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SMOOTH DIAMETER AND EIGENVALUE RIGIDITY IN POSITIVE RICCI CURVATURE

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ABSTRACT. A recent injectivity radius estimate and previous sphere theorems yield the following smooth diameter sphere theorem for manifolds of positive Ricci curvature: For any given m and C there exists a positive constant $\varepsilon = \varepsilon(m, C) > 0$ such that any m-dimensional complete Riemannian manifold with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \leq C$ and diameter $\geq \pi-\varepsilon$ is Lipschitz close and diffeomorphic to the standard unit m-sphere. A similar statement holds when the diameter is replaced by the first eigenvalue of the Laplacian.

Let M be a complete Riemannian manifold of dimension $m \geq 2$ with Ricci curvature $Ricc \geq m-1$. Then, by the Bonnet-Myers theorem, the diameter of M is bounded from above by π, and by Cheng's maximal diameter sphere theorem the maximal value π is attained if and only if M is isometric to the unit m-sphere (see [Chg]).

In contrast to the sectional curvature setting, where Grove and Shiohama (see [GrS]) showed that a complete Riemannian m-manifold with sectional curvature ≥ 1 and diameter $> \pi/2$ must be homeomorphic to the m-dimensional sphere, and where until today no example of an exotic sphere with positive sectional curvature is known, in the case of positive Ricci curvature however the following phenomena arise: First of all, Anderson and Otsu (see [An1], [Ot1]) showed that Cheng's result is not even topologically rigid by constructing, for $m \geq 4$ on closed smooth m-manifolds whose homotopy type is distinct from that of the sphere, Riemannian metrics with $Ricc \geq m-1$ and diameter arbitrarily close to π. Secondly, it is known that many exotic spheres, i.e., smooth manifolds which are homeomorphic, but not diffeomorphic to a standard sphere, carry metrics of positive Ricci curvature (compare [Nash], [Poor], [H], [Ch], [Wr]).

Thus additional assumptions are needed to obtain sphere theorems for positively Ricci curved manifolds, and of particular interest here is the problem of finding conditions which guarantee smooth instead of merely topological rigidity. Contributions to this field of questions have been made in particular by Anderson, Bessa, Brittain, Cai, Colding, Eschenburg, Grove-Petersen, Itokawa, Katsuda, Nakamura, Otsu, Paeng, Perelman, Petersen, Petersen-Zhu, Shiohama, Wilhelm, Wu, Xia, and

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Yamaguchi (see [An2], [Bes], [Br], [Cai], [Co], [E], [GrP], [It], [Kat], [Na], [Ot2], [Pa], [Per], [Pet], [PZ], [Shi1], [Wi], [Wu], [Xia], [Yam]; compare also the survey [Shi2]).

Except for Perelman's topological diameter sphere theorem (see [Per]), stating that for any given m and C there exists a positive constant $\varepsilon = \varepsilon(m, C) > 0$ such that any m-dimensional complete Riemannian manifold with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \geq C$ and diameter $\geq \pi - \varepsilon$ is a twisted sphere, all of these results use further assumptions on additional geometric quantities besides various conditions on curvature and diameter. The purpose of this note is to point out that by combining Perelman's result and previous smooth sphere theorems with a recent injectivity radius estimate one obtains, along with other consequences, the following differentiable sphere theorem which requires only assumptions on curvature and diameter:

1.1 Theorem. *For any given m and C there exists a positive constant $\varepsilon(m, C) > 0$ such that for $\varepsilon \leq \varepsilon(m, C)$ any m-dimensional complete Riemannian manifold M with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \leq C$ and diameter $\geq \pi - \varepsilon$ is diffeomorphic to the standard m-sphere. Moreover, the Lipschitz distance between M and the unit m-sphere $S^m(1)$ converges to zero as $\varepsilon \to 0$.*

Lichnerowicz (see [Li]) showed that the first eigenvalue $\lambda_1(M)$ of the Laplacian of a complete Riemannian m-manifold with Ricci curvature $Ricc \geq m-1$ is bounded from below by m, and by Obata's minimal eigenvalue sphere theorem (see [Ob]) $\lambda_1(M)$ is equal to m if and only if M is isometric to the unit m-sphere. It follows from results of Croke and Cheng (see [Cr], [Chg]) that Theorem 1.1 is equivalent to the following smooth eigenvalue pinching sphere theorem:

