

SKEW PRODUCTS OVER TRANSLATIONS ON \mathbf{T}^d , $d \geq 2$

BASSAM R. FAYAD

(Communicated by Michael Handel)

ABSTRACT. We give an example on \mathbf{T}^4 of a minimal translation R_α and a real analytic function φ , such that the circle-valued skew product extension of R_α by φ has a Lebesgue spectrum in the orthocomplement of the space of eigenfunctions.

1. INTRODUCTION

Let $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, $d \geq 1$, and denote by μ the Haar measure on \mathbf{T}^d . Given a minimal translation R_α on \mathbf{T}^d and a real function φ on \mathbf{R}^d , smooth and \mathbf{Z}^d -periodic, we will consider the map (skew product) $S_{\alpha,\varphi}$:

$$\begin{aligned} \mathbf{T}^d \times \mathbf{T}^1 &\longrightarrow \mathbf{T}^d \times \mathbf{T}^1 \\ (z, s) &\longrightarrow (z + \alpha, s + \varphi(z) \bmod 1). \end{aligned}$$

When $d = 1$, it was proved in [1] that if φ is absolutely continuous, the spectrum of the skew product is singular with respect to the Lebesgue measure.¹ The proof in [1] is based on an improved Denjoy-Koksma inequality implying that $S_{\alpha,\varphi}^{q_n}$ tends uniformly to the identity map on \mathbf{T}^2 as q_n runs through the sequence of denominators of the convergents of α . Hence, $S_{\alpha,\varphi}$ is said to be *rigid* and one easily deduces that its spectrum is purely singular. Here, we want to prove that this is no longer true when $d \geq 2$; namely, we give an example of a skew product over \mathbf{T}^2 with a real analytic-function that is nonrigid (it actually displays “mixing in the fibers”) and we derive from it an example of a skew product over \mathbf{T}^4 that has countable Lebesgue spectrum in the orthocomplement of the space of eigenfunctions. Our argument is essentially based on the construction by J-C. Yoccoz [4] of a minimal translation on \mathbf{T}^2 and a real-analytic complex function φ of \mathbf{T}^2 that give a counterexample, in dimension 2, to the Denjoy-Koksma inequality valid for functions over the circle.

Received by the editors May 30, 2000.

2000 *Mathematics Subject Classification.* Primary 37A25, 37E99.

¹In [1], the authors, P. Gabriel, M. Lemanczyk and P. Liardet, consider skew products over irrational rotations with *circle-valued* functions. They prove that when φ is absolutely continuous with degree 0, the corresponding skew product is rigid. In contrast, if the degree of φ is not zero, and φ' is of bounded variation, the skew product has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions of R_α (see [3]), the best known example being $(x, y) \rightarrow (x + \alpha, x + y)$.

2. MIXING IN THE FIBERS

Take α and α' rationally independent such that the denominators of their convergents, q_n and q'_n , satisfy the following, for $n \geq n_0$:

$$(1) \quad q_n \geq e^{3nq'_{n-1}},$$

$$(2) \quad q'_n \geq e^{3nq_n},$$

and let $S_{\alpha, \alpha', \varphi}$ be the skew product constructed from $R_{\alpha, \alpha'}$ and the real analytic function

$$(3) \quad \varphi(x, y) = 1 + \operatorname{Re} \left(\sum_{j=0}^{\infty} \frac{e^{i2\pi q_j x}}{e^{q_j}} \right) + \operatorname{Re} \left(\sum_{j=0}^{\infty} \frac{e^{i2\pi q'_j y}}{e^{q'_j}} \right).$$

For $l \in \mathbf{Z}$ and $k \in \mathbf{Z}^2$, we denote by $\psi_{k,l} \in L^2(\mathbf{T}^3, \mathbf{C})$ the character $e^{i2\pi \langle k, z \rangle} e^{i2\pi l s}$, where $\langle \cdot, \cdot \rangle$ is the scalar product on \mathbf{R}^2 . In this note we will prove the following estimate:

Proposition 1 (Mixing in the fibers). *Assume $l \neq 0$. Then given any $\epsilon > 0$, we have*

$$\begin{aligned} (\psi_{k,l} \circ S_{\alpha, \alpha', \varphi}^m / \psi_{k,l}) &:= \int_{\mathbf{T}^3} \psi_{k,l} \circ S_{\alpha, \alpha', \varphi}^m(x, y, s) \overline{\psi_{k,l}(x, y, s)} dx dy ds \\ &= O\left(\frac{1}{m^{\frac{1}{3}-\epsilon}}\right) \end{aligned}$$

when m goes to infinity.

