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EXTREME POINTS OF WEAKLY CLOSED $\mathcal{T}(\mathcal{N})$–MODULES

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Abstract. In this paper, we first characterize the rank one operators in the preannihilator $\mathcal{U}_\perp$ of a weakly closed $\mathcal{T}(\mathcal{N})$–module $\mathcal{U}$. Using this characterization for the rank one operators in $\mathcal{U}_\perp$, a complete description of the extreme points of the unit ball $\mathcal{U}_1$ is given. Finally, we show how to apply the techniques of the present paper to other operator systems and characterize their extreme points.

1. Introduction

During the last decade there has been an interest in the Banach space geometry of non–selfadjoint operator algebras. Subjects of investigation there include the extreme point structure of the unit ball [6], [9] and the isometries between such algebras [3], [7], [10]. One possible application of a characterization of extreme points is in the study of isometries. R.V.Kadison [7] used the characterization of extreme points in his study of isometries on C^*–algebras. For the same end, R.L.Moore and T.T.Trent [9] investigated the extreme points for nest algebras. In [1], M.Anoussis and E.G.Katsoulis proved the surprising fact that for any nest with no finite atoms, the convex hull of the unitary elements which are, of course, extreme points would suffice to recover the whole ball. This result was refined and reached its final form by K.R.Davidson [4]. But there is little study about the extreme points of non-algebraic operator systems such as ideals, modules, etc. In this paper, our focus is on weakly closed $\mathcal{T}(\mathcal{N})$–modules $\mathcal{U}$. Since $\mathcal{U}$ is not an algebra in general, we cannot apply the techniques in the main result, Theorem 6 in [9]. Instead, we will apply the techniques of preannihilators to prove the main result of ours.

Let us introduce some notation and terminology. $\mathcal{H}$ represents a complex separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators on $\mathcal{H}$, and $\mathcal{F}(\mathcal{H})$ the set of finite rank operators on $\mathcal{H}$. A nest $\mathcal{N}$ is a chain of closed subspaces of Hilbert space $\mathcal{H}$ containing (0) and $\mathcal{H}$ which is closed under intersection and closed span. If N is an element of a nest $\mathcal{N}$, then N_- denotes the immediate predecessor of N. Similarly we define N_+ as the immediate successor of N. If $\mathcal{N}$ is a nest, then the nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$.

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Suppose that $N \to \widetilde{N}$ is an order homomorphism of $\mathcal{N}$ into itself (that is, $N \leq N^{'}$ implies $\widetilde{N} \leq \widetilde{N'}$). Then the set

$$\mathcal{U} = \{X \in \mathcal{B}(\mathcal{H}) : XN \subseteq \widetilde{N}, \forall N \in \mathcal{N}\}$$

is clearly a weakly closed subset of $\mathcal{B}(\mathcal{H})$ and is easily seen to be a two sided $\mathcal{T}(\mathcal{N})$–module. For each $N \in \mathcal{N}$, define

$$N_* = \wedge\{\widetilde{N'} : N^{'} > N\}$$

and

$$N_\sim = \vee\{N^{'} : \widetilde{N'} < N\}.$$

Since $\mathcal{N}$ is complete, N_* and $N_\sim$ are in $\mathcal{N}$. In [5], S.A.Erdos and S.C.Power gave a complete description of weakly closed $\mathcal{T}(\mathcal{N})$–modules, which we state here as lemmas.

Lemma 1.1 ([5], Theorem 1.5)**.** *Any weakly closed $\mathcal{T}(\mathcal{N})$–module $\mathcal{U}$ is of the form*

$$\mathcal{U} = \{X \in \mathcal{B}(\mathcal{H}) : XN \subseteq \widetilde{N}, \forall N \in \mathcal{N}\},$$

where $N \to \widetilde{N}$ is a left order continuous order homomorphism of $\mathcal{N}$ into $\mathcal{N}$.

