

EXTENSIONS AND EXTREMALITY OF RECURSIVELY GENERATED WEIGHTED SHIFTS

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ABSTRACT. Given an n -step extension $\alpha : x_n, \dots, x_1, (\alpha_0, \dots, \alpha_k)^\wedge$ of a recursively generated weight sequence ($0 < \alpha_0 < \dots < \alpha_k$), and if W_α denotes the associated unilateral weighted shift, we prove that

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } (\lceil \frac{k+1}{2} \rceil + 1)\text{-hyponormal} & (n = 1), \\ W_\alpha \text{ is } (\lceil \frac{k+1}{2} \rceil + 2)\text{-hyponormal} & (n > 1). \end{cases}$$

In particular, the subnormality of an extension of a recursively generated weighted shift is independent of its length if the length is bigger than 1. As a consequence we see that if $\alpha(x)$ is a canonical rank-one perturbation of the recursive weight sequence α , then subnormality and k -hyponormality for $W_{\alpha(x)}$ eventually coincide. We then examine a converse—an “extremality” problem: Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence α and assume that $(k + 1)$ -hyponormality and k -hyponormality for $W_{\alpha(x)}$ coincide. We show that $\alpha(x)$ is recursively generated, i.e., $W_{\alpha(x)}$ is recursive subnormal.

INTRODUCTION

Let \mathcal{H} be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on \mathcal{H} . An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$ and *hyponormal* if $T^*T \geq TT^*$. Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \dots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator T is *subnormal* if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

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for all finite collections $x_0, x_1, \dots, x_k \in \mathcal{H}$ ([Br], [Con, III.1.9]). Using Choleski's algorithm for operator matrices, it is easy to see that this is equivalent to the following positivity test:

$$(0.1) \quad \begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).$$

Condition (0.1) provides a measure of the gap between hyponormality and subnormality, and k -hyponormality has been introduced and studied in an attempt to bridge that gap ([At], [Cu1], [Cu2], [CF1], [CF2], [CF3], [CL1], [CMX], [McCP]). In fact, the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of T , while subnormality requires the validity of (0.1) for all k . If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B , and if we define T to be k -hyponormal whenever the $k \times k$ operator matrix

$$(0.2) \quad M_k(T) := ([T^{*j}, T^i]_{i,j=1}^k)$$

is positive, or equivalently, the $(k + 1) \times (k + 1)$ operator matrix in (0.1) is positive, then the Bram-Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k -hyponormal for every $k \geq 1$ ([CMX]).

If W_α is the weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the moments of W_α are usually defined by $\beta_0 := 1, \beta_{n+1} := \alpha_n \beta_n$ ($n \geq 0$) [Shi]; however, we reserve this term for the sequence $\gamma_n := \beta_n^2$ ($n \geq 0$). A criterion for k -hyponormality can be given in terms of moments ([Cu1, Theorem 4]): if we build a $(k + 1) \times (k + 1)$ Hankel matrix $A(n; k)$ by

$$(0.3) \quad A(n; k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \dots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \dots & \gamma_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \dots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0),$$

then

$$(0.4) \quad W_\alpha \text{ is } k\text{-hyponormal} \iff A(n; k) \geq 0 \quad (n \geq 0).$$

In [Sta], J. Stampfli showed that given $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}$ with $0 < a < b < c$, there always exists a subnormal completion of α , but that for $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ ($a < b < c < d$) such a subnormal completion may not exist.

There are instances where k -hyponormality implies subnormality for weighted shifts. For example, in [CF3] it was shown that if $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ ($a < b < c$), then $W_{\alpha(x)}$ is 2-hyponormal if and only if it is subnormal: more concretely, $W_{\alpha(x)}$ is 2-hyponormal if and only if

$$\sqrt{x} \leq H_2(\sqrt{a}, \sqrt{b}, \sqrt{c}) := \sqrt{\frac{ab(c-b)}{(b-a)^2 + b(c-b)}},$$

in which case $W_{\alpha(x)}$ is subnormal. In this paper we extend the above result to weight sequences of the form $\alpha : x_n, \dots, x_1, (\alpha_0, \dots, \alpha_k)^\wedge$ with $0 < \alpha_0 < \dots < \alpha_k$. Our main results are as follows.

Extensions of Recursively Generated Weighted Shifts. *If $\alpha : x_n, \dots, x_1, (\alpha_0, \dots, \alpha_k)^\wedge$, then*

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 1)\text{-hyponormal} & (n = 1), \\ W_\alpha \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 2)\text{-hyponormal} & (n > 1). \end{cases}$$

In particular, the above theorem shows that the subnormality of an extension of the recursive shift is independent of its length if the length is bigger than 1.

