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ENTROPY, INDEPENDENT SETS AND ANTICHAINS: A NEW APPROACH TO DEDEKIND'S PROBLEM

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ABSTRACT. For n-regular, N-vertex bipartite graphs with bipartition $A \cup B$, a precise bound is given for the sum over independent sets I of the quantity $\mu^{|I\cap A|}\lambda^{|I\cap B|}$. (In other language, this is bounding the *partition function* for certain instances of the *hard-core model*.) This result is then extended to graded partially ordered sets, which in particular provides a simple proof of a well-known bound for Dedekind's Problem given by Kleitman and Markowsky in 1975.

1. INTRODUCTION

"Dedekind's Problem" of 1897 [6] asks for the number $\psi(m)$ of elements in the free distributive lattice on m generators, or equivalently, of antichains in the Boolean algebra $\mathcal{B}_m$. See [11] for some account of the early history of the problem (including, in particular, [6], [4], [18], [7], [13], [8]).

In a 1969 breakthrough, Kleitman [11] used an ingenious elaboration of the basic approach of Hansel [8] to pinpoint the asymptotics of the logarithm of $\psi(m)$:

$$\log \psi(m) < (1 + O((\log m)/\sqrt{m}))\binom{m}{\lfloor m/2 \rfloor}. \tag{1}$$

(All logarithms in this paper are base 2. The easy $\log \psi(m) > \binom{m}{\lfloor m/2 \rfloor}$ was perhaps first observed by Gilbert [7].)

Kleitman's (already not easy) argument was developed considerably further by Kleitman and Markowsky [12] to show

$$\log \psi(m) < (1 + O((\log m)/m))\binom{m}{\lfloor m/2 \rfloor}. \tag{2}$$

Then in 1981, Korshunov [14], using an extremely complicated approach, gave asymptotics for $\psi(m)$ itself. Simpler, though still difficult, arguments for Korshunov's and some related results were later given by Sapozhenko [16].

In this paper we use an entropy approach developed in [10], [9] to give a simple proof of a general result (Theorem 1.5) for graded posets, which essentially includes

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the Kleitman-Markowsky bound (2) as a special case (see Corollary 1.4). The key result is a weighted version (Theorem 1.2) of an earlier result from [9] (Theorem 1.1) bounding the numbers of independent sets in certain bipartite graphs. We now describe these graph results before turning again to posets.

Write $\mathcal{I}(G)$ for the collection of independent sets of a graph G. (An *independent set* is a set of vertices spanning no edges. For graph theory basics see e.g. [2].) Our starting point for the present investigation was the following result from [9].

Theorem 1.1. *If G is an n-regular bipartite graph on N vertices, then*

$$|\mathcal{I}(G)| \leq (2^{n+1}-1)^{N/(2n)}.$$

Notice this is sharp whenever G is a disjoint union of copies of the complete bipartite graph $K_{n,n}$. See [9] for some discussion of related matters. It was suggested in [1] (and conjectured formally in [9]) that Theorem 1.1 is correct for general (i.e. not necessarily bipartite) graphs.

Our first result here is a more general weighted version of Theorem 1.1. This was mainly motivated by the poset applications below, and its slightly fussy statement is what we will need for those applications. Write $d(x)$ for the degree of vertex x.

Theorem 1.2. *Let G be a bipartite graph on $A \cup B$ with $|A| \leq M$ and*

$$d(x)\begin{cases} \leq n & \forall x \in A, \\ \geq n & \forall x \in B. \end{cases}$$

Let

$$\lambda_x = \begin{cases} \mu & \text{if } x \in A, \\ \lambda & \text{if } x \in B, \end{cases}$$

and suppose $\lambda, \mu \geq 1$. Then

$$\sum_{I \in \mathcal{I}(G)} \prod_{x \in I} \lambda_x \leq ((1+\mu)^n + (1+\lambda)^n - 1)^{M/n}. \tag{3}$$

Remarks. 1. This is again sharp for disjoint unions of $K_{n,n}$'s.

2. For general nonnegative weights (or "activities") λ_x on vertices x, the associated *hard-core measure* is the probability distribution on $\mathcal{I}(G)$ given by $\Pr(I) \propto \prod_{x \in I} \lambda_x$, and the quantity on the left-hand side of (3) is the associated "partition function." It seems interesting that this quantity arises naturally in the present context (see the proof of Theorem 1.5).