If x and y are nonzero vectors in $\mathcal{H}$, we define the rank one operator $x \otimes y$ by

$$(x \otimes y)(z) = (z, y)x, \qquad \forall z \in \mathcal{H}.$$

Lemma 1.2 ([5], Lemma 1.1)**.** *Let $\mathcal{U} = \{X \in \mathcal{B}(\mathcal{H}) : XN \subseteq \widetilde{N}, \forall N \in \mathcal{N}\}$, where $N \to \widetilde{N}$ is an order homomorphism of $\mathcal{N}$ into $\mathcal{N}$. Then the following are equivalent:*

1) A rank-one operator $e \otimes f \in \mathcal{U}$.
2) For some $N \in \mathcal{N}$, such that $e \in N$ and $f \in N_\sim^\perp$.
3) For some $M \in \mathcal{N}$, such that $e \in M_*$ and $f \in M^\perp$.

Suppose that $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ the dual space of $\mathcal{X}$. For $\mathcal{S} \subseteq \mathcal{X}$ and $\mathcal{W} \subseteq \mathcal{X}^*$, define

$$\begin{array}{rcl} \mathcal{S}^\perp & = & \{x^* \in \mathcal{X}^* : x^*(x) = 0, \forall x \in \mathcal{S}\}, \\ \mathcal{W}_\perp & = & \{x \in \mathcal{X} : x^*(x) = 0, \forall x^* \in \mathcal{W}\}; \end{array}$$

we will call $\mathcal{S}^\perp$ the annihilator of $\mathcal{S}$ and $\mathcal{W}_\perp$ the preannihilator of $\mathcal{W}$.

The entire paper consists of four sections. Besides this introductory section, section 2 is devoted to the study of rank one operators in the preannihilator $\mathcal{U}_\perp$ of $\mathcal{U}$. In section 3, we characterize the extreme points of the unit ball of $\mathcal{U}$. As an application of the techniques in this paper, we study the extreme points of diagonal-disjoint left ideal $\mathcal{I}$ of a nest algebra $\mathcal{T}(\mathcal{N})$.

2. Some lemmas

Definition 2.1. Let $(\mathcal{X}, \| \cdot \|)$ be a complex normed space, and let $\mathcal{X}_r$ denote the closed ball with center 0 and radius r. For $A \in \mathcal{X}_1$, if $B \in \mathcal{X}$ satisfies

$$\| A \pm B \| \leq 1,$$

then we will say that B is a contractive perturbation of A.

Lemma 2.2. *Let $(\mathcal{X}, \| \cdot \|)$ be a complex normed space and $A \in \mathcal{X}_1$. Then A is not an extreme point of $\mathcal{X}_1$ if and only if A has a nonzero contractive perturbation in $\mathcal{X}$.*

Proof. Suppose that B is a nonzero contractive perturbation of A; then

$$\| A \pm B \| \leq 1.$$

Thus, $A \pm B \in \mathcal{X}_1$ and $A = \frac{(A+B)+(A-B)}{2}, A \neq A+B, A \neq A-B$. Hence A is not an extreme point of $\mathcal{X}_1$.

Suppose, on the contrary, that A is not an extreme point of $\mathcal{X}_1$. Thus, there exist $C, D \in \mathcal{X}_1$ such that $A = \frac{C+D}{2}, A \neq C, A \neq D$. Set $B = \frac{C-D}{2} \neq 0$; we have

$$\| A \pm B \| \leq 1.$$

So B is a nonzero contractive perturbation of A. □

Lemma 2.3. *Suppose that $A, B \in \mathcal{B}(\mathcal{H}), \| A \| \leq 1$ and $\| A \pm B \| \leq 1$. Then there exist bounded operators S and T such that $B = S(I - A^*A)^{\frac{1}{2}}$ and $B = (I - AA^*)^{\frac{1}{2}}T$.*

Proof. The proof proceeds similarly to the proof of [9], Lemma 1. □

Lemma 2.4 ([2], Theorem 1)**.** *Let $A \in \mathcal{B}(\mathcal{H})_1$ and $X \in \mathcal{B}(\mathcal{H})_{\frac{1}{2}}$. Then*

$$\| A \pm (I - AA^*)^{\frac{1}{2}} X (I - A^*A)^{\frac{1}{2}} \| \leq 1.$$

Lemma 2.5. *Let $A \in \mathcal{B}(\mathcal{H})_1$, and suppose that B is a nonzero contractive perturbation of A. For any $Y \in \mathcal{B}(\mathcal{H})$, there exists a nonzero complex number λ_Y such that $\lambda_Y BYB$ is also a contractive perturbation of A. Moreover, A has a nonzero contractive perturbation of the form $(I - AA^*)^{\frac{1}{2}} X (I - A^*A)^{\frac{1}{2}}$ with X an operator in $\mathcal{B}(\mathcal{H})$.*