1.2 Theorem. *For any given m and C there exists a positive constant $\varepsilon'(m, C) > 0$ such that for $\varepsilon \leq \varepsilon'(m, C)$ any m-dimensional complete Riemannian manifold M with Ricci curvature $Ricc \geq m-1$, sectional curvature $K \leq C$ and $\lambda_1(M) \leq m + \varepsilon$ is diffeomorphic to the standard m-sphere. Moreover, the Lipschitz distance between M and the unit m-sphere $S^m(1)$ converges to zero as $\varepsilon \to 0$.*

Theorems 1.1 and 1.2 and sphere theorems as in [Bes] imply that a violation of smooth rigidity in Cheng's maximal diameter or Obata's minimal first eigenvalue sphere theorem can only be achieved by sequences of manifolds whose sectional curvatures blow up and whose injectivity radii converge to zero:

1.3 Corollary. *Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of complete Riemannian m-manifolds with Ricci curvature $Ricc \geq m-1$ whose diameters converge to π or whose first eigenvalues converge to m as n goes to infinity.*

Then either all but finitely many M_n are diffeomorphic to the standard m-sphere, or else $\limsup_{n\to\infty} K_{M_n} = +\infty$ *and* $\lim_{n\to\infty} \mathrm{inj}_{M_n} = 0$.

Theorem 1.1 follows from previous sphere theorems and the following injectivity radius estimate:

1.4 Theorem ([PT])**.** *For each m and any $0 < \delta \leq 1$ there exists a positive constant $i_0(m, \delta) > 0$ such that the injectivity radius i_g of any Riemannian metric g with Ricci curvature $Ricc \geq (m-1)\delta$ and sectional curvature $K \leq 1$ on a simply connected closed m-dimensional manifold with finite second homotopy group is bounded from below by $i_g \geq i_0(m, \delta)$.*

In fact, to prove Theorem 1.1 one observes that since an upper sectional and a lower Ricci curvature bound imply a lower bound for the sectional curvature, Perelman's topological diameter sphere theorem can be invoked and one may assume that the manifolds M in question in Theorem 1.1 are twisted spheres, so that in particular Theorem 1.4 applies to them. With this uniform injectivity radius estimate at hand known differentiable sphere theorems, for example the ones of [Br], [Kat], [Bes], [Pa], or [Yam], now easily yield the theorem's conclusions.

The question remains whether the upper sectional curvature bound in Theorem 1.1 is indeed indispensable, or whether instead the assumptions in Perelman's theorem might already imply diffeomorphism to S^m. To construct Riemannian metrics on exotic spheres which violate smooth rigidity seems in view of Corollary 1.3 actually rather difficult.

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References

[An1] M. T. Anderson, *Metrics of positive Ricci curvature with large diameters*, Man. Math. 68 (1990) 405–415 MR **91g:**53045

[An2] M. T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Inv. Math. 102 (1990) 429–445 MR **92c:**53024

[Br] D. Brittain, *A Diameter Pinching Sphere Theorem for Positive Ricci Curvature*, Preprint (seemingly unpublished, compare however the result's of the proof in [Gr])

[Bes] G. Bessa, *Differentiable sphere theorems for Ricci curvature*, Math. Z. 214 (1993) 245–249 MR **94g:**53035

[Cai] M. Cai, *Rigidity of manifolds with large volume*, Math. Z. 213, No.1 (1993) 17–31 MR **94b:**53071

[Ch] J. Cheeger, *Some examples of manifolds of non-negative curvature*, J. Diff. Geom. 8 (1973) 623–628 MR **49:**6085

[Chg] Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. 143 (1975) 289–297 MR **51:**14170

[Co] T. Colding, *Large manifolds with positive Ricci curvature*, Invent. Math. 124 (1996) 193–214 MR **96k:**53068

[Cr] C. B. Croke, *An eigenvalue pinching theorem*, Invent. Math. 68 (1982) 253–256 MR **84a:**58084

[E] J.-H. Eschenburg, *Diameter, volume, and topology for positive Ricci curvature*, J. Diff. Geom. 33 (1991) 743–747 MR **92b:**53054

[Gr] K. Grove, *Metric differential geometry*, Differential geometry (Lyngby, 1985), 171–227, Lecture Notes in Math., 1263, Springer, Berlin-New York, 1987 MR **88i:**53075