This *mixing in the fibers* already eliminates the rigidity encountered when $d = 1$. Before we prove the proposition we show how the estimation of the rate of “mixing” enclosed in it enables us to construct a skew product with a Lebesgue component in its spectrum:

Assume $R_{\hat{\alpha}} = R_{\alpha_1, \alpha'_1, \alpha_2, \alpha'_2}$ is a minimal translation on \mathbf{T}^4 and the couples (α_i, α'_i) , $i = 1, 2$, both satisfy (1) and (2). We will denote by (x_1, x'_1, x_2, x'_2) the coordinates on \mathbf{T}^4 . Let $\varphi_1(x_1, x'_1)$ and $\varphi_2(x_2, x'_2)$ be as in (3), and define on \mathbf{T}^4 the real analytic function $\hat{\varphi} = \varphi_1 + \varphi_2$.

Define now on \mathbf{T}^5 the skew product $\hat{S} = S_{\hat{\alpha}, \hat{\varphi}}$. Let $H = L^2(\mathbf{T}^5, \mathbf{C})$ be the set of complex functions on \mathbf{T}^5 that are L^2 with respect to the Haar measure. By $U_{\hat{S}}$ we refer to the unitary operator on H associated to \hat{S} :

$$\begin{aligned} H &\longrightarrow H, \\ f &\longrightarrow f \circ \hat{S}. \end{aligned}$$

(The operator $U_{\hat{S}}$ is unitary because \hat{S} preserves the Haar measure on \mathbf{T}^5 .)

The space H decomposes under the action of $U_{\hat{S}}$ into a countable sum of *orthogonal invariant subspaces* H_l , $l \in \mathbf{Z}$, where H_l designates the subspace containing functions of the form $a(x_1, x'_1, x_2, x'_2) e^{i2\pi l s}$.

Finally, from Proposition 1, it follows that

Theorem 1. *For any $l \in \mathbf{Z}^*$, $U_{\hat{S}}$ has a Lebesgue spectrum on H_l .*

Proof. For $(k_1, k'_1, k_2, k'_2) \in \mathbf{Z}^4$, let $\xi_{k_1, k'_1, k_2, k'_2, l}$ be the character

$$e^{i2\pi(k_1 x_1 + k'_1 x'_1 + k_2 x_2 + k'_2 x'_2 + l s)}.$$

We immediately have

$$\begin{aligned} |c_m| &:= \left| (U_{\hat{S}}^m \xi_{k_1, k'_1, k_2, k'_2, l} / \xi_{k_1, k'_1, k_2, k'_2, l}) \right| \\ &= \left| \left(\psi_{k_1, k'_1, l} \circ S_{\alpha_1, \alpha'_1, \varphi_1}^m / \psi_{k_1, k'_1, l} \right) \left(\psi_{k_2, k'_2, l} \circ S_{\alpha_2, \alpha'_2, \varphi_2}^m / \psi_{k_2, k'_2, l} \right) \right| \\ &= O\left(\frac{1}{m^{\frac{2}{3}} - 2\epsilon}\right) \end{aligned}$$

when $l \neq 0$ and m goes to infinity. The latter is satisfied for any $\epsilon > 0$, hence the series $\sum |c_m|^2$ converges and the spectral measure of $\xi_{k_1, k'_1, k_2, k'_2, l}$ is absolutely continuous with respect to Lebesgue (by definition, the numbers c_m are the Fourier coefficients of this measure). Furthermore, by the so-called ‘‘purity law’’ on the unitary operators arising from cocycles, we conclude that $U_{\hat{S}}$ has a Lebesgue spectrum on H_l , when $l \neq 0$ (the purity law, see [2], states that the spectral type of $U_{\hat{S}}$ on H_l is pure, i.e. either discrete, or continuous and purely singular, or equivalent to Lebesgue). \square

Proof of Proposition 1. Assume α , α' and φ satisfy (1), (2) and (3). From now on, we will denote the Birkhoff sums of φ with respect to $R_{\alpha, \alpha'}$ by

$$\varphi_m(x, y) := \sum_{k=0}^{m-1} \varphi(R_{\alpha, \alpha'}^k(x, y)).$$

We have

$$(4) \quad \left| (\psi_{k, l} \circ S_{\alpha, \alpha', \varphi}^m / \psi_{k, l}) \right| = \left| \int_{\mathbf{T}^2} e^{i2\pi l \varphi_m(x, y)} dx dy \right|,$$

and we will derive the estimation of Proposition 1 from the large oscillations of $\varphi_m(x, y)$ for *all* integers m , large enough. We underlined the last quantifier because there lies the difference with the one-dimensional case, the oscillations of φ_m being large in one or in the other direction x and y depending on whether m is far from q_n or far from q'_n . Indeed, a direct computation on the Birkhoff sums of φ_m implies the following, where $\|\cdot\|_{C^3}$ denotes a norm on the space of real functions on \mathbf{R}^2 of class C^3 and \mathbf{Z}^2 -periodic.