Proof. By Lemma 2.3, there exist operators S and T for which $B = S(I - A^*A)^{\frac{1}{2}} = (I - AA^*)^{\frac{1}{2}}T$. Thus

$$BYB = (I - AA^*)^{\frac{1}{2}} TYS (I - A^*A)^{\frac{1}{2}}.$$

If we choose λ_Y such that $\| \lambda_Y TYS \| \leq 1/2$, then the first statement follows Lemma 2.4. Set $Y = B^*$. Since $B \neq 0$, we have $BB^*B \neq 0$. Thus, there exists some operator X such that $(I - AA^*)^{\frac{1}{2}} X (I - A^*A)^{\frac{1}{2}}$ is a nonzero contractive perturbation of A. □

Lemma 2.6. *Let $\mathcal{U}$ be the weakly closed $\mathcal{T}(\mathcal{N})$–module determined by the left order continuous order homomorphism $N \to \widetilde{N}$. For any $N \in \mathcal{N}$, $X \in \mathcal{B}(\mathcal{H})$, then the operator $P(N)X(I - P(N_\sim)) \in \mathcal{U}$.*

Proof. For any $N' \in \mathcal{N}$, we consider separately two cases.

Case 1. $\widetilde{N'} < N$. We have

$$N' \subseteq N_\sim = \vee\{G : \widetilde{G} < N\}.$$

Thus, $(I - P(N_\sim))P(N') = 0$, so $P(N)X(I - P(N_\sim))N' = (0) \subseteq \widetilde{N'}$.

Case 2. $\widetilde{N'} \geq N$. We have

$$P(N)X(I - P(N_\sim))N' \subseteq N \subseteq \widetilde{N'}.$$

Hence for any $N' \in \mathcal{N}$, no matter in Case 1 or 2, we have $P(N)X(I - P(N_\sim))N' \subseteq \widetilde{N'}$. Therefore, the operator $P(N)X(I - P(N_\sim)) \in \mathcal{U}$. □

For the purpose of this paper, we give another form of Theorem 2.4 in [5]. Let $\mathcal{B}(\mathcal{H})_*$ be the set of w^*–continuous linear functionals on $\mathcal{B}(\mathcal{H})$. Then $\mathcal{B}(\mathcal{H})_* \cong \mathcal{C}_1(\mathcal{H})$, where $\mathcal{C}_1(\mathcal{H})$ is the ideal of trace class operators on $\mathcal{H}$.

Proposition 2.7. *Let $\mathcal{U}$ be the weakly closed $\mathcal{T}(\mathcal{N})$–module determined by the left order continuous order homomorphism $N \to \widetilde{N}$ and set*

$$\mathcal{W} = \{X \in \mathcal{B}(\mathcal{H}) : (I - P(N_\sim))XP(N) = 0, \forall N \in \mathcal{N}\}.$$

Then $\rho \in \mathcal{B}(\mathcal{H})_*$ annihilates $\mathcal{U}$ if and only if ρ is of the form

$$\rho(\cdot) = tr(X\cdot),$$

where X is a trace class operator in $\mathcal{W}$.

Proof. Necessity. If $\rho \in \mathcal{B}(\mathcal{H})_* \cong \mathcal{C}_1(\mathcal{H})$, there exists an operator $X \in \mathcal{C}_1(\mathcal{H})$ such that $\rho(\cdot) = tr(X\cdot)$ and ρ annihilates $\mathcal{U}$. For any $Y \in \mathcal{B}(\mathcal{H})$ and $N \in \mathcal{N}$, by Lemma 2.6 the operator $P(N)YP(N_\sim)^\perp \in \mathcal{U}$. Thus

$$tr(P(N_\sim)^\perp XP(N)Y) = tr(XP(N)YP(N_\sim)^\perp) = 0, \quad \forall Y \in \mathcal{B}(\mathcal{H});$$

