

## ON THE COHOMOLOGY OF GENERALIZED HOMOGENEOUS SPACES

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ABSTRACT. We observe that work of Gugenheim and May on the cohomology of classical homogeneous spaces  $G/H$  of Lie groups applies verbatim to the calculation of the cohomology of generalized homogeneous spaces  $G/H$ , where  $G$  is a finite loop space or a  $p$ -compact group and  $H$  is a “subgroup” in the homotopical sense.

We are interested in the cohomology  $H^*(G/H; R)$  of a generalized homogeneous space  $G/H$  with coefficients in a commutative Noetherian ring  $R$ . Here  $G$  is a “finite loop space” and  $H$  is a “subgroup”. More precisely,  $G$  and  $H$  are homotopy equivalent to  $\Omega BG$  and  $\Omega BH$  for path connected spaces  $BG$  and  $BH$ , and  $G/H$  is the homotopy fiber of a based map  $f : BH \rightarrow BG$ . We always assume this much, and we add further hypotheses as needed. Such a framework of generalized homogeneous spaces was first introduced by Rector [10], and a more recent framework of  $p$ -compact groups has been introduced and studied extensively by Dwyer and Wilkerson [4] and others.

We ask the following question: How similar is the calculation of  $H^*(G/H; R)$  to the calculation of the cohomology of classical homogeneous spaces of compact Lie groups? When  $R = \mathbf{F}_p$  and  $H$  is of maximal rank in  $G$ , in the sense that  $H^*(H; \mathbf{Q})$  and  $H^*(G; \mathbf{Q})$  are exterior algebras on the same number of generators, the second author has studied the question in [8, 9]. There, the fact that  $H^*(BG; R)$  need not be a polynomial algebra is confronted and results similar to the classical theorems of Borel and Bott [2, 3] are nevertheless proven. The purpose of this note is to begin to answer the general question without the maximal rank hypothesis, but under the hypothesis that  $H^*(BG; R)$  and  $H^*(BH; R)$  are polynomial algebras.

In fact, we shall not do any new mathematics. Rather, we shall merely point out that work of the first author [7] that was done before the general context was introduced goes far towards answering the question. Essentially the following theorem was announced in [7] and proven in [5]. We give a brief sketch of its proof and then return to a discussion of its applicability to the question on hand. Let  $BT^n$  be a classifying space of an  $n$ -torus  $T^n$ .

**Theorem 1.** *Assume the following hypotheses.*

- (i)  $\pi_1(BG)$  acts trivially on  $H^*(G/H; R)$ .

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- (ii)  $R$  is a PID and  $H_*(BG; R)$  is of finite type over  $R$ .
- (iii)  $H^*(BG; R)$  is a polynomial algebra.
- (iv) There is a map  $e : BT^n \rightarrow BH$  such that  $H^*(BT^n; R)$  is a free  $H^*(BH; R)$ -module via  $e^*$ .

Then  $H^*(G/H; R)$  is isomorphic as an  $R$ -module to  $\text{Tor}_{H^*(BG; R)}(R, H^*(BH; R))$ , regraded by total degree. Moreover, there is a filtration on  $H^*(G/H; R)$  such that its associated bigraded  $R$ -algebra is isomorphic to  $\text{Tor}_{H^*(BG; R)}(R, H^*(BH; R))$ .

*Proof.* The first two hypotheses ensure that  $H^*(G/H; R)$  is isomorphic to the differential torsion product  $\text{Tor}_{C^*(BG; R)}(R, C^*(BH; R))$ . (See, for example, [5, pp. 21–25]. The second hypothesis allows Lemma 3.2 there to be applied with  $\mathbf{Z}$  replaced by  $R$ , thus allowing the finite type over  $\mathbf{Z}$  hypothesis assumed there to be replaced by the finite type over  $R$  hypothesis assumed here.) Therefore there is an Eilenberg-Moore spectral sequence that converges from  $\text{Tor}_{H^*(BG; R)}(R, H^*(BH; R))$  to  $H^*(G/H; R)$ . The conclusion of the theorem is that this spectral sequence collapses at  $E_2$  with trivial additive extensions, but not necessarily trivial multiplicative extensions. The last hypothesis and a comparison of spectral sequences argument essentially due to Baum [1] shows that the conclusion holds in general if it holds when  $BH = BT^n$ . (See [5, pp. 37–38].) Here the strange result [5, 4.1] gives that there is a morphism

$$g : C^*(BT^n; R) \rightarrow H^*(BT^n; R)$$

of differential algebras such that  $g$  induces the identity map on cohomology and annihilates all  $\cup_1$ -products.

