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NILPOTENCY DEGREE OF COHOMOLOGY RINGS IN CHARACTERISTIC 3

PHAM ANH MINH

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To Phuong and Nin

Abstract. The purpose of this note is to provide a 3-group $\mathcal{G}$ whose mod-3 cohomology ring has a nilpotent element $\xi \in H^*(\mathcal{G})$ satisfying $\xi^3 \neq 0$.

1. Statement of the main result

For every p-group G, denote by $H^*(G)$ the mod-p cohomology algebra of G. We are now interested in the nilpotency degrees of elements of $H^*(G)$. For the case $p = 2$, in [1], [2], it was shown that, given any positive integer n, there exists a 2-group whose cohomology ring has elements of nilpotency degree $n+1$. However, the methods given there do not generalize to the case of odd characteristic p. In this case, it seems that, until now, we do not have any example of elements of $H^*(G)$ having nilpotency degrees greater than p.

The purpose of this note is to provide a 3-group $\mathcal{G}$ whose cohomology ring has a nilpotent element $\xi \in H^*(\mathcal{G})$ satisfying $\xi^3 \neq 0$. The group $\mathcal{G}$ is obtained as follows. Let

$$\mathbb{E} = \langle a_1, a_2 | a_1^3 = a_2^3 = [a_1, a_2]^3 = [a_1, [a_1, a_2]] = [a_2, [a_1, a_2]] = 1\rangle$$

be the extraspecial 3-group of order 3^3 and of exponent 3, and let

$$K = C_3^4 = \langle a_3, \ldots, a_6 | a_i^3 = [a_i, a_j] = 1\rangle$$

be the elementary abelian 3-group of rank 4. Set $G = \mathbb{E} \times K$. Define $u_i \in H^1(G) = \mathrm{Hom}(G, \mathbb{Z}/3), v_i \in H^2(G), 1 \leq i \leq 6$, by

$$u_i(a_j) = \delta_{ij},$$
$$v_i = \beta u_i,$$

with δ_{ij} the Kronecker symbol and β the Bockstein homomorphism. For $3 \leq i \leq 6$, consider u_i, v_i as elements of $H^*(K)$ via the restriction map. So, by the Künneth formula,

$$H^*(K) = \Lambda[u_3, \ldots, u_6] \otimes \mathbb{F}_3[v_3, \ldots, v_6],$$

and $H^*(G)$ may be identified with

$$H^*(\mathbb{E}) \otimes H^*(K).$$

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Consider the central extension

$$(\mathcal{G}) \qquad 0 \to \mathbb{Z}/3 \xrightarrow{i} \mathcal{G} \to G \to 1$$

corresponding to the cohomology class $z = v_1 + u_3u_4 + u_5u_6 \in H^2(G)$. We will prove

Theorem. *There exists a nilpotent element $\xi \in H^2(\mathcal{G})$ satisfying $\xi^3 \neq 0$.*

However, for $p > 3$, our method could not be applied. So, the existence of a p-group G having a cohomology class of nilpotency degree greater than $p > 3$ still remains an open problem. Besides, we could not provide any example of cohomology classes of a 3-group having nilpotency degrees greater than a given n, although it is known that, for any $p \geq 2$ and for any p-group G of order p^m, nilpotency degrees of nilpotent elements of $H^*(G)$ are bounded above by p^{m-1} (see [5]).

2. Proof of the main result

Via the restriction map, for $1 \leq i \leq 2$, u_i and v_i can be considered as elements of $H^*(\mathbb{E})$. Set $c = [a_1, a_2]$ and let $A_0 = \langle a_1, c\rangle, A_1 = \langle a_1a_2, c\rangle, A_2 = \langle a_1^2a_2, c\rangle, A_3 = \langle a_2, c\rangle$ be elementary abelian subgroups of $\mathbb{E}$. Set $s_0 = \mathrm{Res}^{\mathbb{E}}_{A_0}(u_1), s_i = \mathrm{Res}^{\mathbb{E}}_{A_i}(u_2)$, $1 \leq i \leq 3$, and let $t_j = \beta s_j, 0 \leq j \leq 3$. Let u be a generator of $H^1(\langle c\rangle)$ and set $v = \beta u \in H^2(\langle c\rangle)$. It follows that

$$H^*(\langle c\rangle) = \Lambda[u] \otimes \mathbb{F}_3[v].$$

So, by Künneth formula, we have

$$H^*(A_i) = \Lambda[s_i, u] \otimes \mathbb{F}_3[t_i, v],$$

$0 \leq i \leq 3$.