then

$$P(N_\sim)^\perp XP(N) = 0, \qquad \forall N \in \mathcal{N}.$$

So

$$X \in \mathcal{W} \cap \mathcal{C}_1(\mathcal{H}).$$

Sufficiency. If $X \in \mathcal{W} \cap \mathcal{C}_1(\mathcal{H})$, let $e \otimes f$ be any rank one operator of $\mathcal{U}$. Then, by Lemma 1.2, there exists some element $N \in \mathcal{N}$ such that $e \in N$ and $f \in N_\sim^\perp$. Therefore,

$$e \otimes f = P(N)(e \otimes f)P(N_\sim)^\perp$$

and so

$$\begin{aligned} tr(X(e \otimes f)) &= tr(XP(N)(e \otimes f)P(N_\sim)^\perp) \\ &= tr(P(N_\sim)^\perp XP(N)(e \otimes f)) = 0. \end{aligned}$$

Since the map $tr(X\cdot)$ is linear and w^*–continuous, it follows from [5] Lemma 1.2 and Corollary 1.6 that

$$tr(XA) = 0, \qquad \forall A \in \mathcal{U}.$$

So $\rho(\cdot) = tr(X\cdot)$ annihilates $\mathcal{U}$. □

Proposition 2.7 proves $\mathcal{U}_\perp = \mathcal{W} \cap \mathcal{C}_1(\mathcal{H})$; now we can give a characterization of a rank one operator in $\mathcal{U}_\perp$.

Theorem 2.8. *Let $\mathcal{U}$ be the weakly closed $\mathcal{T}(\mathcal{N})$–module determined by the left order continuous order homomorphism $N \to \widetilde{N}$. Then, a rank one operator $x \otimes y \in \mathcal{U}_\perp$ if and only if, for some $N \in \mathcal{N}$, $x \in N$ and $y \in \widetilde{N}^\perp$.*

Proof. Sufficiency. For any $N' \in \mathcal{N}$, we consider separately two cases.

Case 1. $N' \leq \widetilde{N}$. We have

$$(x \otimes y)N' = P(N)(x \otimes y)P(\widetilde{N})^\perp N' = (0) \subseteq N'_\sim.$$

Case 2. $N' > \widetilde{N}$. We have

$$N \subseteq N'_\sim = \vee\{G : \widetilde{G} < N'\};$$

then

$$(x \otimes y)N' = P(N)(x \otimes y)P(\widetilde{N})^\perp N' \subseteq N \subseteq N'_\sim.$$

It follows from Case 1, Case 2 and Proposition 2.7 that the rank one operator $x \otimes y \in \mathcal{W} \cap \mathcal{F}(\mathcal{H}) \subseteq \mathcal{W} \cap \mathcal{C}_1(\mathcal{H}) = \mathcal{U}_\perp$.

Necessity. Suppose that a rank one operator $x\otimes y \in \mathcal{U}_\perp = \mathcal{W}\cap\mathcal{C}_1(\mathcal{H})$. It follows from the definition of $N_\sim$ that the mapping $N \to N_\sim$ is an order homomorphism of $\mathcal{N}$ into $\mathcal{N}$. Since

$$\mathcal{W} = \{X \in \mathcal{B}(\mathcal{H}) : XN \subseteq N_\sim, \forall N \in \mathcal{N}\},$$

by Lemma 1.2 there exists an element $N \in \mathcal{N}$ such that

$$x \in N \quad \text{and} \quad y \in (\vee\{N' : N'_\sim < N\})^\perp.$$

Now we will prove $\widetilde{N} \subseteq \vee\{N' : N'_\sim < N\}$.

Since $(\widetilde{N})_\sim = \vee\{G : \widetilde{G} < \widetilde{N}\} \subseteq \vee\{G : G < N\} = N_- \subseteq N$, we consider separately two cases.