3. We conjecture that the assumption "$\lambda, \mu \geq 1$" in Theorem 1.2 can be relaxed to "$\lambda, \mu \geq 0$."

We now turn to posets. (For general poset background see [17].) Recall that an *antichain* in a poset is a set of pairwise incomparable elements. We write $\mathcal{A}(P)$ for the set of antichains of P and $a(P) = |\mathcal{A}(P)|$.

A poset P is *graded* if there is some $r : P \to \mathbf{Z}$ (a *rank function*) such that $y \lessdot x \Rightarrow r(x) = r(y) + 1$. We will usually take P to be graded by $\{1, \ldots, k\}$ (that is, the range of r is $\{1, \ldots, k\}$) for some k, set $r^{-1}(i) = P_i$, and refer to the P_i's as the *levels* of P. Note that we identify a poset with its ground set when we can do so without causing trouble.

Define $d_{\text{down}}(x) = |\{y : y \lessdot x\}|$ and $d_{\text{up}}(x)$ similarly.

Theorem 1.3. *Let P be a graded poset with levels $P_1, \dots, P_k$, with $|P_k| \le M$, and assume*

$$\begin{aligned} d_{\rm up}(x) &\geq n \ \forall x \in P_1 \cup \cdots \cup P_{k-1}, \\ d_{\rm down}(x) &\leq n \ \forall x \in P_2 \cup \cdots \cup P_k. \end{aligned}$$

Then $a(P) \le (k2^n - (k-1))^{M/n}$.

The case $k = 2$ is (a slight generalization of) Theorem 1.1, and the extremal examples for Theorem 1.3 generalize the disjoint unions of $K_{n,n}$'s which are extreme for Theorem 1.1; namely, for $n|M$, let P consist of M/n disjoint copies of the graded poset with n elements at each of k levels and $y < x$ whenever $r(y) < r(x)$.

As we will see, [10] follows easily from Theorem 1.3:

Corollary 1.4. $\log \psi(m) \le (1 + \frac{2\log(m+1)}{m})\binom{m}{\lfloor m/2 \rfloor}$.

Our proof of Theorem 1.3 actually gives the more general weighted version (which is again exact for the P's described above):

Theorem 1.5. *Assume the hypotheses of Theorem 1.3. Let $\lambda_1, \dots, \lambda_k \ge 1$ and set $\lambda_x = \lambda_i$ whenever $x \in P_i$. Then*

$$\sum_{I \in \mathcal{A}(P)} \prod_{x \in I} \lambda_x \le (\sum_{i=1}^{k} (1+\lambda_i)^n - (k-1))^{M/n}.$$

Here we again expect that "$\lambda_i \ge 1$" can be relaxed to "$\lambda_i \ge 0$," as would follow from the corresponding relaxation of Theorem 1.2 conjectured above.

Theorem 1.5 is proved by induction on k, the base case $k = 2$—namely Theorem 1.2—being in fact the main step. The proof of Theorem 1.2 is given in Section 3 following a brief entropy review in Section 2, and the induction step for Theorem 1.5 is given in Section 4. Finally, Section 5 gives the easy derivation of Corollary 1.4.

2. Entropy

Here we briefly review relevant entropy background. (This is mainly copied from [9]. For more thorough discussions see [15], [5].)

In what follows $\mathbf{X}$, $\mathbf{Y}$, etc. are discrete random variables (r.v.'s), which in our usage are allowed to take values in any countable (here always finite) set. As stated earlier, we always take $\log = \log_2$.

As usual, H is the (binary) entropy function,

$$H(\alpha) = \alpha \log(1/\alpha) + (1-\alpha)\log(1/(1-\alpha)).$$

The *entropy* of r.v. $\mathbf{X}$ is

$$H(\mathbf{X}) = \sum_x p(x) \log \frac{1}{p(x)} ,$$

where we write $p(x)$ for $\Pr(\mathbf{X} = x)$ (and extend this convention in natural ways below). The *conditional entropy* of $\mathbf{X}$ given $\mathbf{Y}$ is

$$H(\mathbf{X}|\mathbf{Y}) = \mathbf{E} H(\mathbf{X}|\mathbf{Y} = y) = \sum_y p(y) \sum_x p(x|y) \log \frac{1}{p(x|y)} .$$

For a random vector $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n)$ (note this is also a r.v.), we have

$$H(\mathbf{X}) = H(\mathbf{X}_1) + H(\mathbf{X}_2|\mathbf{X}_1) + \cdots + H(\mathbf{X}_n|\mathbf{X}_1, \dots, \mathbf{X}_{n-1}). \tag{4}$$