It follows from [4] that $H^*(\mathbb{E})$ is detected by the A_i's. According to [3], we have

Lemma 1. (i) *There exist elements $U_1, U_2 \in H^2(\mathbb{E})$ such that:*

(ia) *v_1, v_2, U_1, U_2 is a basis of $H^2(\mathbb{E})$. Furthermore, we have*

$$\begin{array}{llll}
u_1|_{A_0} = s_0, & u_1|_{A_1} = s_1, & u_1|_{A_2} = 2s_2, & u_1|_{A_3} = 0,\\
v_1|_{A_0} = t_0, & v_1|_{A_1} = t_1, & v_1|_{A_2} = 2t_2, & v_1|_{A_3} = 0,\\
u_2|_{A_0} = 0, & u_2|_{A_1} = s_1, & u_2|_{A_2} = s_2, & u_2|_{A_3} = s_3,\\
v_2|_{A_0} = 0, & v_2|_{A_1} = t_1, & v_2|_{A_2} = t_2, & v_2|_{A_3} = t_3,\\
U_1|_{A_0} = s_0u, & U_1|_{A_1} = s_1u + t_1, & U_1|_{A_2} = 2s_2u + t_2, & U_1|_{A_3} = 0,\\
U_2|_{A_0} = 0, & U_2|_{A_1} = 2s_1u + t_1, & U_2|_{A_2} = 2s_2u + 2t_2, & U_2|_{A_3} = s_3u;
\end{array}$$

(ib) *$v_1^2, v_1v_2, v_2^2, v_1U_1, v_1U_2, v_2U_1, v_2U_2$ is a basis of $H^4(\mathbb{E})$ and $U_1^2 = v_1U_2$, $U_2^2 = v_2U_1$, $U_1U_2 = v_1v_2$.*

(ii) *v_1u_1, v_1u_2 is linearly independent in $H^3(\mathbb{E})$.*

Lemma 2. *Let X be an element of $H^*(G)$ of degree $n \leq 2$. Then $X = 0$ provided that one of the following conditions is satisfied:*

(i) *$Xv_1 = 0$;*

(ii) *$Xz = 0$.*

Proof. The case $n = 1$ is trivial. For $n = 2$, write $X = a_{2,0} + a_{1,1} + a_{0,2}$ with $a_{i,j} \in H^i(\mathbb{E}) \otimes H^j(K)$. If $Xv_1 = 0$, then $a_{2,0}v_1 = a_{1,1}v_1 = a_{0,2}v_1 = 0$, hence $a_{0,2} = 0$, and, by Lemma 1, $a_{2,0} = a_{1,1} = 0$; so $X = 0$. If $Xz = 0$, then

$$\begin{aligned} a_{2,0}v_1 &= 0, \\ a_{2,0}(u_3u_4 + u_5u_6) + a_{0,2}v_1 &= 0, \\ a_{1,1}v_1 = a_{0,2}(u_3u_4 + u_5u_6) = a_{1,1}(u_3u_4 + u_5u_6) &= 0, \end{aligned}$$

hence, it follows from what we just proved that $a_{2,0} = a_{1,1} = a_{0,2} = 0$, so $X = 0$. The lemma follows.

Lemma 3. *The cup-product with v_1^2 is an injective map from $H^2(\mathbb{E})$ to $H^6(\mathbb{E})$.*

Proof. If $v_1^2(\lambda_1 v_1 + \lambda_2 v_2 + \mu_1 U_1 + \mu_2 U_2) = 0$ with $\lambda_i, \mu_i \in \mathbb{Z}/3$, then restricting to $A_j, 0 \le j \le 3$ yields $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. The lemma follows.

For elements $X, Y, \dots$ of $H^*(G)$, denote by $(X, Y, \dots)$ the ideal of $H^*(G)$ generated by $X, Y, \dots$. We have

Lemma 4. *Let X be an element of $H^1(G)$ with $X\beta z \in (z)$. Then $X = 0$.*

Proof. Write $X\beta z = Yz$ with $Y \in H^2(G)$. Since X and βz are free of v_1, it follows that $Yv_1 = 0$. By Lemma 2, $Y = 0$, so $X\beta z = 0$. Write $X = \sum_{i=1}^6 \lambda_i u_i$. A direct verification shows that $\lambda_1 = \cdots = \lambda_6 = 0$. The lemma follows.

Lemma 5. $v_1^2 v_2 \notin (z, \beta z)$.

