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INTERPOLATION IN NEST ALGEBRA MODULES

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Abstract. Let $\mathcal{A}$ be a nest algebra and $\mathcal{L}at\mathcal{A}$ its invariant projection (or subspace) lattice. In this paper, using order homomorphisms of $\mathcal{L}at\mathcal{A}$, we give necessary and sufficient conditions on bounded linear operators X and Y on a Hilbert space to guarantee the existence of an operator A in a certain $\mathcal{A}$-module such that $AX = Y$.

Introduction

In the study of a reflexive operator algebra $\mathcal{A}$, it is useful to know for which vectors x and y there exists $A \in \mathcal{A}$ satisfying $Ax = y$. E.C. Lance initiated the study of such problems and gave necessary and sufficient conditions on vectors x and y to guarantee the existence of an operator A in a given nest algebra Alg$\mathcal{N}$ such that $Ax = y$, where x and y are vectors in a Hilbert space H and $\mathcal{N}$ is a nest. In addition, Lance specified the minimum norm such an operator can have. A. Hopenwasser [4] extended Lance's result to the case where the nest $\mathcal{N}$ is replaced by an arbitrary commutative invariant projection lattice. Munch [8] restricted the operator A to be a Hilbert-Schmidt operator in nest algebra Alg$\mathcal{N}$. In [1] the authors considered the problem of finding A so that $Ax = y$ and A is required to be in certain ideals contained in Alg$\mathcal{N}$. All the results mentioned above are in the case of *"vector interpolation"*.

There is another problem related to the one above: for a given operator algebra $\mathcal{A}$ on a Hilbert space H, for which bounded operators X and Y on H does there exist an A in $\mathcal{A}$ satisfying $AX = Y$? In [5] the authors solved this problem in nest algebra and gave a sufficient and necessary condition. We call this case *operator interpolation.* It is clear that the *"operator interpolation"* problem includes the case of the *"vector interpolation"* problem. Indeed, if we denote by $x \otimes m^*$ the rank-one operator defined by the equation $x \otimes m^*(u) = \langle m, u\rangle x$, and if we set $X = x \otimes u^*$ and $Y = y \otimes u^*$, then the equations $AX = Y$ and $Ax = y$ represent the same restriction on A.

In this paper, we consider the operator interpolation problem in the case of a nest algebra module. We follow the suggestion hinted in [5] and use the order

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homomorphisms of the invariant projection lattice to solve the operator interpolation problem and give a sufficient and necessary condition in following Theorem 1, extending the main result in [5].

Interpolation in nest algebra modules

Let H be a separable Hilbert space, $B(H)$ the set of all bounded linear operators on H and $\mathcal{A}$ an algebra in $B(H)$.

First, we give some notations and definitions.

A projection $P \in B(H)$ is *invariant* under $\mathcal{A}$ if $AP = PAP$ for any $A \in \mathcal{A}$. A closed subspace S of H is *invariant* under $\mathcal{A}$ if $AS \subseteq S$ for any $A \in \mathcal{A}$. A *nest* $\mathcal{N}$ is a family of closed subspaces of a Hilbert space H totally ordered by inclusion. $\mathcal{N}$ is *complete* if it contains (0) and H and contains the join (closed linear span) and meet (intersection) of every subfamily.

Since there is a one-to-one correspondence between closed subspaces and projections, a complete *nest* is also described as a strongly closed, linearly ordered collection of projections on H, containing 0 and the identity.

A *nest algebra* corresponding a nest $\mathcal{N}$ is the set of all bounded linear operators on H that leave each projection (or subspace) in $\mathcal{N}$ invariant.

If $\mathcal{A}$ is a nest algebra, $\mathcal{L}at\mathcal{A}$ denotes the invariant projection (or subspace) lattice. In this paper all nests we consider are always complete.

