}{2}} \int_0^{\pi} |x + e^{it}|^c f(t)dt \, .$$

Then $F(x)$ achieves its absolute maximum at $x = 1$.

Proof. (a) We note that, with the replacement of t by $t + \frac{\pi}{2}$, the integral becomes

$$-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\left(t + \frac{\pi}{2}\right) \sin t \, dt = \int_0^{\frac{\pi}{2}} \left(f\left(\frac{\pi}{2} - t\right) - f\left(\frac{\pi}{2} + t\right)\right) \sin t \, dt \, ,$$

which, from the monotonicity assumption on f, is clearly non-negative.

(b) A simple calculation shows that

$$F'(x) = \frac{c(1 - x^2)}{(1 + x^2)^2} \int_0^{\pi} g(t) \cos t \, dt \, , \tag{2.2}$$

where $g(t) = \left(1 + \frac{2x}{1+x^2} \cos t\right)^{\frac{c}{2}-1} f(t)$.

We observe that, for $c \ge 2$, the function $g(t)$ is monotonically decreasing on $[0, \pi]$ for all $0 \le x < \infty$. Hence, from part (a), we conclude that the integral in (2.2) is positive. That the absolute maximum of F occurs at $x = 1$ now follows from the fact that $F'(x) = 0$ only if $x = 1$, $F'(x) > 0$ for $0 \le x < 1$, and $F'(x) < 0$ for $x > 1$. $\square$

We now come to the proof of Theorem 1.

Proof. For a given $p(z)$, we consider the quantity

$$v(p) = \int_{-\pi}^{\pi} \frac{|p(e^{it})|^c \, w(t) \, dt}{\prod_{j=1}^n (1 + |z_j|^2)^{\frac{c}{2}}} \, .$$

Using Corollary 1 and then applying Lemma 2.2(b) inductively on each $|z_j|$, we deduce that

$$v(p) \le v(p^*) \le v(P^*) \, ,$$

where

$$v(P^*) = 2^{\frac{-nc}{2}} \int_{-\pi}^{\pi} |(e^{it}+1)^n|^c w(t)dt\,.$$

This establishes Theorem 1. □

3. An alternative proof of (1.1)

The proof is essentially based on the following:

Lemma 3.1. *Let A be a subset of $N = \{1, 2, \dots, n\}$ and $\tilde{A}$ be the complement $N \backslash A$. Define*

$$d(A) = \prod_{j\in A} |z_j| \text{ and } d(\tilde{A}) = \prod_{j\in \tilde{A}} |z_j|\,.$$

Then

$$d^2(A) + d^2(\tilde{A}) \le \|p\|_2^2\,.$$

Proof. Let $B(z, A) = \prod_{j\in A} \frac{z-z_j}{1-z\bar{z}_j}$ (we note that $B(z, A)$ has a removable singularity at z_j in case $|z_j| = 1$). Then,

$$|B(z, A)| = 1 \text{ if } |z| = 1\,, \tag{3.1}$$

and we can write

$$p(z) = B(z, A)P_A(z)\,, \tag{3.2}$$

where

$$P_A(z) = \prod_{j\in A} (1 - z\bar{z}_j) \prod_{j\in \tilde{A}} (z - z_j) = d_0 z^n + \cdots + d_n \tag{3.3}$$

with $d_0 = \prod_{j\in A} (-\bar{z}_j)$ and $d_n = \prod_{j\in \tilde{A}} (-z_j)$. From (3.1), (3.2), and Parseval's identity,

$$\|p\|_2^2 = \|P_A\|_2^2 = \sum_{j=0}^{n} |d_j|^2 \ge d^2(A) + d^2(\tilde{A})\,. \tag{3.4}$$

□

We now prove (1.1).

Proof of (1.1). We first make a simple and crucial observation about the product $\prod_{j=1}^{n} (1 + |z_j|^2)$:

(a) Its summand consists of 2^n terms (some of them may be repeated if the z_j are not distinct) and

(b) it can be rewritten as

$$\prod_{j=1}^{n} (1 + |z_j|^2) = \frac{1}{2} \sum_{i=1}^{2^n} d^2(A_i) + d^2(\tilde{A}_i)\,, \tag{3.5}$$

where A_i runs through all the subsets of N.

Applying (3.4) to (3.5), the desired conclusion follows. □

We observe that if equality holds in (1.1), then the equality holds for (3.4) for every subset A of N. In particular, it implies that P_A consists of two terms, from which one easily deduces that

$$p(z) = z^n + a\,.$$

A direct computation shows that $|a| = 1$.

From the inequality

$$\left(\frac{a^t + b^t}{2}\right)^{\frac{1}{t}} \le \left(\frac{a^s + b^s}{2}\right)^{\frac{1}{s}} \text{ if } s \ge t \text{ and } a, b \ge 0\,,$$

we deduce the following:

Corollary 2. *Let* $0 < c \le 2$. *Then*

$$\prod_{j=1}^{n} (1 + |z_j|^c) \le 2^{n-\frac{c}{2}} \|p\|_2^c\,.$$

References

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