Some useful inequalities are

$$H(\mathbf{X}) \leq \log|\,\mathrm{range}(\mathbf{X})|,$$

$$H(\mathbf{X}|\mathbf{Y}) \leq H(\mathbf{X}), \tag{5}$$

and more generally,

$$\text{if } \mathbf{Y} \text{ determines } \mathbf{Z}, \text{ then } H(\mathbf{X}|\mathbf{Y}) \leq H(\mathbf{X}|\mathbf{Z}),$$

$$H(\mathbf{X}_1, \ldots, \mathbf{X}_n) \leq \sum H(\mathbf{X}_i) \tag{6}$$

(e.g. by (4) and (5)). We will use these facts without reference in what follows.

We will occasionally need the formula

$$H(\mathbf{X}|\mathbf{X} \neq 0) = \frac{H(\mathbf{X}) - H(\Pr(\mathbf{X}=0))}{1 - \Pr(\mathbf{X}=0)} \tag{7}$$

(derived by rearranging $H(\mathbf{X}) = H(\mathbf{1}_{\{\mathbf{X}=0\}}) + (1 - \Pr(\mathbf{X}=0))H(\mathbf{X}|\mathbf{X} \neq 0)$).

Finally, we need one less classical result, due to J. Shearer (see [3, p. 33]). For random vector $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $A \subseteq [n]$, set $\mathbf{X}_A = (\mathbf{X}_i : i \in A)$. Shearer's Lemma, which in particular generalizes (6), is

Lemma 2.1. *Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be a random vector and $\mathcal{A}$ a collection of subsets (possibly with repeats) of $[n]$, with each element of $[n]$ contained in at least m members of $\mathcal{A}$. Then*

$$H(\mathbf{X}) \leq \frac{1}{m} \sum_{A \in \mathcal{A}} H(\mathbf{X}_A).$$

(The version stated in [3] is less general, but the proof given there yields Lemma 2.1.)

3. Proof of Theorem 1.2

Set

$$Z = Z(G, \lambda) = \sum_{I \in \mathcal{I}(G)} \prod_{x \in I} \lambda_x.$$

For the proof of Theorem 1.2 we first realize $\log Z$ as the entropy of a random vector, giving this construction in more generality than will be needed below. Thus, for the time being, $G = (V, E)$ is an arbitrary graph and $\lambda : V \to [1, \infty)$ an arbitrary assignment of weights to the vertices.

With each $v \in V$ we associate a set $S_v \ni 0$ and nonnegative weights $\alpha_v(s)$, $s \in S_v$, such that

$$\alpha_v(0) = 1, \ \sum_{s \neq 0} \alpha_v(s) = \lambda_v,$$

and the r.v. $\mathbf{X}_v$ given by

$$\Pr(\mathbf{X}_v = s) = \alpha_v(s) / \sum_{s'} \alpha_v(s')$$

satisfies $H(\mathbf{X}_v) = \log(1 + \lambda_v)$. (This is possible if and only if $\lambda_v \geq 1$, with the "only if" following from

$$\log(1 + \lambda_v) = H(\mathbf{X}_v) \geq H(\Pr(\mathbf{X}_v = 0)) = H((1 + \lambda_v)^{-1}).)$$

Say that a vector $(s_v : v \in V) \in \prod S_v$ is *independent* if $\{v : s_v \neq 0\} \in \mathcal{I}(G)$.

Finally, let $\mathbf{Y} = (\mathbf{Y}_v : v \in V)$ be chosen from the independent vectors in $\prod S_v$ with $\Pr(\mathbf{Y} = (s_v)) \propto \prod \alpha_v(s_v)$. (Equivalently, if we take $\mathbf{X}_v$'s independent with distributions as above, then $\mathbf{Y}$ is $(\mathbf{X}_v : v \in V)$ conditioned on independence.) We assert that

$$H(\mathbf{Y}) = \log Z. \tag{8}$$

Proof. Let $\mathbf{I} = \{v \in V : \mathbf{Y}_v \neq 0\}$. Thus $\mathbf{I}$ is an r.v. taking values in $\mathcal{I}(G)$ with $\Pr(I) := \Pr(\mathbf{I} = I) = \prod_{v\in I} \lambda_v / Z$. (In other words, $\mathbf{I}$ is chosen according to the hard-core measure corresponding to λ.)