Canonical Rank-One Perturbations. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty = (\alpha_0, \dots, \alpha_k)^\wedge$. If $W_{\alpha'}$ is a perturbation of W_α at the j -th weight, then*

$$W_{\alpha'} \text{ is subnormal} \iff \begin{cases} W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 1)\text{-hyponormal} & (j = 0), \\ W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 2)\text{-hyponormal} & (j \geq 1). \end{cases}$$

Extremality Criterion. *Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence α . If $(k + 1)$ -hyponormality and k -hyponormality for $W_{\alpha(x)}$ coincide, then $\alpha(x)$ is recursively generated, i.e., $W_{\alpha(x)}$ is recursive subnormal.*

1. EXTENSIONS OF RECURSIVELY GENERATED SHIFTS

C. Berger’s characterization of subnormality for unilateral weighted shifts (cf. [Hal], [Con, III.8.16]) states that W_α is subnormal if and only if there exists a Borel probability measure μ support in $[0, \|W_\alpha\|^2]$, with $\|W_\alpha\|^2 \in \text{supp } \mu$, such that

$$\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \geq 0.$$

Given an initial segment of weights $\alpha : \alpha_0, \dots, \alpha_m$, the sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ ($i = 0, \dots, m$) is said to be *recursively generated* by α if there exist $r \geq 1$ and $\varphi_0, \dots, \varphi_{r-1} \in \mathbb{R}$ such that

$$\gamma_{n+r} = \varphi_0 \gamma_n + \dots + \varphi_{r-1} \gamma_{n+r-1} \quad \text{for all } n \geq 0,$$

where $\gamma_0 := 1$, $\gamma_n := \hat{\alpha}_0^2 \cdots \hat{\alpha}_{n-1}^2$ ($n \geq 1$). In this case the weighted shift $W_{\hat{\alpha}}$ with a weight sequence $\hat{\alpha}$ is said to be *recursively generated* (or simply *recursive*). If

$$g(t) := t^r - (\varphi_{r-1} t^{r-1} + \dots + \varphi_0),$$

then g has r distinct real roots $0 \leq s_0 < \dots < s_{r-1}$ ([CF2, Theorem 3.9]). Let

$$V := \begin{pmatrix} 1 & 1 & \dots & 1 \\ s_0 & s_1 & \dots & s_{r-1} \\ \vdots & \vdots & & \vdots \\ s_0^{r-1} & s_1^{r-1} & \dots & s_{r-1}^{r-1} \end{pmatrix}$$

and let

$$\begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix} := V^{-1} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

If $W_{\hat{\alpha}}$ is a recursively generated subnormal shift, then the Berger measure of $W_{\hat{\alpha}}$ is of the form

$$\mu := \rho_0 \delta_{s_0} + \dots + \rho_{r-1} \delta_{s_{r-1}}.$$

Given an initial segment of weights

$$\alpha : \alpha_0, \dots, \alpha_{2k} \quad (k \geq 0),$$

suppose $\hat{\alpha} \equiv (\alpha_0, \dots, \alpha_{2k})^\wedge$, i.e., $\hat{\alpha}$ is recursively generated by α . Write

$$\mathbf{v}_n := \begin{pmatrix} \gamma_n \\ \vdots \\ \gamma_{n+k} \end{pmatrix} \quad (0 \leq n \leq k+1).$$

Then $\{\mathbf{v}_0, \dots, \mathbf{v}_{k+1}\}$ is linearly dependent in \mathbb{R}^{k+1} . Now the *rank* of α is defined by the smallest integer i ($1 \leq i \leq k+1$) such that \mathbf{v}_i is a linear combination of $\mathbf{v}_0, \dots, \mathbf{v}_{i-1}$. Since $\{\mathbf{v}_0, \dots, \mathbf{v}_{i-1}\}$ is linearly independent, there exists a unique i -tuple $\varphi \equiv (\varphi_0, \dots, \varphi_{i-1}) \in \mathbb{R}^i$ such that $\mathbf{v}_i = \varphi_0 \mathbf{v}_0 + \dots + \varphi_{i-1} \mathbf{v}_{i-1}$, or equivalently,

$$\gamma_j = \varphi_{i-1} \gamma_{j-1} + \dots + \varphi_0 \gamma_{j-i} \quad (i \leq j \leq k+i),$$

which says that $(\alpha_0, \dots, \alpha_{k+i})$ is recursively generated by $(\alpha_0, \dots, \alpha_i)$. In this case, W_α is said to be *i*-recursive (cf. [CF3, Definition 5.14]).