Case 1. $(\widetilde{N})_\sim < N$. We have $\widetilde{N} \in \{N' : N'_\sim < N\}$, so

$$\widetilde{N} \subseteq \vee\{N' : N'_\sim < N\}.$$

Case 2. $(\widetilde{N})_\sim = N$. We have

$$(\widetilde{N})_\sim = \vee\{G : \widetilde{G} < \widetilde{N}\} = \vee\{G : G < N\} = N_- = N. \tag{1}$$

For any $G \in \mathcal{N}$ such that $G < N$, it follows from the discussion just before Case 1 that

$$(\widetilde{G})_\sim \leq G < N.$$

Thus, $\widetilde{G} \in \{N' : N'_\sim < N\}$ and

$$\vee\{\widetilde{G} : \widetilde{G} < \widetilde{N}\} \subseteq \vee\{\widetilde{G} : G < N\} \subseteq \vee\{N' : N'_\sim < N\}. \tag{2}$$

Since the mapping $M \to \widetilde{M}$ of $\mathcal{N}$ into $\mathcal{N}$ is left order continuous and

$$N = \vee\{G : \widetilde{G} < \widetilde{N}\} = \vee\{G : G < N\}$$

by equation (1), we have

$$\begin{aligned}\vee\{\widetilde{G} : \widetilde{G} < \widetilde{N}\} &= (\vee\{G : \widetilde{G} < \widetilde{N}\})^\sim = \widetilde{N}\\ &= (\vee\{G : G < N\})^\sim = \vee\{\widetilde{G} : G < N\}.\end{aligned}$$

It follows from (2) that

$$\widetilde{N} \subseteq \vee\{N' : N'_\sim < N\}.$$

By Case 1 and Case 2, we obtain

$$(\vee\{N' : N'_\sim < N\})^\perp \subseteq \widetilde{N}^\perp.$$

Therefore, $x \in N$ and $y \in \widetilde{N}^\perp$. □

3. Extreme points of the unit ball of $\mathcal{U}$

In this section, we will characterize the extreme points of the unit ball of $\mathcal{U}$.

Proposition 3.1. *Suppose that $\mathcal{U}$ is the weakly closed $\mathcal{T}(\mathcal{N})$–module determined by the left order continuous order homomorphism $N \to \widetilde{N}$, and that $B \in \mathcal{B}(\mathcal{H})$. If $BXB = 0$ for every rank one operator X in $\mathcal{U}_\perp$, then there exists an element $N_0 \in \mathcal{N}$ such that $B = P(N_{0*})BP(N_0)^\perp$.*

Proof. For any $N \in \mathcal{N}\backslash(0)$, if $B\mid_N \neq 0$, we may choose $x \in N$ with $Bx \neq 0$. If $\widetilde{N} = \mathcal{H}$, $B^* \mid_{\widetilde{N}^\perp} = 0$. If $\widetilde{N} \neq \mathcal{H}$, for any $y \in \widetilde{N}^\perp$, it follows from Theorem 2.8 that the rank one operator $x \otimes y \in \mathcal{U}_\perp$. Hence,

$$Bx \otimes B^*y = B(x \otimes y)B = 0.$$

We obtain $B^*y = 0$ and $B^* \mid_{\widetilde{N}^\perp} = 0$. Therefore, for any $N \in \mathcal{N}$, either $B\mid_N = 0$ or $B^* \mid_{\widetilde{N}^\perp} = 0$.

Set

$$N_0 = \vee\{N' \in \mathcal{N} : B\mid_{N'} = 0\}.$$

Thus,

$$B\mid_{N_0} = 0 \quad \text{and} \quad B\mid_{N'} \neq 0, \quad \forall N' > N_0.$$

So

$$B = BP(N_0)^\perp \quad \text{and} \quad B^* \mid_{\widetilde{N'}^\perp} = 0, \quad \forall N' > N_0.$$

Thus, for any $N' > N_0$, we have $B^* = B^*P(\widetilde{N'})$ and $B = P(\widetilde{N'})B$. So,

$$B = P(\widetilde{N'})BP(N_0)^\perp, \qquad \forall N' > N_0.$$

Following the definition of N_*, we obtain $B = P(N_{0*})BP(N_0)^\perp$ by taking the limit in the above equation. □

Theorem 3.2. *Suppose that $\mathcal{U}$ is the weakly closed $\mathcal{T}(\mathcal{N})$–module determined by the left order continuous order homomorphism $N \to \widetilde{N}$. If $A \in \mathcal{U}_1$ and A is not an extreme point in $\mathcal{U}_1$, then there exists in $\mathcal{U}$ a nonzero rank one contractive perturbation of A.*

Proof. By Lemma 2.2, there exists a nonzero operator B in $\mathcal{U}$ such that

$$\| A \pm B \| \leq 1.$$

Hence by Lemma 2.5, for any $X \in \mathcal{U}_\perp$, there exists a nonzero complex number λ_X such that

$$\| A \pm \lambda_X BXB \| \leq 1.$$

If there is a rank one operator $X = x \otimes y \in \mathcal{U}_\perp$ such that $\lambda_X BXB \neq 0$, now we will prove that the rank one operator $\lambda_X BXB \in \mathcal{U}$.