Definition 1. Let $\mathcal{A}$ be a nest algebra. We call ϕ an order homomorphism of $\mathcal{L}at\mathcal{A}$ into itself if $E \leq F$ implies $\phi(E) \leq \phi(F)$ for any $E, F \in \mathcal{L}at\mathcal{A}$. Denote

$$\text{Hom } \mathcal{L}at\mathcal{A} = \{\phi : \phi \text{ is an order homomorphism from } \mathcal{L}at\mathcal{A} \text{ into } \mathcal{L}at\mathcal{A}\}$$

and for any $\phi \in \text{Hom}\mathcal{L}at\mathcal{A}$,

$$\mathcal{U}_\phi = \{G \in B(H) : \phi(E)^\perp G E = 0, \text{ for all } E \in \mathcal{L}at\mathcal{A}\}, \quad \phi \in \text{ Hom}\mathcal{L}at\mathcal{A}.$$

We know that $\mathcal{U}_\phi$ is a weakly closed $\mathcal{A}$-module ([6]).

Definition 2 (see [6])**.** The weakly closed $\mathcal{A}$-module $\mathcal{U}$ is said to be determined by the order homomorphism ϕ if $\mathcal{U} = \mathcal{U}_\phi$.

We define

$$H[\mathcal{U}] = \{\phi \in \text{ Hom}\mathcal{L}at\mathcal{A} : \ \phi \text{ determines } \mathcal{U}\}.$$

By [6], we know that for a given weakly closed $\mathcal{A}$-module $\mathcal{U}$, there is an element η in $H[\mathcal{U}]$ such that $\eta(0) = 0$.

First, we introduce a theorem of Douglas [3] and an extension theorem (in [5]).

Theorem D ([3])**.** *Let Y and X be bounded operators on a Hilbert space H. The following statements are equivalent:*

(1) $\mathrm{r}(Y^*) \subseteq \mathrm{r}(X^*)$;
(2) $Y^*Y \leq \lambda^2 X^*X$, *for some* $\lambda \geq 0$;
(3) *there exists a bounded operator A on H so that $AX = Y$.*

Here $\mathrm{r}(T)$ *means the range of operator T. Moreover if the above conditions are valid, operator A is unique and*:

(a) $\|A\|^2 = \inf\{u : Y^*Y \leq uX^*X\}$;
(b) $\ker(Y^*) = \ker(A^*)$; *and*
(c) $\mathrm{r}(A^*) \subseteq \mathrm{r}(X)^-$.

Theorem E ([5]). *Let H, K and L be Hilbert spaces, and let N be a subspace of H. Let A and D be operators in $B(N,K)$ and in $B(H,L)$ respectively and suppose that, for every vector x in N, we have $\|Ax\| \leq k\|Dx\|$ for some fixed positive constant k. Then there exists an operator $\widetilde{A}$ in $B(H,K)$, such that:*

a) $\widetilde{A}x = Ax$ for every $x \in N$;
b) $\|\widetilde{A}x\| \leq k\|Dx\|$ for every $x \in H$.

Theorem 1. *Let $\mathcal{A}$ be a nest algebra, $\mathcal{U}$ a weakly closed $\mathcal{A}$-module, and ϕ be in $H[\mathcal{U}]$ satisfying $\phi(0) = 0$. Let X and Y be bounded operators on H. The following are equivalent:*

(1) *there exists an operator A in $\mathcal{U}$ such that $AX = Y$.*
(2) $\sup\{\frac{\|(\phi(E))^{\perp}Yf\|}{\|E^{\perp}Xf\|}$*: $f \in H$, $E \in \mathcal{L}at\mathcal{A}\} = k < \infty$.*

[*We use the convention $0/0 = 0$, when necessary.*] *Moreover if case* (2) *holds, we may choose the operator A so that $\|A\| = k$.*

Remark. When ϕ is an identity homomorphism, then Theorem 1 turns into Theorem 2 in [5].

Proof. (1) $\Longrightarrow$ (2): Suppose there exists the indicated operator A in $\mathcal{U}$, such that $AX = Y$. For any projections E in $\mathcal{L}at\mathcal{A}$, and for f in H, we have

$$\begin{aligned}\|\phi(E)^{\perp}Yf\| &= \|\phi(E)^{\perp}AXf\| = \|\phi(E)^{\perp}A(I-E+E)Xf\| \\ &\leq \|\phi(E)^{\perp}AEXf\| + \|\phi(E)^{\perp}AE^{\perp}Xf\|.\end{aligned}$$

Since $\phi(E)^{\perp}AE = 0$, for every E in $\mathcal{L}at\mathcal{A}$, we have

$$\begin{aligned}\|\phi(E)^{\perp}Yf\| &\leq \|\phi(E)^{\perp}AE^{\perp}Xf\| \leq \|\phi(E)^{\perp}A\|.\|E^{\perp}Xf\| \\ &\leq \|A\|\ \|E^{\perp}Xf\|.\end{aligned}$$

Therefore $\|A\|$ may play the role of the desired constant k.