We begin with:

Lemma 1.1 ([CF2, Propositions 2.3, 2.6, and 2.7]). *Let $A, B \in M_n(\mathbb{C})$, $\tilde{A}, \tilde{B} \in M_{n+1}(\mathbb{C})$ ($n \geq 1$) be such that*

$$\tilde{A} = \begin{pmatrix} A & * \\ * & * \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} * & * \\ * & B \end{pmatrix}.$$

Then we have:

- (i) *If $A \geq 0$ and if \tilde{A} is a flat extension of A (i.e., $\text{rank}(\tilde{A}) = \text{rank}(A)$), then $\tilde{A} \geq 0$.*
- (ii) *If $A \geq 0$ and $\tilde{A} \geq 0$, then $\det(A) = 0$ implies $\det(\tilde{A}) = 0$.*
- (iii) *If $B \geq 0$ and $\tilde{B} \geq 0$, then $\det(B) = 0$ implies $\det(\tilde{B}) = 0$.*

Lemma 1.2. *If $\alpha \equiv (\alpha_0, \dots, \alpha_k)^\wedge$, then*

$$(1.2.1) \quad W_\alpha \text{ is subnormal} \iff W_\alpha \text{ is } (\lfloor \frac{k}{2} \rfloor + 1)\text{-hyponormal}.$$

In the cases where W_α is subnormal and $i := \text{rank}(\alpha)$, we have $\alpha = (\alpha_0, \dots, \alpha_{2i-2})^\wedge$.

Proof. We only need to establish the sufficiency condition in (1.2.1). Let $i := \text{rank}(\alpha)$. Since W_α is *i*-recursive, [CF1, Proposition 5.15] implies that the subnormality of W_α follows after we verify that $A(0, i-1) \geq 0$ and $A(1, i-1) \geq 0$. Now observe that $i-1 \leq \lfloor \frac{k}{2} \rfloor + 1$ and

$$A(j, \lfloor \frac{k}{2} \rfloor + 1) = \begin{pmatrix} A(j, i-1) & * \\ * & * \end{pmatrix} \quad (j = 0, 1),$$

so the positivity of $A(0, i-1)$ and $A(1, i-1)$ is a consequence of the $(\lfloor \frac{k}{2} \rfloor + 1)$ -hyponormality of W_α . For the second assertion, observe that $\det A(n, i) = 0$ for all $n \geq 0$. By assumption $A(n, i+1) \geq 0$, so by Lemma 1.1 (ii) we have $\det A(n, i+1) = 0$, which says that $(\alpha_0, \dots, \alpha_{2i-1}) \subset (\alpha_0, \dots, \alpha_{2i-2})^\wedge$. By iteration we obtain $(\alpha_0, \dots, \alpha_k) \subset (\alpha_0, \dots, \alpha_{2i-2})^\wedge$, and therefore $(\alpha_0, \dots, \alpha_k)^\wedge = (\alpha_0, \dots, \alpha_{2i-2})^\wedge$. This proves the lemma. \square

In what follows, and for notational convenience, we shall set $x_{-j} := \alpha_j$ ($0 \leq j \leq k$).

Theorem 1.3 (Subnormality Criterion). *If $\alpha : x_n, \dots, x_1, (\alpha_0, \dots, \alpha_k)^\wedge$, then*

$$(1.3.1) \quad W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 1)\text{-hyponormal} & (n = 1), \\ W_\alpha \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 2)\text{-hyponormal} & (n > 1). \end{cases}$$

Furthermore, in the cases where the above equivalence holds, if $\text{rank}(\alpha_0, \dots, \alpha_k) = i$, then

$$(1.3.2) \quad W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } i\text{-hyponormal} & (n = 1), \\ W_\alpha \text{ is } (i + 1)\text{-hyponormal} & (n > 1). \end{cases}$$

In fact,

$$\begin{cases} x_1 = H_i(x_0, \dots, x_{2-2i}), \\ x_2 = H_i(x_1, \dots, x_{3-2i}), \\ \dots\dots\dots \\ x_{n-1} = H_i(x_{n-2}, \dots, x_{n-2i}), \\ x_n \leq H_i(x_{n-1}, \dots, x_{n-2i+1}), \end{cases}$$

where H_i is the modulus of i -hyponormality [CF3, Proposition 3.4 and (3.4)], i.e.,

$$H_i(\alpha) := \sup\{x > 0 : W_{x\alpha} \text{ is } i\text{-hyponormal}\}.$$

Therefore, $W_\alpha = W_{x_n(x_{n-1}, \dots, x_{n-2i+1})^\wedge}$.