It follows from Theorem 2.8 that there exists $N_0 \in \mathcal{N}$ such that $x \in N_0$ and $y \in \widetilde{N_0}^\perp$. For any $N \in \mathcal{N}$, we consider separately two cases.

Case 1. $N \leq N_0$. We have $\widetilde{N} \leq \widetilde{N_0}$, and $BN \subseteq \widetilde{N} \subseteq \widetilde{N_0}$. Hence

$$\begin{aligned} \lambda_X BXBN &= \lambda_X BP(N_0)(x \otimes y)P(\widetilde{N_0})^\perp BN \\ &\subseteq \lambda_X BP(N_0)(x \otimes y)P(\widetilde{N_0})^\perp \widetilde{N_0} = (0) \subseteq \widetilde{N}. \end{aligned}$$

Case 2. $N > N_0$. We have $\widetilde{N} \geq \widetilde{N_0}$, so

$$\begin{aligned} \lambda_X BXBN &= \lambda_X BP(N_0)(x \otimes y)P(\widetilde{N_0})^\perp BN \\ &\subseteq BN_0 \subseteq \widetilde{N_0} \subseteq \widetilde{N}. \end{aligned}$$

By Case 1 and Case 2, we obtain that $\lambda_X BXBN \subseteq \widetilde{N}$, for any $N \in \mathcal{N}$. Hence the rank one operator $\lambda_X BXB \in \mathcal{U}$, and $\lambda_X BXB$ is a nonzero rank one contractive perturbation of A in $\mathcal{U}$.

Suppose, on the contrary, that $BXB = 0$ for each rank one X in $\mathcal{U}_\perp$. By Proposition 3.1, there exists an element $N_0 \in \mathcal{N}$ such that

$$B = P(N_{0*})BP(N_0)^\perp.$$

Since $B \neq 0$, we may choose a nonzero vector x with $Bx \neq 0$. Likewise, we may choose a nonzero vector y with $B^*y \neq 0$. Therefore, the rank one operator $C = B(x \otimes y)B = Bx \otimes B^*y \neq 0$ and by Lemma 1.2, we have

$$C = B(x \otimes y)B = P(N_{0*})B(x \otimes y)BP(N_0)^{\perp} \in \mathcal{U}.$$

It follows from Lemma 2.5 that there exists a nonzero complex number $\lambda_{x\otimes y}$ such that $\lambda_{x\otimes y}C$ is a nonzero rank one contractive perturbation of A in $\mathcal{U}$. The proof is complete. □

Now we are in the position to characterize the extreme point structure of $\mathcal{U}_1$ completely.

Theorem 3.3. *Suppose that $\mathcal{U}$ is the weakly closed $\mathcal{T}(\mathcal{N})$–module determined by the left order continuous order homomorphism $N \to \widetilde{N}$. If $A \in \mathcal{U}_1$, then A is extreme in $\mathcal{U}_1$ if and only if for any $N \in \mathcal{N}$, either $N \cap ran(I - AA^*)^{\frac{1}{2}}$ or $N_{\sim}^{\perp} \cap ran(I - A^*A)^{\frac{1}{2}} = (0)$.*

Proof. Sufficiency. Suppose that A is not an extreme point in $\mathcal{U}_1$. By Theorem 3.2, there exists a nonzero rank one operator $x \otimes y$ of $\mathcal{U}$ such that

$$\| A \pm x \otimes y \| \leq 1.$$

By Lemma 2.3, there exist bounded operators S and T such that

$$x \otimes y = S(I - A^*A)^{\frac{1}{2}} = (I - AA^*)^{\frac{1}{2}}T.$$

Hence, $x \in ran(I - AA^*)^{\frac{1}{2}}$ and $y \in ran(I - A^*A)^{\frac{1}{2}}$. Furthermore, since $x \otimes y \in \mathcal{U}$ and by Lemma 1.2, there exists an element $N \in \mathcal{N}$ with $x \in N, y \in N_{\sim}^{\perp}$. So

$$x \in N \cap ran(I - AA^*)^{\frac{1}{2}} \quad \text{and} \quad y \in N_{\sim}^{\perp} \cap ran(I - A^*A)^{\frac{1}{2}}.$$

This is a contradiction, so A is an extreme point of $\mathcal{U}_1$.