(2) $\Longrightarrow$ (1): Let X and Y be fixed bounded linear operators. We first consider a slightly simpler case. We show by induction that for an arbitrary finite subnest, $\mathcal{N}_{[n]} = \{0, E_1, E_2, \cdots, E_n, I\} \subseteq \mathcal{L}at\mathcal{A}$, under the assumption

$$\sup\{\frac{\|(\phi(E))^{\perp}Yf\|}{\|E^{\perp}Xf\|}\colon\ f \in H,\ E \in \mathcal{N}_{[n]}\} = k < \infty,$$

there exists an operator $A_{[n]} \in \mathcal{U}_{\phi[n]}$ satisfying $A_{[n]}X = Y$, where

$$\mathcal{U}_{\phi[n]} = \{G \in B(H) : \phi(E)^{\perp}GE = 0, E \in \mathcal{N}_{[n]}\}.$$

First step. If there are no projections in the subnest other than 0 and I, by assumption (2):

if $E = I$, we have $\|\phi(I)^{\perp}Yf\| \leq \|I^{\perp}Xf\| = 0$; hence $Y = \phi(I)Y$;
if $E = 0$, $\mathrm{r}(Y^*) \subseteq \mathrm{r}(X^*)$, we have

$$\|\phi(0)^{\perp}Yf\| \leq k\|Xf\| \Longrightarrow \|Yf\|^2 \leq k^2\|Xf\|^2.$$

It is apparent that $Y^*Y \leq k^2X^*X$. By Theorem D, we have $\mathrm{r}(Y^*) \subseteq \mathrm{r}(X^*)$. By Douglas' theorem, we obtain an operator B such that $BX = Y$. Set $A_{[0]} = \phi(I)B$; obviously, $A_{[0]}$ belongs to

$$\mathcal{U}_{[0]} = \{G \in B(H) : \phi(E)^{\perp}GE = 0, E = 0, E = I\},$$

and $A_{[0]}X = \phi(I)BX = \phi(I)Y = Y$. The fact that $\|A_{[0]}\|$ can be taken equal to k is a consequence of condition (a) of Theorem D.

Second step. Now, suppose that the statement is true for arbitrary finite subnests of $\mathcal{L}at\mathcal{A}(=\mathcal{N})$ with no more than $n-1$ nontrivial projections.

Let $\{0, E_1, E_2, \cdots, E_n, I\} = \mathcal{N}_{[n]}$ be a finite subnest (of $\mathcal{L}at\mathcal{A}$) with n nontrivial projections; correspondingly, there exists $\mathcal{U}_{\phi[n]}$.

Denote

$$X_1 = E_1^{\perp}X \qquad \text{and} \qquad Y_1 = \phi(E_1)^{\perp}Y.$$

We have for each $j = 2, 3, ..., n$,

$$X_1^* E_j^{\perp} = (E_1^{\perp}X)^* E_j^{\perp} = X^* E_1^{\perp} E_j^{\perp} = X^* E_j^{\perp}$$

and

$$Y_1^* \phi(E_j)^{\perp} = (\phi(E_1)^{\perp}Y)^* \phi(E_j)^{\perp} = Y^* \phi(E_1)^{\perp}\phi(E_j)^{\perp} = Y^*\phi(E_j)^{\perp}.$$

In the last equality, we use $\phi(E_1) \leq \phi(E_j)$, since ϕ is an order homomorphism.