Proof. Consider the $(k + 1) \times (l + 1)$ ‘‘Hankel’’ matrix $A(n; k, l)$ defined by (cf. [CL1])

$$A(n; k, l) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \dots & \gamma_{n+l} \\ \gamma_{n+1} & \gamma_{n+2} & \dots & \gamma_{n+1+l} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \dots & \gamma_{n+k+l} \end{pmatrix} \quad (n \geq 0).$$

Case 1 ($\alpha : x_1, (\alpha_0, \dots, \alpha_k)^\wedge$): Let $\hat{A}(n; k, l)$ and $A(n; k, l)$ denote the Hankel matrices corresponding to the weight sequences $(\alpha_0, \dots, \alpha_k)^\wedge$ and α , respectively. Suppose W_α is $(\lfloor \frac{k+1}{2} \rfloor + 1)$ -hyponormal. Then by Lemma 1.2, $W_{(\alpha_0, \dots, \alpha_k)^\wedge}$ is subnormal. Observe that

$$A(n + 1; m, m) = x_1^2 \hat{A}(n; m, m) \quad \text{for all } n \geq 0 \text{ and all } m \geq 0.$$

Thus it suffices to show that $A(0; m, m) \geq 0$ for all $m \geq \lfloor \frac{k+1}{2} \rfloor + 2$. Also observe that if \tilde{B} denotes the $(k - 1) \times k$ matrix obtained by eliminating the first row of a $k \times k$ matrix B , then

$$\tilde{A}(0; m, m) = x_1^2 \hat{A}(0; m - 1, m) \quad \text{for all } m \geq \lfloor \frac{k+1}{2} \rfloor + 2.$$

Therefore, for every $m \geq \lfloor \frac{k+1}{2} \rfloor + 2$, $A(0; m, m)$ is a flat extension of

$$A(0; \lfloor \frac{k+1}{2} \rfloor + 1, \lfloor \frac{k+1}{2} \rfloor + 1).$$

This implies $A(0; m, m) \geq 0$ for all $m \geq \lfloor \frac{k+1}{2} \rfloor + 2$ and therefore W_α is subnormal.

Case 2 ($\alpha : x_n, \dots, x_1, (\alpha_0, \dots, \alpha_k)^\wedge$): As in Case 1, let $\hat{A}(n; k, l)$ and $A(n; k, l)$ denote the Hankel matrices corresponding to the weight sequences $(\alpha_0, \dots, \alpha_k)^\wedge$

and α , respectively. Observe that $\det \hat{A}(n; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0$ for all $n \geq 0$. Suppose W_α is $([\frac{k+1}{2}] + 2)$ -hyponormal. Observe that

$$A(n + 1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = x_1^2 \cdots x_n^2 \hat{A}(1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1),$$

so that

$$(1.3.3) \quad \det A(n + 1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0.$$

Also observe that

$$A(n - 1; [\frac{k+1}{2}] + 2, [\frac{k+1}{2}] + 2) = \begin{pmatrix} x_2^2 \cdots x_n^2 & & \\ & * & \\ & & A(n + 1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) \end{pmatrix}.$$

Since W_α is $([\frac{k+1}{2}] + 1)$ -hyponormal, it follows from Lemma 1.1 (iii) and (1.3.3) that $\det A(n - 1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = 0$. Note that

$$A(n - 1; [\frac{k+1}{2}] + 1, [\frac{k+1}{2}] + 1) = x_1^2 \cdots x_n^2 \begin{pmatrix} \frac{1}{x_1^2} & \hat{\gamma}_0 & \cdots & \hat{\gamma}_{[\frac{k+1}{2}]+1} \\ \hat{\gamma}_0 & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{[\frac{k+1}{2}]+2} \\ \vdots & \vdots & & \vdots \\ \hat{\gamma}_{[\frac{k+1}{2}]+1} & \hat{\gamma}_{[\frac{k+1}{2}]+2} & \cdots & \hat{\gamma}_{2[\frac{k+1}{2}]+2} \end{pmatrix},$$

where $\hat{\gamma}_j$ denotes the moments corresponding to the weight sequence $(\alpha_0, \dots, \alpha_k)^\wedge$. Therefore x_1 is determined uniquely by $\{\alpha_0, \dots, \alpha_k\}$ such that $(x_1, \alpha_0, \dots, \alpha_{k-1})^\wedge = x_1(\alpha_0, \dots, \alpha_k)^\wedge$. More precisely, if $i := \text{rank}(\alpha)$ and $\varphi_0, \dots, \varphi_{i-1}$ denote the coefficients of recursion in $(\alpha_0, \dots, \alpha_k)^\wedge$, then

$$x_1 = H_i[(\alpha_0, \dots, \alpha_k)^\wedge] = \left[\frac{\varphi_0}{\hat{\gamma}_{i-1} - \varphi_{i-1}\hat{\gamma}_{i-2} - \cdots - \varphi_1\hat{\gamma}_0} \right]^{\frac{1}{2}}$$