Proof. Assume that $v_1^2 v_2 \in (z, \beta z)$. Then there exist elements $a_{i,j}, b_{i,j} \in H^i(\mathbb{E}) \otimes H^j(K)$ satisfying

$$\begin{aligned} v_1^2 v_2 &= \sum a_{i,j} z + \sum b_{i,j} \beta z \\ &= \sum a_{i,j}(v_1 + u_3u_4 + u_5u_6) + \sum b_{i,j}(v_3u_4 - v_4u_3 + v_5u_6 - v_6u_5). \end{aligned} \tag{1}$$

By decomposing (1), we get

$$v_1^2 v_2 = a_{4,0} v_1, \tag{2}$$

$$0 = a_{4,0}(u_3u_4 + u_5u_6) + a_{2,2}v_1, \tag{3}$$

$$0 = a_{2,2}(u_3u_4 + u_5u_6) + a_{0,4}v_1 + b_{2,1}(v_3u_4 - v_4u_3 + v_5u_6 - v_6u_5). \tag{4}$$

It follows from (3) and Lemma 2 that $a_{4,0}$ contains v_1 as a factor. By Lemma 3, $a_{4,0} = v_2 v_1$. Hence, by (3) and Lemma 2, $a_{2,2} = -v_2(u_3u_4 + u_5u_6)$. So, from (4), we get

$$2v_2u_3u_4u_5u_6 = a_{0,4}v_1 + b_{2,1}(v_3u_4 - v_4u_3 + v_5u_6 - v_6u_5),$$

a contradiction. The lemma follows.

With some abuse of notation, we consider elements of $H^*(G)$ as elements of $H^*(\mathcal{G})$ via the inflation map. Set $\xi = U_1 \in H^2(\mathcal{G})$. The first part of the theorem follows from

Lemma 6. *ξ is nilpotent.*

Proof. It follows from Lemma 1 that $\xi^2 = v_1 U_2$; hence

$$\xi^3 = v_1 U_1 U_2 = v_1^2 v_2.$$

So $\xi^9 = (v_1^3)^2 v_2^3$. Since, in $H^*(\mathcal{G})$, $z = 0$, we have $v_1^3 = (-u_3u_4 - u_5u_6)^3 = 0$. It follows that $\xi^9 = 0$. Therefore ξ is nilpotent. The lemma is proved.

The proof of the theorem is completed by the following.

Lemma 7. *$\xi^3 \neq 0$ in $H^*(\mathcal{G})$.*

Proof. Assume that $\zeta = \xi^3 = v_1^2 v_2 = 0$ in $H^*(\mathcal{G})$. So $\zeta \in \operatorname{ImInf}_{\mathcal{G}}^G$. Set $Z = i(\mathbb{Z}/3) \subset \mathcal{G}$. Denote by $\{E_r, d_r\}$ the Hochschild-Serre spectral sequence corresponding to the extension $(\mathcal{G})$. So

$$E_2 = H^*(G) \otimes H^*(Z).$$

Let s be a generator of $H^1(Z)$ and set $t = \beta s \in H^2(Z)$. s can be chosen such that $d_2(s) = z$. t is then transgressive and $d_3(t) = \beta z$. So, by Lemma 5, $\zeta \neq 0$ in $E_4^{6,0}$. Since $\zeta \in \operatorname{ImInf}_{\mathcal{G}}^G$, it must be hit by d_4 or d_5. If $\zeta = d_4(a \otimes st)$ with $a \otimes st \in E_4^{2,3}$, then

$$\begin{aligned} d_2(a \otimes st) &= (a \otimes t) d_2(s) \\ &= az \otimes t \\ &= 0 \end{aligned}$$

in $E_2^{4,2}$; so $az = 0$ in $H^*(G)$, hence $a = 0$ by Lemma 2, a contradiction. If $\zeta = d_5(b \otimes t^2)$ with $b \otimes t^2 \in E_5^{1,4}$, then

$$\begin{aligned} d_3(b \otimes t^2) &= (b \otimes 1) d_3(t^2) \\ &= (b \otimes 1)(2\beta z \otimes t) \\ &= 2b\beta z \otimes t \\ &= 0 \end{aligned}$$

in $E_3^{4,2}$; so $b\beta z \in (z)$, hence $b = 0$ by Lemma 4, a contradiction. Therefore $\zeta \neq 0$. The lemma follows.

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Department of Mathematics, College of Sciences, University of Hue, Dai hoc Khoa hoc, Hue, Vietnam

E-mail address: paminh@dng.vnn.vn

Current address: 53 Craig Road, Stockport SK4 2AP, England