Necessity. Suppose that there exists an element $N \in \mathcal{N}$ with

$$N \cap ran(I - AA^*)^{\frac{1}{2}} \neq (0) \quad \text{and} \quad N_{\sim}^{\perp} \cap ran(I - A^*A)^{\frac{1}{2}} \neq (0).$$

Thus we may choose nonzero vectors x and y such that

$$0 \neq (I - AA^*)^{\frac{1}{2}}x \in N \quad \text{and} \quad 0 \neq (I - A^*A)^{\frac{1}{2}}y \in N_{\sim}^{\perp}.$$

Therefore, by Lemma 1.2, the nonzero rank one operator

$$\begin{aligned} B &= (I - AA^*)^{\frac{1}{2}}x \otimes (I - A^*A)^{\frac{1}{2}}y \\ &= (I - AA^*)^{\frac{1}{2}}(x \otimes y)(I - A^*A)^{\frac{1}{2}} \in \mathcal{U}. \end{aligned}$$

It follows from Lemma 2.4 that B is a nonzero contractive perturbation of A in $\mathcal{U}$, provided the norms of x and y are sufficiently small. By Lemma 2.2, A is not an extreme point of $\mathcal{U}_1$. This is a contradiction. □

Theorem 3.3 shows that each unitary element in $\mathcal{U}$ is an extreme point of $\mathcal{U}_1$. Would the convex hull of the set of unitary elements in $\mathcal{U}$ suffice to recover the whole ball of $\mathcal{U}$? We plan to investigate the problem in a future paper.

4. Applications

Let $\mathcal{T}(\mathcal{N})$ be a nest algebra. $\mathcal{D} = \mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^*$ is called the diagonal of $\mathcal{T}(\mathcal{N})$. Suppose $\mathcal{I}$ is a (left) ideal of $\mathcal{T}(\mathcal{N})$. If $\mathcal{D} \cap \mathcal{I} = (0)$, then we say that $\mathcal{I}$ is diagonal-disjoint. The Jacobson radical $\mathcal{R}_{\mathcal{N}}$, Larson ideal $\mathcal{R}_{\mathcal{N}}^{\infty}$ and the left ideal $\mathcal{J}_{\mathcal{N}} = \{T \in \mathcal{T}(\mathcal{N}) : s - \lim_{\mathcal{F}} \triangle_{\mathcal{F}}(T) = (0)\}$ are all diagonal-disjoint (left) ideals of $\mathcal{T}(\mathcal{N})$ and $\mathcal{R}_{\mathcal{N}} \subseteq \mathcal{R}_{\mathcal{N}}^{\infty} \subseteq \mathcal{J}_{\mathcal{N}}$.

Lemma 4.1 ([8], Theorem 2.2 and Remark 3.8)**.** *Suppose that $\mathcal{I}$ is a diagonal-disjoint (left) ideal containing the radical $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{T}(\mathcal{N})$. Then a rank one operator $x \otimes y$ belongs to $\mathcal{I}$ if and only if there exists an element $N \in \mathcal{N}$ such that $x \in N$ and $y \in N^{\perp}$.*

Proposition 4.2. *Suppose that $\mathcal{T}(\mathcal{N})$ is the nest algebra with respect to a nest $\mathcal{N}$ and that $B \in \mathcal{B}(\mathcal{H})$. If $BXB = 0$ for each rank one operator X in $\mathcal{T}(\mathcal{N})$, then there exists an element $N_0 \in \mathcal{N}$ such that $B = P(N_0)BP(N_0)^{\perp}$.*

Proof. If $B = 0$, we can choose $N_0 = \mathcal{H}$ in this case. Now we suppose that $B \neq 0$, and set

$$N_0 = \vee\{N^{'} \in \mathcal{N} : B\mid_{N'} = 0\}.$$

Thus we have $B\mid_{N_0} = 0$ and $B = BP(N_0)^{\perp}$. Since $B \neq 0$, $N_0 \neq \mathcal{H}$. From the definition of N_0, $B\mid_{N'} \neq 0$ for any $N^{'} > N_0$. Now we consider separately two cases.