Consequently for each $j = 2, 3, ..., n$,

$$\begin{aligned}
\|\phi(E_j)^{\perp}Y_1 f\| &= \|\phi(E_1)^{\perp}\phi(E_j)^{\perp}Y_1 f\| \\
&= \|\phi(E_j)^{\perp}\phi(E_1)^{\perp}Y_1 f\| \\
&= \|\phi(E_j)^{\perp}Y f\| \\
&\leq k\|E_j^{\perp}Xf\| = k\|E_j^{\perp}E_1^{\perp}Xf\| \\
&= k\|E_j^{\perp}X_1 f\| \text{ [by (2)].}
\end{aligned}$$

Thus, it follows that assumption (2) holds for the operators X_1 and Y_1 with respect to the subnest $\mathcal{N}'_{[n]} = \{0, E_2, ..., E_n, I\}$. Using the inductive hypothesis, there exists an operator A' in $\mathcal{U}'_{\phi[n]}$ such that $\|A'\| \leq k$, and $A'X_1 = Y_1$, where

$$\mathcal{U}'_{\phi[n]} = \{G \in B(H) : \phi(E)^{\perp}GE = 0, E \in \mathcal{N}'_{[n]}\}.$$

Since $\phi(E_1)^{\perp}Y = (\phi(E_1)^{\perp})^2 Y = \phi(E_1)^{\perp}Y_1$, we have

$$\phi(E_1)^{\perp}Y = Y_1 = A'X_1 = A'E_1^{\perp}X = \phi(E_1)^{\perp}A'E_1^{\perp}X.$$

Set $B = \phi(E_1)^{\perp}A'E_1^{\perp}$. Since the projections in $\mathcal{N}_{[n]}$ all commute with each other, it is clear that B is in

$$\mathcal{U}_{\phi_{[n]}} = \{G \in B(H) : \phi(E)^{\perp}GE = 0, E \in \mathcal{N}_{[n]}\}.$$

Indeed, (i) if $E = E_1$, then

$$\phi(E_1)^{\perp}BE = \phi(E_1)^{\perp}A'E_1^{\perp}E_1 = 0.$$

Also (ii) if $E = E_j$ for each $j = 2, 3, ..., n$, then

$$\begin{aligned}
\phi(E_j)^{\perp}BE_j &= \phi(E_j)^{\perp}\phi(E_1)^{\perp}A'E_1^{\perp}E_j \\
&= \phi(E_1)^{\perp}.(\phi(E_j)^{\perp}A'E_j)E_1^{\perp} = 0.
\end{aligned}$$

From above, B is in $\mathcal{U}_{\phi[n]}$. Moreover,

$$BX = \phi(E_1)^{\perp}A'E_1^{\perp}X = \phi(E_1)^{\perp}A'X_1 = \phi(E_1)^{\perp}Y$$

and $\|B\| = \|\phi(E_1)^{\perp}A'E_1^{\perp}\| \leq \|A'\| \leq k$.

Next: $\mathrm{r}(Y^*\phi(E_1)) \subseteq \mathrm{r}(Y^*) \subseteq \mathrm{r}(X^*)$ follows from choosing $E = 0$ in part (2). (Since $\phi(0) = 0^\perp = I$.) Theorem D then asserts the existence of an operator A_0 for which $A_0X = \phi(E_1)Y$. For any vector f, we have

$$\begin{aligned}\|A_0Xf\|^2 + \|BXf\|^2 &= \|\phi(E_1)Yf\|^2 + \|\phi(E_1)^\perp Yf\|^2\\ &= \|Yf\|^2 \le k^2\|Xf\|^2;\end{aligned}$$

therefore

$$\begin{aligned}\|A_0Xf\|^2 &\le k^2\|Xf\|^2 - \|BXf\|^2 = k^2\langle Xf, Xf\rangle - \langle BXf, BXf\rangle\\ &= \langle k^2IXf, Xf\rangle - \langle B^*BXf, Xf\rangle\\ &= \langle (k^2I - B^*B)Xf, Xf\rangle\\ &\le \|(k^2I - B^*B)^{1/2}Xf\|^2.\end{aligned}$$

So we have the form

$$\|A_0g\|^2 \le \|Dg\|^2,\ g \in \mathrm{r}(X)^- \text{ and } D = (k^2I - B^*B)^{1/2}.$$

Applying Theorem E with $N = \mathrm{r}[X]^-$, $K = \mathrm{r}[\phi(E_1)]$ and $L = H$, $A_0 \in B(N, K)$, $D \in B(H, H)$, we see that there is an operator $\widetilde{A}$ in $B(H, \mathrm{r}[\phi(E_1)])$ such that $\|\widetilde{A}x\| \le \|Dx\|$ for all x in H, and such that $\widetilde{A}Xf = A_0Xf$ for all f in H. Finally set $A_{[n]} = \widetilde{A} + B$. It is the $A_{[n]}$ that we are looking for.