Now the general theory of differential torsion products of [5] kicks in. In modern language, implicit in the discussion of [6, p. 70], there is a model category structure on the category of  $A$ -modules for any DGA  $A$  over  $R$  such that every right  $A$ -module  $M$  admits a cofibrant approximation of a very precise sort. Namely, for any  $HA$ -free resolution  $X \otimes_R HA \rightarrow HM$  of  $HM$ , there is a cofibrant approximation  $P = X \otimes_R A \rightarrow M$ . Grading is made precise in the cited sources. The essential point is that  $P$  is not a bicomplex but rather has differential with many components. When  $HA$  is a polynomial algebra and  $M = R$ , we can take  $X$  to be an exterior algebra with one generator for each polynomial generator of  $HA$ . Here, assuming that  $A$  has a  $\cup_1$ -product that satisfies the Hirsch formula ( $\cup_1$  is a graded derivation), [5, 2.2] specifies the required differential explicitly in terms of  $\cup_1$ -products. Using  $g$  to replace  $C^*(BT^n; R)$  by  $H^*(BT^n; R)$  in our differential torsion product, we see that the differential torsion product  $\text{Tor}_{C^*(BG; R)}(R, H^*(BT^n; R))$  is computed by exactly the same chain complex as the ordinary torsion product  $\text{Tor}_{H^*(BG; R)}(R, H^*(BT^n; R))$ . (See [5, 2.3].) The conclusion follows.  $\square$

Hypotheses (i) and (ii) in Theorem 1 are reasonable and not very restrictive. Hypothesis (iii) is intrinsic to the method at hand. Note that  $H^*(BG; R)$  can have infinitely many polynomial generators, so that  $G$  need not be finite. The key hypothesis is (iv). Here the following homotopical version of a theorem of Borel is relevant. It was first noticed by Rector [10, 2.2] that Baum's proof [1] of Borel's theorem is purely homotopical. A generalized variant of Baum's proof is given in [5, pp. 40–42]. That proof applies directly to give the following theorem. We state it for  $H$  and  $G$  as in the first paragraph. However, we are interested in its applicability to  $T^n$  and  $H$  in Theorem 1, and we restate it as a corollary in that special case.

**Theorem 2.** *Let  $R$  be a field and assume the following hypotheses.*

- (i)  $\pi_1(BG)$  acts trivially on  $H^*(G/H; R)$ .
- (ii)  $H^*(BH; R)$  and  $H^*(BG; R)$  are polynomial algebras on the same finite number of generators.
- (iii)  $H^*(G/H; R)$  is a finite dimensional  $R$ -module.

Then  $H^*(G/H; R) \cong R \otimes_{H^*(BG; R)} H^*(BH; R)$  as an algebra and

$$H^*(BH; R) \cong H^*(BG; R) \otimes_R H^*(G/H; R)$$

as a left  $H^*(BG; R)$ -module. In particular,  $H^*(BH; R)$  is  $H^*(BG; R)$ -free.

**Corollary 3.** *Let  $R$  be a field and assume given a map  $e : BT^n \rightarrow BH$  that satisfies the following properties, where  $H/T^n$  is the fiber of  $e$ .*

- (i)  $\pi_1(BH)$  acts trivially on  $H^*(H/T^n; R)$ .
- (ii)  $H^*(BH; R)$  is a polynomial algebra on  $n$  generators.
- (iii)  $H^*(H/T^n; R)$  is a finite dimensional  $R$ -module.

Then  $H^*(H/T^n; R) \cong R \otimes_{H^*(BH; R)} H^*(BT^n; R)$  as an algebra and

$$H^*(BT^n; R) \cong H^*(BH; R) \otimes_R H^*(H/T^n; R)$$

as a left  $H^*(BH; R)$ -module. In particular,  $H^*(BT^n; R)$  is  $H^*(BH; R)$ -free.

When Corollary 3 applies, its conclusion gives hypothesis (iv) of Theorem 1. We comment briefly on applications to the integral and  $p$ -compact settings for the study of generalized homogeneous spaces.

*Remark 4.* A counterexample of Rector [10] shows that not all finite loop spaces  $H$  have (integral) maximal tori. When  $H$  does have a maximal torus, hypothesis (iii) of the corollary holds by definition. Assuming that  $H$  is simply connected, [9, 3.11] describes for which primes  $p$   $H^*(BH; \mathbf{Z})$  is  $p$ -torsion free, so that  $H^*(BH; \mathbf{F}_p)$  is a polynomial algebra. If  $R$  is the localization of  $\mathbf{Z}$  at the primes  $p$  for which  $H^*(H; \mathbf{Z})$  is  $p$ -torsion free, then  $H^*(BH; R)$  is also a polynomial algebra, and  $H^*(BT; R)$  is a free  $H^*(BH; R)$ -module. That is, hypothesis (iv) of Theorem 1 holds for the localization of  $\mathbf{Z}$  away from the finitely many “bad primes” for which  $H^*(BH; \mathbf{F}_p)$  is not a polynomial algebra on  $n$  generators.

*Remark 5.* In the  $p$ -compact setting, taking  $R = \mathbf{F}_p$ , Dwyer and Wilkerson [4, 8.13, 9.7] prove that if  $H$  is connected,  $BH$  is  $\mathbf{F}_p$ -complete,  $H^*(H; \mathbf{F}_p)$  is finite dimensional, and  $H^*(H; \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}$  is an exterior algebra on  $n$  generators, then there is a map  $e : BT^n \rightarrow BH$  such that  $H^*(H/T^n; \mathbf{F}_p)$  is finite dimensional. Here Corollary 3 applies whenever  $H^*(BH; \mathbf{F}_p)$  is a polynomial algebra on  $n$  generators.

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