(cf. [CF3, (3.4)]). Continuing this process we can see that x_1, \dots, x_{n-1} are determined uniquely by a telescoping method such that

$$(x_{n-1}, \dots, x_{n-1-k})^\wedge = x_{n-1}, \dots, x_1, (\alpha_0, \dots, \alpha_k)^\wedge$$

and $W_{(x_{n-1}, \dots, x_{n-1-k})^\wedge}$ is subnormal. Therefore, after $(n - 1)$ steps, Case 2 reduces to Case 1. This completes the proof of the first assertion. For the second assertion, note that if $\text{rank}(\alpha_0, \dots, \alpha_k) = i$, then

$$\det \hat{A}(n; i, i) = 0.$$

Now applying the above argument with i in place of $[\frac{k+1}{2}] + 1$ gives that x_1, \dots, x_{n-1} are determined uniquely by $\alpha_0, \dots, \alpha_{2i-2}$ such that $W_{(x_{n-1}, \dots, x_{n-2i-1})^\wedge}$ is subnormal. Thus the second assertion immediately follows. Finally, observe that the preceding argument also establishes the remaining assertions. \square

Remark 1.4. (a) From Theorem 1.3 we note that the subnormality of an extension of a recursive shift is independent of its length, if the length is bigger than 1.

(b) In Theorem 1.3, “ $[\frac{k+1}{2}]$ ” cannot be relaxed to “ $[\frac{k}{2}]$ ”. For example, consider the following weight sequences:

- (i) $\alpha : \sqrt{\frac{1}{2}}, (\sqrt{\frac{3}{2}}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$ with $\varphi_0 = 0$;
- (ii) $\alpha' : \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$.

Observe that α equals α' . Then a straightforward calculation shows that W_α (and hence $W_{\alpha'}$) is 2-hyponormal but not 3-hyponormal (and hence, not subnormal). Note that $k = 3$ and $n = 1$ in (i) and $k = 2$ and $n = 2$ in (ii).

(c) Note that the second assertion of Theorem 1.3 does *not* imply that if $\text{rank}(\alpha_0, \dots, \alpha_k) = i$, then (1.3.2) holds in general. Theorem 1.3 says only that when W_α is $(\lfloor \frac{k+1}{2} \rfloor + 1)$ -hyponormal ($n = 1$), i -hyponormality and subnormality coincide, and that when W_α is $(\lfloor \frac{k+1}{2} \rfloor + 2)$ -hyponormal ($n > 1$), $(i + 1)$ -hyponormality and subnormality coincide. For example consider the weight sequence

$$\hat{\alpha} \equiv (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}, 2)^\wedge \quad \text{with } \varphi_0 = 0 \text{ (here } \varphi_1 = 0 \text{ also)}.$$

Since $(\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}) \subset (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}})^\wedge$, we can see that $\text{rank}(\alpha) = 2$. Put

$$\beta \equiv 1, (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}, 2)^\wedge.$$

If (1.3.2) held true without assuming (1.3.1), then 2-hyponormality would imply subnormality for W_β . However, a straightforward calculation shows that W_β is 2-hyponormal but not 3-hyponormal (and hence not subnormal). In fact, $\det A(n, 2) = 0$ for all $n \geq 0$ except for $n = 2$ and $\det A(2, 2) = 160 > 0$, while since

$$\varphi_3 = -\frac{\alpha_3^2 \alpha_4^2 (\alpha_5^2 - \alpha_4^2)}{\alpha_4^2 - \alpha_3^2} = -102 \quad \text{and} \quad \varphi_4 = \frac{\alpha_4^2 (\alpha_5^2 - \alpha_3^2)}{\alpha_4^2 - \alpha_3^2} = 34$$

(so that $\alpha_6 = \sqrt{\varphi_4 - \frac{\varphi_3}{\alpha_5^2}} = \sqrt{\frac{17}{2}}$), we have that

$$\det A(1, 3) = \det \begin{pmatrix} 1 & 2 & 6 & 20 \\ 2 & 6 & 20 & 68 \\ 6 & 20 & 68 & 272 \\ 20 & 68 & 272 & 2312 \end{pmatrix} = -3200 < 0.$$