Case 1. $N_{0+} \neq N_0$. We can choose $x \in N_{0+} \ominus N_0$ with $Bx \neq 0$. For any $y \in (N_{0+})_{-}^{\perp} = N_0^{\perp}$, the rank one operator $x \otimes y \in \mathcal{T}(\mathcal{N})$ and $B(x \otimes y)B = 0$. Thus $B^*y = 0$ for any $y \in N_0^{\perp}$, that is, $B^*\mid_{N_0^{\perp}} = 0$. Hence $B^* = B^*P(N_0)$ and $B = P(N_0)B$, so $B = P(N_0)BP(N_0)^{\perp}$.

Case 2. $N_{0+} = N_0$. Choose a decreasing sequence $\{N_k\}$ in $\mathcal{N}$ with $N_k > N_0$ and $\{P(N_k)\}$ converging to $P(N_0)$ in the strong operator topology. For $N_k > N_0$, we have $B\mid_{N_k} \neq 0$, and $B^*\mid_{N_k^{\perp}} = 0$. So $B^* = B^*P(N_k)$, $B = P(N_k)B$. Thus,

$$B = P(N_k)BP(N_0)^{\perp}, \qquad k = 1, 2, \cdots .$$

By taking the limit, we obtain $B = P(N_0)BP(N_0)^{\perp}$. ☐

Proposition 4.3. *Suppose that $\mathcal{I}$ is a diagonal-disjoint left ideal containing the radical $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{T}(\mathcal{N})$. If $A \in \mathcal{I}_1$ and A is not an extreme point in $\mathcal{I}_1$, then there exists in $\mathcal{I}$ a nonzero rank one contractive perturbation of A.*

Proof. By Lemma 2.2, there exists in $\mathcal{I}$ a nonzero operator B such that

$$\parallel A \pm B \parallel \leq 1.$$

It follows from Lemma 2.5 that for any $X \in \mathcal{T}(\mathcal{N})$, there exists a nonzero complex number λ_X such that

$$\parallel A \pm \lambda_X BXB \parallel \leq 1.$$

Since $\mathcal{I}$ is a left ideal of $\mathcal{T}(\mathcal{N})$ and $B \in \mathcal{I}$, the operator $\lambda_X BXB$ belongs to $\mathcal{I}$. If there is a rank one operator X in $\mathcal{T}(\mathcal{N})$ such that $\lambda_X BXB \neq 0$, then the theorem is proved.

Suppose, on the contrary, that $BXB = 0$ for each rank one operator X in $\mathcal{T}(\mathcal{N})$. By Proposition 4.2 and using the similar argument as in the proof of Theorem 3.2, one can complete the proof of the theorem. ☐

Proposition 4.3 plays the same role as Theorem 3.2 in the study of extreme points, so we can give a characterization of the extreme points of the unit ball $\mathcal{I}_1$.

Theorem 4.4. *Suppose that* $\mathcal{I}$ *is a diagonal-disjoint left ideal containing the radical* $\mathcal{R}_{\mathcal{N}}$ *of* $\mathcal{T}(\mathcal{N})$*. If* $A \in \mathcal{I}_1$*, then* A *is an extreme point in* $\mathcal{I}_1$ *if and only if for each element* $N \in \mathcal{N}$*, either* $N \cap ran(I - AA^*)^{\frac{1}{2}} = (0)$ *or* $N^{\perp} \cap ran(I - A^*A)^{\frac{1}{2}} = (0)$.

Proof. It follows from Proposition 4.3 and a similar argument as in the proof of Theorem 3.3. We leave these details to the interested readers. □

Remark 4.5. Since $\mathcal{R}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}}^{\infty}$ and $\mathcal{J}_{\mathcal{N}}$ are diagonal-disjoint (left) ideals containing the radical $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{T}(\mathcal{N})$, then Theorem 4.4 also characterizes the extreme point structure of the unit balls of these important (left) ideals in $\mathcal{T}(\mathcal{N})$.

Remark 4.6. All results in section 4 are also true for diagonal-disjoint right ideals containing the radical $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{T}(\mathcal{N})$.

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