[GrP] K. Grove and P. Petersen, *A pinching theorem for homotopy spheres*, J.A.M.S. 3, 3 (1990) 671-677 MR **91e:**53040

[GrS] K. Grove and K. Shiohama, *A generalized sphere theorem*, Ann. Math. 106 (1977) 201-211 MR **58:**18268

[H] H. Hernández-Andrade, *A class of compact manifolds with positive Ricci curvature*, Differential Geometry, Proc. Symp. Pure Math. 28 (1975) 73–87 MR **52:**1565

[It] Y. Itokawa, *The topology of certain Riemannian manifolds with positive Ricci curvature*, J. Diff. Geom. 18 (1983) 151–155 MR **84i:**53044

[Kat] A. Katsuda, *Gromov's convergence theorem and its applications*, Nagoya Math. J. 100 (1985) 11-48; Erratum: Nagoya Math. J. 114 (1989) 173-174 MR **87e:**53067; MR **90e:**53057

[Li] A. Lichnerowicz, *Géométrie des groupes de transformations*, Dunod, Paris (1958) MR **23:**A1329

[Na] G. Nakamura, *Diameter sphere theorems for manifolds of positive Ricci curvature*, Dissertation, Nagoya University 1989

[Nash] J. C. Nash, *Positive Ricci curvature on fiber bundles*, J. Diff. Geom. 14 (1979) 241–265 MR **81k:**53039

[Ob] M. Obata, *Certain conditions for a Riemannian manifold to be isometric to a sphere*, J. Math. Soc. Japan 14 (1962) 333–340 MR **25:**5479

[Ot1] Y. Otsu, *On manifolds of positive Ricci curvature with large diameter*, Math. Z. 206 (1991) 255–264 MR **91m:**53033

[Ot2] Y. Otsu, *On manifolds of small excess*, Amer. J. Math. 115 (1993) 1229–1280 MR **95i:**53046

[Pa] S.-H. Paeng, *A sphere theorem under a curvature perturbation. I; II*, Kyushu J. Math. 50 (1996) 459–470; Kyushu J. Math. 52 (1998) 439–454 MR **97m:**53068; MR **99k:**53081

[Per] G. Perelman, *A diameter sphere theorem for manifolds of positive Ricci curvature*, Math. Z. 218 (1995) 595–596 MR **96f:**53056

[Pet] P. Petersen, *Small excess and Ricci curvature*, J. Geom. Anal. 1,4 (1991) 383–387 MR **93a:**53031

[Poor] W. A. Poor, *Some exotic spheres with positive Ricci curvature*, Math. Annalen 216 (1975) 245–252 MR **53:**3945

[PT] A. Petrunin and W. Tuschmann, *Diffeomorphism Finiteness, Positive Pinching, and Second Homotopy*, Geom. and Funct. Anal. (GAFA) 9 (1999) 736–774 MR **2000k:**53031

[PZ] P. Petersen and S. Zhu, *An excess sphere theorem*, Ann. Sci. Ec. Norm. Super., IV. Ser. 26, No.2 (1993) 175–188 MR **94b:**53077

[Shi1] K. Shiohama, *A sphere theorem for manifolds of positive Ricci curvature*, Trans. Amer. Math. Soc. 275 (1983) 811–819 MR **84c:**53041

[Shi2] K. Shiohama, *Sphere Theorems*, F.J.E. Dillen and L.C.A. Verstraelen (edts.), Handbook of Differential Geometry, Vol. I, Elsevier Science B.V., Amsterdam 2000, 865–903 MR **2001c:**53051

[Wi] F. Wilhelm, *On radius, systole, and positive Ricci curvature*, Math. Z. 218 (1995) 597–602 MR **96d:**53040

[Wr] D. Wraith, *Exotic spheres with positive Ricci curvature*, J. Diff. Geom. 46 (1997) 638–649 MR **98i:**53058

[Wu] J.-Y. Wu, *A diameter pinching sphere theorem for positive Ricci curvature*, Proc. A.M.S. 107,3 (1989) 797–802 MR **90h:**53045

[Xia] C. Xia, *Rigidity and sphere theorem for manifolds with positive Ricci curvature*, Manuscr. Math. 85, No.1 (1994) 79–87 MR **95j:**53057

[Yam] T. Yamaguchi, *Lipschitz convergence of manifolds of positive Ricci curvature with large volume*, Math. Ann. 284 (1989) 423–436 MR **90c:**53114

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