(d) On the other hand, Theorem 1.3 does show that if $\alpha \equiv (\alpha_0, \dots, \alpha_k)$ is such that $\text{rank}(\alpha) = i$ and $W_{\hat{\alpha}}$ is subnormal with associated Berger measure μ , then $W_{\hat{\alpha}}$ has an n -step $(i + 1)$ -hyponormal extension $W_{x_n, \dots, x_1, \hat{\alpha}}$ ($n \geq 2$) if and only if $\frac{1}{i^n} \in L^1(\mu)$,

$$x_{j+1} = \left[\frac{\varphi_0}{\gamma_{i-1}^{(j)} - \varphi_{i-1} \gamma_{i-2}^{(j)} - \dots - \varphi_1 \gamma_0^{(j)}} \right]^{\frac{1}{2}} \quad (0 \leq j \leq n - 2),$$

and

$$x_n \leq \left[\frac{\varphi_0}{\gamma_{i-1}^{(n-1)} - \varphi_{i-1} \gamma_{i-2}^{(n-1)} - \dots - \varphi_1 \gamma_0^{(n-1)}} \right]^{\frac{1}{2}},$$

where $\varphi_0, \dots, \varphi_{i-1}$ denote the coefficients of recursion in $(\alpha_0, \dots, \alpha_{2i-2})^\wedge$ and $\gamma_m^{(j)}$ ($0 \leq m \leq i - 1$) are the moments corresponding to the weight sequence $(x_j, \dots, x_1, \alpha_0, \dots, \alpha_{k-j})^\wedge$ with $\gamma_m^{(0)} = \gamma_m$.

We now observe that the determination of k -hyponormality and subnormality for canonical rank-one perturbations of recursive shifts falls within the scope of the theory of extensions.

Corollary 1.5. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty = (\alpha_0, \dots, \alpha_k)^\wedge$. If $W_{\alpha'}$ is a perturbation of W_α at the j -th weight, then*

$$W_{\alpha'} \text{ is subnormal} \iff \begin{cases} W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 1)\text{-hyponormal} & (j = 0), \\ W_{\alpha'} \text{ is } (\lfloor \frac{k+1}{2} \rfloor + 2)\text{-hyponormal} & (j \geq 1). \end{cases}$$

Proof. Observe that if $j = 0$, then $\alpha' = x, (\alpha_1, \dots, \alpha_{k+1})^\wedge$ and if instead $j \geq 1$, then $\alpha' = \alpha_0, \dots, \alpha_{j-1}, x, (\alpha_{j+1}, \dots, \alpha_{j+k+1})^\wedge$. Thus the result immediately follows from Theorem 1.3. □

2. EXTREMALITY OF RECURSIVELY GENERATED SHIFTS

In Corollary 1.5, we showed that if $\alpha(x)$ is a canonical rank-one perturbation of a recursive weight sequence, then subnormality and k -hyponormality for the corresponding shift eventually coincide. In this section we consider a converse.

Problem 2.1 (Extremality Problem). Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence α . If there exists $k \geq 1$ such that $(k + 1)$ -hyponormality and k -hyponormality for the corresponding shift $W_{\alpha(x)}$ coincide, does it follow that $\alpha(x)$ is recursively generated?

In [CF3], the following extremality criterion was established.

Lemma 2.2 (Extremality Criterion [CF3, Theorem 5.12, Proposition 5.13]). *Let α be a weight sequence and let $k \geq 1$.*

- (i) *If W_α is k -extremal (i.e., $\det A(j, k) = 0$ for all $j \geq 0$), then W_α is recursive subnormal.*
- (ii) *If W_α is k -hyponormal and if $\det A(i_0, j_0) = 0$ for some $i_0 \geq 0$ and some $j_0 < k$, then W_α is recursive subnormal.*

In particular, Lemma 2.2 (ii) shows that if W_α is subnormal and if $\det A(i_0, j_0) = 0$ for some $i \geq 0$ and some $j \geq 0$, then W_α is recursive subnormal.

We now answer Problem 2.1 affirmatively.

Theorem 2.3. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a weight sequence and let $\alpha_j(x)$ be a canonical perturbation of α in the j -th weight. Write*

$$\mathfrak{H}_k := \{x \in \mathbb{R}^+ : W_{\alpha_j(x)} \text{ is } k\text{-hyponormal}\}.$$

If $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ for some $k \geq 1$, and if $x \in \mathfrak{H}_k$, then $\alpha_j(x)$ is recursively generated, i.e., $W_{\alpha_j(x)}$ is recursive subnormal.

Proof. Suppose $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ and let $H_k := \sup_x \mathfrak{H}_k$. To avoid triviality we assume $\alpha_{j-1} < x < \alpha_{j+1}$.

Case 1 ($j = 0$): In this case, clearly H_k^2 is the nonzero root of the equation $\det A(0, k) = 0$ and for $x \in (0, H_k]$, $W_{\alpha_0(x)}$ is k -hyponormal. By assumption $H_k = H_{k+1}$, so $W_{\alpha_0(H_{k+1})}$ is $(k + 1)$ -hyponormal. The result now follows from Lemma 2.2 (ii).