Lemma 1. *For any $m \in [e^{nq_n}, e^{nq'_n}]$, we have*

$$\varphi_m(x, y) = \frac{m}{e^{q_n}} \cos(2\pi q_n x) + h^{(m)}(x) + \phi_m(y),$$

where $\phi_m(y)$ denotes the Birkhoff sums of the ‘‘ y -part’’ of φ , and where $h^{(m)}$ satisfies, for any m in the above interval, $\|h^{(m)}\|_{C^3} = O(q_n)$.

There is of course an equivalent expression when $m \in [e^{nq'_n}, e^{(n+1)q_{n+1}}]$ in which we interchange x and q_n with y and q'_n respectively.

Proof of the lemma. For any $m \in \mathbf{N}$, we have

$$\varphi_m(x, y) = \operatorname{Re} \left(\frac{\sum_{k=1}^{\infty} \frac{X(m, q_k)}{e^{q_k}} e^{i2\pi q_k x} + \sum_{k=1}^{\infty} \frac{Y(m, q'_k)}{e^{q'_k}} e^{i2\pi q'_k y}}{Y(m, q'_k) e^{q'_k} e^{i2\pi q'_k y}} \right),$$

where

$$X(m, q_k) = \frac{1 - e^{i2\pi m q_k \alpha}}{1 - e^{i2\pi q_k \alpha}}, \quad Y(m, q'_k) = \frac{1 - e^{i2\pi m q'_k \alpha'}}{1 - e^{i2\pi q'_k \alpha'}}.$$

We will need the following simple inequalities:

- (5) For all $k \in \mathbf{N}^*$, and any $m \in \mathbf{N}$, $|X(m, q_k)| \leq m$;
 (6) for $k < n$, and any $m \in \mathbf{N}$, $|X(m, q_k)| \leq q_n$;
 (7) for $m \leq e^{nq'_n}$, $|X(m, q_n) - m| = o(1)$.

Proof of (5), (6) and (7). First,

$$X(m, q_k) = \sum_{j=0}^{m-1} e^{i2\pi j q_k \alpha},$$

so the first inequality is trivial.

For the other inequalities remember that the denominators of the convergents of α satisfy

$$(8) \quad \|q_{n-1}\alpha\| \leq \|k\alpha\|, \quad \forall k < q_n,$$

and

$$(9) \quad \frac{1}{2q_n} \leq \frac{1}{q_{n-1} + q_n} \leq \|q_{n-1}\alpha\| < \frac{1}{q_n},$$

where $\|\cdot\|$ denotes the distance to the closest integer.

Next, notice that for any k and for any m

$$|X(m, q_k)| \leq \frac{2}{|1 - e^{i2\pi q_k \alpha}|}.$$

Then using the inequality $\sin(\pi u) \geq 2u$, when $0 \leq u \leq \frac{1}{2}$, we have

$$\frac{2}{|1 - e^{i2\pi q_k \alpha}|} = \frac{1}{\sin \pi \|q_k \alpha\|} \leq \frac{1}{2\|q_k \alpha\|},$$

so, if $k < n$, we have from (8) and the left-hand side in (9) that this last term is bounded by q_n . Hence, (6) is proved.

For (7), we again use

$$X(m, q_n) = \sum_{j=0}^{m-1} e^{i2\pi j q_n \alpha},$$

since $\|q_n \alpha\| \leq \frac{1}{q_{n+1}} \leq \frac{1}{3e^{nq'_n}}$, one has for $j \leq e^{nq'_n}$

$$e^{i2\pi j q_n \alpha} = 1 + O(e^{-2nq'_n}),$$

which immediately leads to (7).