First: we have $A_{[n]} \in \mathcal{U}_{\phi[n]}$. We only need to check that $\widetilde{A}$ is in $\mathcal{U}_{\phi[n]}$.

For (i) if $E = E_1$, since $\widetilde{A}E_1H \subseteq \mathrm{r}[\phi(E_1)]$, then $\phi(E_1)^\perp\widetilde{A}E_1 = 0$.

For (ii) if $E = E_j, i = 2, 3, ..., n$, then we have

$$\begin{aligned}\phi(E_i)^\perp\widetilde{A}E_i &= \phi(E_1)^\perp\phi(E_i)^\perp\widetilde{A}E_i \quad (\text{since } E_i > E_1 \text{ so } \phi(E_i)^\perp \le \phi(E_1)^\perp)\\ &= \phi(E_i)^\perp\phi(E_1)^\perp\widetilde{A}E_i.\end{aligned}$$

Since $\mathrm{r}[\widetilde{A}E_i] \subseteq \mathrm{r}[\phi(E_1)]$, we have $\phi(E_1)\widetilde{A}E_i = 0$ and $\phi(E_i)^\perp\widetilde{A}E_i = 0$, from which we have $\widetilde{A}$ in $\mathcal{U}_{\phi[n]}$.

Second: we have

$$\begin{aligned}A_{[n]}X &= \widetilde{A}X + BX = A_0X + BX\\ &= \phi(E_1)^\perp Y + \phi(E_1)Y = Y.\end{aligned}$$

Third: the facts $\mathrm{r}[\widetilde{A}] \subseteq \mathrm{r}[\phi(E_1)]$ and $\mathrm{r}[B] \subseteq \mathrm{r}[\phi(E_1)^\perp]$ mean that for any vector x,

$$\begin{aligned}\|A_{[n]}x\|^2 &= \|\widetilde{A}x + Bx\|^2 = \|\widetilde{A}x\|^2 + \|Bx\|^2 \le \|Dx\|^2 + \|Bx\|^2\\ &= \langle (k^2I - B^*B)^{1/2}x, (k^2I - B^*B)^{1/2}x\rangle + \|Bx\|^2\\ &= \langle (k^2I - B^*B)x, x\rangle + \|Bx\|^2\\ &= k^2\|x\|^2 - \|Bx\|^2 + \|Bx\|^2 = k^2\|x\|,\end{aligned}$$

so

$$(*) \qquad \|A_{[n]}\| \le k.$$

This completes the proof for any subnest of $\mathcal{L}at\mathcal{A}$ with n nontrivial projections and by induction for all finite subnests.

The remainder of the proof is the same as [5].

Consider a maximal chain $\mathcal{M}$ of finite subnests of $\mathcal{L}at\mathcal{A}$, ordered by inclusion. The net $\{A_{\mathcal{F}} \in \mathcal{U}_{\phi(\mathcal{F})} : \mathcal{F} \in \mathcal{M}\}$ is bounded (by $(*)$) and will therefore have a weak limit point, say A. Clearly, $AX = Y$. Furthermore, because, for any projection E in $\mathcal{L}at\mathcal{A}$, E will eventually lie in some finite subnest in the chain $\mathcal{M}$, we have $A \in \mathcal{U}$. This completes the proof that (2) implies (1).

Example. For any ϕ in $H[u], \phi$ satisfies condition (2) of the above theorem but $\phi(0) \neq 0$. There need not exist an A in $\mathcal{U}$ such that $AX = Y$. For example, let $\mathcal{N}$ be a nest given by $\mathcal{N} = \{0, E_1, E_2, ..., E_n, I\}$ and $n > 1$. Set $\mathcal{A} = \text{Alg}\mathcal{N}$; by [2] we know that $\mathcal{L}at\mathcal{A} = \mathcal{N}$. Let $\phi(E) = E_1$ for all E in $\mathcal{N}$, $Y = E_1$ and $X = 0$. We have

$$\sup\{\frac{\|\phi(E)^{\perp}Yf\|}{\|E^{\perp}Xf\|};\ E \in \mathcal{N}, f \in H\} = 0;$$

therefore, ϕ satisfies condition (2), but for any G in $B(H), GX = 0 \neq E_1, GX \neq Y$.

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