Case 2 ($j \geq 1$): Let $A_x(n, k)$ denote the Hankel matrix corresponding to $\alpha_j(x)$. Since $W_{\alpha_j(x)}$ is $(k + 1)$ -hyponormal for $x \in \mathfrak{H}_k$, we have that $A_x(n, k + 1) \geq 0$ for all $n \geq 0$ and all $x \in \mathfrak{H}_k$. Observe that if $n \geq j + 1$, then

$$A_x(n, k) = \alpha_0^2 \cdots \alpha_{j-1}^2 x^2 \begin{pmatrix} \tilde{\gamma}_{n-j-1} & \cdots & \tilde{\gamma}_{n-j-1+k} \\ \vdots & & \vdots \\ \tilde{\gamma}_{n-j-1+k} & \cdots & \tilde{\gamma}_{n-j-1+2k} \end{pmatrix},$$

where $\tilde{\gamma}_*$ denotes the moments corresponding to the subsequence $\alpha_{j+1}, \alpha_{j+2}, \dots$. Therefore for $n \geq j + 1$, the positivity of $A_x(n, k)$ is independent of the values of $x > 0$. This gives

$$W_{\alpha_j(x)} \text{ is } k\text{-hyponormal} \iff A_x(n, k) \geq 0 \text{ for all } n \leq j.$$

Write

$$\mathfrak{H}_k^{(i)} := \left\{ x : \det A_x(i, k) \geq 0 \text{ and } \alpha_{j-1} < x < \alpha_{j+1} \right\} \quad (0 \leq i \leq j)$$

and

$$H_k^{(i)} = \sup_x \mathfrak{H}_k^{(i)} \quad (0 \leq i \leq j).$$

Since $\det A_x(i, k)$ is a polynomial in x we have $\det A_{H_k^{(i)}}(i, k) = 0$. Observe that

$$\bigcap_{i=0}^j \mathfrak{H}_k^{(i)} = \mathfrak{H}_k \quad \text{and} \quad \max_{0 \leq i \leq j} H_k^{(i)} = H_k.$$

Since \mathfrak{H}_k is a closed interval, by [CL2, Theorem 2.11] it follows that $H_k \in \mathfrak{H}_k$, say $H_k = H_k^{(p)}$ for some $0 \leq p \leq j$. Then $\det A_{H_k^{(p)}}(p, k) = 0$ and $W_{\alpha(H_k^{(p)})}$ is $(k + 1)$ -hyponormal. Therefore it follows from Lemma 2.2 (ii) that W_α is recursive subnormal. This completes the proof. \square

We conclude this section with two corollaries of independent interest.

Corollary 2.4. *With the notations in Theorem 2.3, if $j \geq 1$ and $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ for some k , then \mathfrak{H}_k is a singleton set.*

Proof. By [CL2, Theorem 2.2],

$$\mathfrak{H}_\infty := \{x \in \mathbb{R}^+ : W_{\alpha_j(x)} \text{ is subnormal}\}$$

is a singleton set. By Theorem 2.3, we have that $\mathfrak{H}_k = \mathfrak{H}_\infty$. \square

Corollary 2.5. *If W_α is a nonrecursive shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and if $\alpha(x)$ is a canonical rank-one perturbation of α , then for every $k \geq 1$ there always exists a gap between k -hyponormality and $(k + 1)$ -hyponormality for $W_{\alpha(x)}$. More concretely, if we let*

$$\mathfrak{H}_k := \{x : W_{\alpha(x)} \text{ is } k\text{-hyponormal}\},$$

then $\{\mathfrak{H}_k\}_{k=1}^\infty$ is a strictly decreasing nested sequence of closed intervals in $(0, \infty)$ except when the perturbation occurs in the first weight. In that case, the intervals are of the form $(0, H_k]$.

Proof. Straightforward from Theorem 2.3. \square

3. SOME REVEALING EXAMPLES

We now illustrate our results with two examples. Consider $\alpha(y, x) : \sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$, where $a < b < c$. Without loss of generality, we assume $a = 1$. Observe that

$$H_2(1, \sqrt{b}, \sqrt{c}) = \sqrt{\frac{bc - b^2}{1 + bc - 2b}}$$

and

$$\left(H_2(\sqrt{x}, 1, \sqrt{b})\right)^2 = \frac{x(b-1)}{(x-1)^2 + (b-1)} := f(x).$$

Thus $W_{\alpha(y,x)}$ is 2-hyponormal if and only if $0 < x \leq \frac{bc-b^2}{1+bc-2b}$ and $0 < y \leq f(x)$. To completely describe the region $\mathcal{R} := \{(x, y) : W_{\alpha(y,x)} \text{ is 2-hyponormal}\}$, we study the graph of f . Observe that

$$f'(x) = \frac{(b-1)(b-x^2)}{(b-2x+x^2)^2} > 0 \quad \text{and} \quad f''(x) = \frac{2(b-1)(2b-3bx+x^3)}{(b-2x+x^2)^3}.$$