Returning to the proof of the lemma, we want to find a bound, when $m \leq e^{nq'_n}$, to the C^2 norm of

$$h^{(m)}(x) = \operatorname{Re} \left(\sum_{k=1}^{\infty} \frac{X(m, q_k)}{e^{q_k}} e^{i2\pi q_k x} \right) - \frac{m}{e^{q_n}} \cos(2\pi q_n x).$$

If we consider the second derivatives of the sum above we have from (5)

$$\left| \operatorname{Re} \left(\sum_{k=n+1}^{\infty} \frac{X(m, q_k)}{e^{q_k}} (2\pi q_k)^2 e^{i2\pi q_k x} \right) \right| \leq m \sum_{k=n+1}^{\infty} \frac{(2\pi q_k)^2}{e^{q_k}},$$

which implies, since $m \leq e^{nq'_n}$, and $q_k \geq e^{3nq'_n}$ for $k \geq n+1$,

$$(10) \quad = o(1).$$

From (6), it follows that

$$(11) \quad \left| \operatorname{Re} \left(\sum_{k=1}^{n-1} \frac{X(m, q_k)}{e^{q_k}} (2\pi q_k)^2 e^{i2\pi q_k x} \right) \right| = O(q_n).$$

Finally, (7) implies that

$$\left| \frac{X(m, q_n) - m}{e^{q_n}} (2\pi q_n)^2 \right| = o(1),$$

and we obtain the required bound on the second derivative of $h^{(m)}$, which is clearly also valid for $h^{(m)}$ and its first derivative. \square

Assume now $m \in [e^{nq_n}, e^{nq'_n}]$. In light of the lemma we have just stated, our problem is reduced to majorizing, for $l \in \mathbf{Z}^*$, the absolute value of the integral

$$I_m(l) = \int_{\mathbf{T}} e^{i2\pi l \left[\frac{m}{e^{q_n}} \cos(2\pi q_n x) + h^{(m)}(x) \right]} dx,$$

with the hypothesis $\|h^{(m)}\|_{C^3} \leq q_n$.

Whenever $n \geq \frac{2}{\epsilon}$, one has for $m \in [e^{nq_n}, e^{nq'_n}]$

$$(12) \quad a_m := \frac{m}{e^{q_n}} \geq m^{1-\frac{\epsilon}{2}}.$$

With a slight abuse of notation we will write $\varphi_m(x)$ for the function $a_m \cos(2\pi q_n x) + h^{(m)}(x)$.

First, we break down the integral to avoid the zeros of $\sin(2\pi q_n x)$:

$$\begin{aligned} I_m(l) &= \sum_{k=0}^{2q_n-1} \int_{\frac{k}{2q_n}}^{\frac{k+1}{2q_n}} e^{i2\pi l \varphi_m(x)} dx \\ &= \sum_{k=0}^{2q_n-1} \int_{\frac{k}{2q_n}}^{u_k} e^{i2\pi l \varphi_m(x)} dx + \sum_{k=0}^{2q_n-1} \int_{u_k}^{v_k} e^{i2\pi l \varphi_m(x)} dx + \sum_{k=0}^{2q_n-1} \int_{v_k}^{\frac{k+1}{2q_n}} e^{i2\pi l \varphi_m(x)} dx, \end{aligned}$$

where

$$u_k = \frac{k}{2q_n} + \frac{1}{4q_n} \left(\frac{1}{m} \right)^{\frac{1}{3}}, \quad v_k = \frac{k+1}{2q_n} - \frac{1}{4q_n} \left(\frac{1}{m} \right)^{\frac{1}{3}}.$$

We then have

$$(13) \quad |I_m(l)| \leq m^{-\frac{1}{3}} + \sum_{k=0}^{2q_n-1} |I_m^k(l)|$$

where

$$I_m^k(l) := \int_{u_k}^{v_k} e^{i2\pi l \varphi_m(x)} dx, \quad k = 0, \dots, 2q_n - 1.$$

Denote

$$\bar{\varphi}_m(x) := \frac{\varphi_m(x)}{a_m} = \cos(2\pi q_n x) + \frac{h^{(m)}(x)}{a_m}.$$

(The function $\bar{\varphi}_m$ is not a Birkhoff sum.)