Notice that on $\{\mathbf{I} = I\}$ the r.v.'s $\mathbf{Y}_v$ are independent with

$$\mathbf{Y}_v \begin{cases} \equiv 0 & \text{if } v \notin I, \\ \sim (\mathbf{X}_v | \mathbf{X}_v \neq 0) & \text{if } v \in I. \end{cases}$$

In particular we find, using (7), that for $v \in I$, $H(\mathbf{Y}_v | \mathbf{I} = I) = \log \lambda_v$. Thus

$$\begin{aligned} H(\mathbf{Y}) &= H(\mathbf{I}) + H(\mathbf{Y}|\mathbf{I}) \\ &= \sum_I \Pr(I)[\log(1/\Pr(I)) + H(\mathbf{Y}|\mathbf{I} = I)] \\ &= \log Z. \end{aligned}$$

□

Proof of Theorem 1.2. Let G and the values λ_x be as in the statement of the theorem, and $\mathbf{Y}$ as above. We must show that

$$H(\mathbf{Y}) \leq (M/n)\log((1+\mu)^n + (1+\lambda)^n - 1). \tag{9}$$

The proof of this is similar to the proof of Theorem 1.1 in [9]. Denote by $N(v)$ the set of neighbors of vertex v and by Q_v the event $\{\mathbf{Y}_w = 0 \ \forall w \in N(v)\}$, and set $q_v = \Pr(Q_v)$.

Letting v run over A, and again writing $\mathbf{Y}_W$ for $(\mathbf{Y}_w : w \in W)$, we have, using Lemma 2.1,

$$\begin{aligned} H(\mathbf{Y}) &= H(\mathbf{Y}_A|\mathbf{Y}_B) + H(\mathbf{Y}_B) \\ &\leq \sum_v [H(\mathbf{Y}_v|\mathbf{Y}_B) + \frac{1}{n}H(\mathbf{Y}_{N(v)})]. \end{aligned}$$

We may rewrite the entropy terms in the sum as

$$H(\mathbf{Y}_v|\mathbf{Y}_B) = q_v H(\mathbf{Y}_v|Q_v) = q_v \log(1+\mu)$$

and

$$H(\mathbf{Y}_{N(v)}) = H(q_v) + (1-q_v)H(\mathbf{Y}_{N(v)}|\overline{Q}_v),$$

and we will show below that

$$H(\mathbf{Y}_{N(v)}|\overline{Q}_v) \leq \log((1+\lambda)^{d(v)} - 1) \leq \log((1+\lambda)^n - 1). \tag{10}$$

Combining the preceding identities and bounds gives

$$H(\mathbf{Y}) \leq \sum [q_v \log(1+\mu) + (1/n)\{H(q_v) + (1-q_v)\log((1+\lambda)^n - 1)\}]. \tag{11}$$

The contribution of v,

$$\frac{1}{n}\log((1+\lambda)^n - 1) + \frac{1}{n}[H(q_v) + q_v(n\log(1+\mu) - \log((1+\lambda)^n - 1))],$$

is maximized at

$$q_v = \frac{2^T}{2^T+1} = \frac{(1+\mu)^n}{(1+\mu)^n + (1+\lambda)^n - 1}\ , \tag{12}$$

where $T = n\log(1+\mu) - \log((1+\lambda)^n - 1)$, and inserting this value of q_v in (11) gives (9). (The final calculation can be avoided by observing that when G is $K_{n,n}$, (11) gives away nothing and (12) gives the actual value of q_v for every v.)

To complete the proof of the theorem, we need to verify (the first inequality in) (10). Let $\mathbf{T} = \mathbf{I} \cap N(v)$ and observe that

$$H(\mathbf{Y}_{N(v)}|\mathbf{T} = T) = |T|\log\lambda.$$

Thus, setting $p_T = \Pr(\mathbf{T} = T|\overline{Q}_v)$, we have

$$H(\mathbf{Y}_{N(v)}|\overline{Q}_v) = \sum_T p_T(\log(1/p_T) + |T|\log\lambda).$$

That this is at most $\log((1+\lambda)^{d(v)} - 1)$ is an instance of the following more general inequality, which is itself just an example of the nonnegativity of information-theoretical divergence. □

Proposition 3.1. *For any $\lambda \geq 0$, $\mathcal{S} \subseteq 2^B$ (B any set), and probability distribution p on $\mathcal{S}$,*

$$\sum_{S\in\mathcal{S}} p_S(\log(1/p_S) + |S|\log\lambda) \leq \log\sum_{S\in\mathcal{S}} \lambda^{|S|}. \tag{13}$$

Proof. Set $W = \sum_{S\in\mathcal{S}} \lambda^{|S|}$ and $q_S = \lambda^{|S|}/W$. Then the left-hand side of (13) may be rewritten as

$$\sum_{S\in\mathcal{S}} p_S \log\frac{q_S}{p_S} + \log W \leq \log W.$$

□

4. Proof of Theorem 1.5

We proceed by induction on k. As noted earlier, the case $k = 2$ is Theorem 1.2, so we assume $k \geq 3$.