Note that $b-2x+x^2 = (b-1) + (1-x)^2 > 0$ and $f'(\sqrt{b}) = 0$. To consider the sign of f'' , we let $g(x) := 2b-3bx+x^3$. Then $g'(\sqrt{b}) = 0$, $g(0) = 2b > 0$, $g(1) = -b+1 < 0$, and $g''(x) > 0$ ($x > 0$). Hence there exists $x_0 \in (0, 1)$ such that $f''(x_0) = 0$, $f''(x) > 0$ on $0 < x < x_0$, and $f''(x) < 0$ on $x_0 < x \leq 1$. We investigate which of the two values x_0 or $\tilde{H} := H_2(1, \sqrt{b}, \sqrt{c})^2$ is bigger. By a simple calculation, we have

$$g(\tilde{H}) = \frac{(-1+b)b \cdot g_1(b, c)}{(1-2b+bc)^3},$$

where

$$g_1(b, c) = -(2-10b+17b^2-11b^3+b^4+3bc-9b^2c+9b^3c-3b^3c^2+b^2c^3).$$

For notational convenience we let $b := 1+h$, $c := 1+h+k$. Then

$$g_1(b, c) = 2h^5 + (3h^3 + 3h^4)k + (-1 - 2h - h^2)k^3.$$

If h is sufficiently small (i.e., b is sufficiently close to 1), then $g_1 < 0$, i.e., $\tilde{H} > x_0$. If k is sufficiently small (i.e., b is sufficiently close to c), then $g_1 > 0$, i.e., $\tilde{H} < x_0$. Thus, if $\tilde{H} \leq x_0$, then f is concave up on $x \leq \tilde{H}$. If $\tilde{H} > x_0$, then $(x_0, f(x_0))$ is an inflection point. Thus, f is concave up on $0 < x < x_0$ and concave down on $x_0 < x \leq \tilde{H}$. Moreover, $W_{\alpha(y,x)}$ is 2-hyponormal if and only if $(x, y) \in \{(x, y) | 0 \leq y \leq f(x), 0 < x \leq \tilde{H}\}$, and $W_{\alpha(y,x)}$ is k -hyponormal ($k \geq 3$) if and only if $x = \tilde{H}$ and $0 \leq y \leq f(\tilde{H})$.

Example 3.1 ($b = 2$, $c = 3$).

$$f(x) = \frac{x}{1+(1-x)^2}.$$

The graph of \mathcal{R} is given in Figure 1; notice that f is concave up in this case.

Example 3.2 ($b = \frac{11}{10}$, $c = 10$).

$$f(x) = \frac{x}{11-20x+10x^2}.$$

The graph of \mathcal{R} is given in Figure 2; in this case, f has an inflection point at $x_0 \approx 0.85821$.

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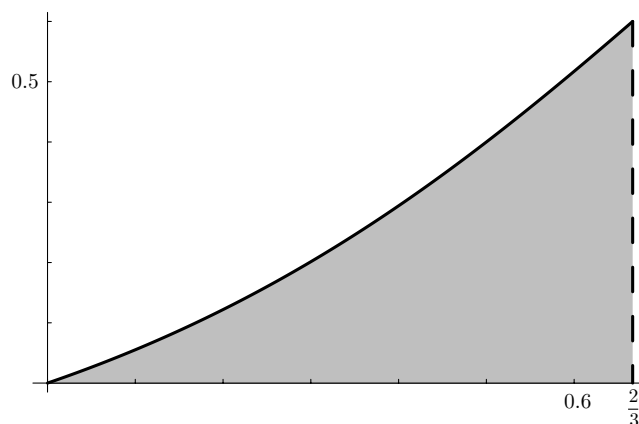


FIGURE 1.

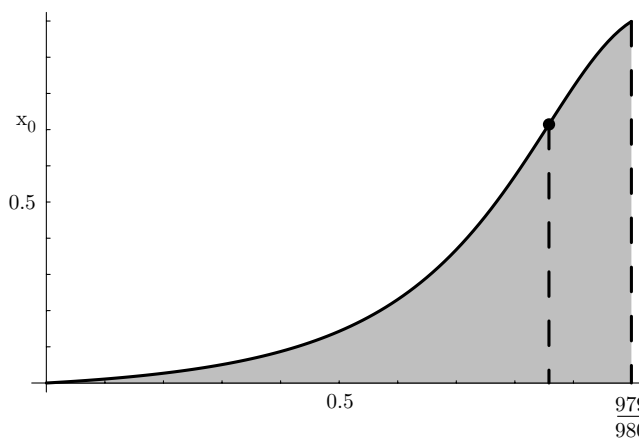


FIGURE 2.

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