Notice that, for $x \in [u_k, v_k]$, $2\pi q_n x \in [k\pi + \frac{\pi}{2}m^{-\frac{1}{3}}, (k+1)\pi - \frac{\pi}{2}m^{-\frac{1}{3}}]$, which implies

$$(14) \quad |\sin(2\pi q_n x)| \geq m^{-\frac{1}{3}}.$$

Since $\|h^{(m)}\|_{C^3} \leq q_n \leq \log m$, on one hand, we have for any $m \in [e^{nq_n}, e^{nq'_n}]$:

$$(15) \quad \|\bar{\varphi}_m(x)\|_{C^2} \leq 4\pi^2 q_n^2 + q_n \leq (\log m)^2,$$

and on the other hand, using (12) and (14), we have that $\bar{\varphi}'_m(x)$ is of constant sign when $x \in [u_k, v_k]$ and

$$(16) \quad |\bar{\varphi}'_m(x)| \geq 2\pi q_n m^{-\frac{1}{3}} - \frac{\log m}{a_m} \geq m^{-\frac{1}{3}}.$$

Hence, in the expression of $I_m^k(l)$ we can make the change of variable $s = \bar{\varphi}_m(x)$, and obtain

$$|I_m^k(l)| = \left| \int_{\bar{\varphi}_m(u_k)}^{\bar{\varphi}_m(v_k)} \frac{e^{i2\pi l a_m s}}{\bar{\varphi}'_m(\bar{\varphi}_m^{-1}(s))} ds \right|.$$

If now we integrate by parts, having (16) in mind we obtain

$$(17) \quad |I_m^k(l)| \leq m^{\frac{1}{3}} \pi l a_m + \frac{1}{2\pi l a_m} \left| \int_{\bar{\varphi}_m(u_k)}^{\bar{\varphi}_m(v_k)} \frac{\bar{\varphi}_m''}{[\bar{\varphi}'_m]^3}(\bar{\varphi}_m^{-1}(s)) e^{i2\pi l a_m s} ds \right|.$$

Going back to the initial variable we have

$$\begin{aligned} \left| \int_{\bar{\varphi}_m(u_k)}^{\bar{\varphi}_m(v_k)} \frac{\bar{\varphi}_m''}{[\bar{\varphi}'_m]^3}(\bar{\varphi}_m^{-1}(s)) e^{i2\pi l a_m s} ds \right| &= \left| \int_{u_k}^{v_k} \frac{\bar{\varphi}_m''(x)}{[\bar{\varphi}'_m]^2(x)} e^{i2\pi l a_m \bar{\varphi}_m(x)} dx \right| \\ &\leq \frac{1}{2q_n} m^{\frac{2}{3}} \|\bar{\varphi}_m\|_{C^2} \\ &\leq \frac{1}{2q_n} m^{\frac{2}{3}} [\log m]^2. \end{aligned}$$

Hence, (17) becomes

$$|I_m^k(l)| \leq \frac{m^{\frac{1}{3}}}{\pi l a_m} + \frac{1}{4\pi l a_m q_n} m^{\frac{2}{3}} [\log m]^2.$$

On the right-hand side of this inequality, the second term is clearly dominant and (12) implies

$$|I_m^k(l)| \leq \frac{1}{l q_n} m^{-\frac{1}{3} + \frac{\epsilon}{2}} [\log m]^2 \leq \frac{1}{q_n} m^{-\frac{1}{3} + \epsilon},$$

for any $m \in [e^{nq_n}, e^{nq'_n}]$, n large enough. Coming back to (13), we obtain

$$|I_m(l)| \leq m^{-\frac{1}{3}} + 2m^{-\frac{1}{3} + \epsilon} \leq 3m^{-\frac{1}{3} + \epsilon}.$$

When $m \in [e^{nq'_n}, e^{(n+1)q_{n+1}}]$, we proceed in the same way integrating along y in (4). Proposition 1 is hence proved. \square

ACKNOWLEDGMENTS

I wish to thank J-P. Thouvenot for his attention to this note, as well as the referee for useful comments.

REFERENCES

- [1] P. Gabriel, M. Lemańczyk, and P. Liardet. Ensemble d'invariants pour les produits croisés de Anzai. *Mémoire de la Société Mathématique de France. Nouvelle Série*, **47**, (1991), 102 pp. MR **93b**:28042
- [2] Henry Helson. Cocycles on the circle. *J. Operator Theory*, **16** (1986), 189-199. MR **88f**:22020
- [3] A. Iwanik, M. Lemańczyk, and D. Rudolph. Absolutely continuous cocycles over irrational rotations. *Israel J. Math.*, **83**, (1993), p. 73–95. MR **94i**:58108
- [4] J-C. Yoccoz. *Petits diviseurs en dimension 1*. Astérisque (1982), Appendix 1.

DEPARTMENT OF MATHEMATICS, MCALLISTER BUILDING, PENN STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802-6401

E-mail address: `fayadb@math.psu.edu`