Set $\mathcal{A}(P) = \mathcal{A}$ and for $I \in \mathcal{A}$ set $w(I) = \prod_{x\in I}\lambda_x$. (So we should bound $\sum_{I\in\mathcal{A}} w(I)$.)

For $X \subseteq P_k$ let $M(X) = |\{y \in P_{k-1} : y \not< x\ \forall x \in X\}|$. Then the poset $P_X := \{z \in P_1 \cup \cdots \cup P_{k-1} : z \not< x\ \forall x \in X\}$ (with the order inherited from P) satisfies the hypotheses of Theorem 1.5 with k replaced by $k-1$ and M by $M(X)$ (and weights $\lambda_1, \ldots, \lambda_{k-1}$). So by induction,

$$\sum\{w(I) : I \in \mathcal{A},\ I \cap P_k = X\} \leq \lambda_k^{|X|}\left[\sum_{i=1}^{k-1}(1+\lambda_i)^n - (k-2)\right]^{M(X)/n}. \tag{14}$$

Set $\lambda = \left[\sum_{i=1}^{k-1}(1+\lambda_i)^n - (k-2)\right]^{1/n} - 1$ (≥ 1). Then summing (14) over X we have

$$\sum_{I\in\mathcal{A}} w(I) \leq \sum_{X\subseteq P_k} \lambda_k^{|X|}(1+\lambda)^{M(X)}. \tag{15}$$

So setting $A = P_k$, $B = P_{k-1}$, $E(G) = \{\{x,y\} : x \in A, y \in B, x > y\}$, and $\mu = \lambda_k$ (and $\lambda = \lambda$), and applying Theorem 1.2, we find that the right-hand side of (15) is

$$\begin{aligned}\sum_{I\in\mathcal{I}(G)} \mu^{|I\cap A|}\lambda^{|I\cap B|} &\leq [(1+\mu)^n + (1+\lambda)^n - 1]^{M/n} \\ &= [\sum_{i=1}^{k}(1+\lambda_i)^n - (k-1)]^{M/n}. \quad \square\end{aligned}$$

5. Proof of Corollary 1.4

Say that a poset Q is a *relaxation* of the poset P if P and Q have the same ground set and $y \lessdot x$ in Q implies $y \lessdot x$ in P. This clearly implies $a(Q) \geq a(P)$, so Theorem 1.3 contains Corollary 1.4 via the following easy observation.

Proposition 5.1. *For each m there is a poset P^m graded by $\{0, 1, \ldots, m\}$, satisfying the hypotheses of Theorem 1.3 with $M = \binom{m}{\lfloor m/2\rfloor}$ and $n = \lceil m/2\rceil$, and containing some relaxation of $\mathcal{B}_m$ as a subposet.*

Proof. For a poset P let $D(P)$ be the directed graph whose arcs are the cover relations of P ($y \to x$ in $D(P)$ iff $y \lessdot x$ in P). Of course $D(P)$ determines P and vice versa. The desired P^m is obtained from $\mathcal{B}_m$ via the following three steps:

Step 1. Delete arcs from $D(\mathcal{B}_m)$ until all in-degrees are at most $\lceil m/2\rceil$. (That is, for each x at level $i > \lceil m/2\rceil$ we delete $i - \lceil m/2\rceil$ arcs entering x.)

Step 2. For $i > \lceil m/2\rceil$ add $\binom{m}{\lceil m/2\rceil} - \binom{m}{i}$ new elements to level i.

Step 3. For $i \geq \lceil m/2\rceil$ add arcs between levels i and $i+1$ so that all degrees between these two levels are $\lceil m/2\rceil$. (It is easy to see that this only requires that the number of new elements at level $i+1$ be at least $\lceil m/2\rceil$.) $\square$

Added in proof

N. Pippenger (*Entropy and enumeration of Boolean functions*, IEEE Trans. Info. Th. **45** (1999), 2096–2100) uses an approach akin to that of [11], but based on entropy, to prove the slightly weaker $\log\psi(m) < (1 + O(\log^{3/2} m/m^{1/4}))\binom{m}{\lfloor m/2\rfloor}$.

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