PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 2, Pages 609–611
S 0002-9939(01)06089-0
Article electronically published on June 19, 2001

ACTIONS OF FINITELY GENERATED GROUPS ON THE UNIVERSAL MENGER CURVE

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(Communicated by Alan Dow)

ABSTRACT. We prove that every finitely generated group acts effectively on the universal Menger curve.

The purpose of this note is to prove the following theorem on the universal Menger curve μ^1. A description of the n-dimensional universal Menger compactum μ^n ($n \geq 1$) is given below. For more information, the reader is referred to [1], [4], [5], [8], [9], etc.

Theorem. *Every finitely generated group acts on the universal Menger curve μ^1, and, in particular, is isomorphic to a subgroup of the homeomorphism group of μ^1.*

The algebraic structure of the homeomorphism group $H(\mu^n)$ ($n \geq 1$) has been studied by several authors. For example, $H(\mu^n)$ is simple ([3], [7]), and has the trivial homology ([12]). The above theorem presents a certain "universality" of $H(\mu^1)$. The result was announced in [8].

Here we briefly describe a construction (called the *Lefschetz construction*) of the n-dimensional universal Menger compactum μ^n ($n \geq 1$). Let D^{2n+1} be the $(2n+1)$-dimensional ball with a combinatorial triangulation L. We define a sequence of compact PL manifolds $(M_i)_{i \geq 0}$ as follows. Let $M_0 = D^{2n+1}$, $L_0 = L$, and $M_1 = \mathrm{st}(|L_0^{(n)}|, \beta^2 L_0)$, where $\beta^2 L_0$ is the second barycentric subdivision of L_0 and $|L_0^{(n)}|$ is the polyhedron associated with the n-skeleton of L_0. Further, let $L_1 = \beta^2 L_0 | M_1$, the restriction of $\beta^2 L_0$ to the submanifold M_1. Inductively, let $M_{i+1} = \mathrm{st}(|L_i^{(n)}|, \beta^2 L_i)$ and $L_{i+1} = \beta^2 L_i | M_{i+1}$. The intersection $\bigcap_{i=1}^{\infty} M_i$ is denoted by μ^n and is called the *n-dimensional universal Menger compactum*. It has been known that every compact 0-dimensional group acts freely on μ^n for each $n \geq 1$ ([2], [6], [10], [11], etc.).

The following result is crucial for the proof of the above theorem, the proof of which is outlined on p. 58 of [5]. A locally compact separable metric space, each point of which has a neighbourhood homeomorphic to μ^n, is called a *μ^n-manifold*.

Proposition. *The Freudenthal compactification of each* (*noncompact*) *connected μ^1-manifold is homeomorphic to μ^1.*

Proof of the theorem. Let G be a group with a finite set S of generators. We may assume at the outset that G is an infinite group, $S = \{s^{-1} | s \in S\}$ and the

Received by the editors February 10, 2000 and, in revised form, August 8, 2000.
2000 *Mathematics Subject Classification.* Primary 57S05, 54F50.
Key words and phrases. The universal Menger curve, homeomorphism group.

unit element does not belong to S. Consider the Cayley graph $K = K(G, S)$ of G with respect to S: The set of vertices of K is equal to G, and two vertices $g_1, g_2 \in G \subset K$ are joined by an edge of K if and only if $g_2^{-1} g_1 \in S$. It is a connected one-dimensional simplicial complex and the left multiplication of G on the vertex set G induces the canonical simplicial and effective action on K. The set of all edges of K is denoted by $E(K)$.

Let $\widetilde{K}$ be a 3-manifold which collapses onto K such that:

(1) there exists a handle-body decomposition $\widetilde{K} = \bigcup_{g \in G} h_g^0 \cup \bigcup_{e \in E(K)} h_e^1$, where each h_g^0 is a 0-handle and each h_e^1 is a 1-handle ($g \in G,\ e \in E(K)$), and

(2) $h_g^0 \cap h_e^1 \neq \varnothing$ if and only if g is an end point of e (see Figure 1).

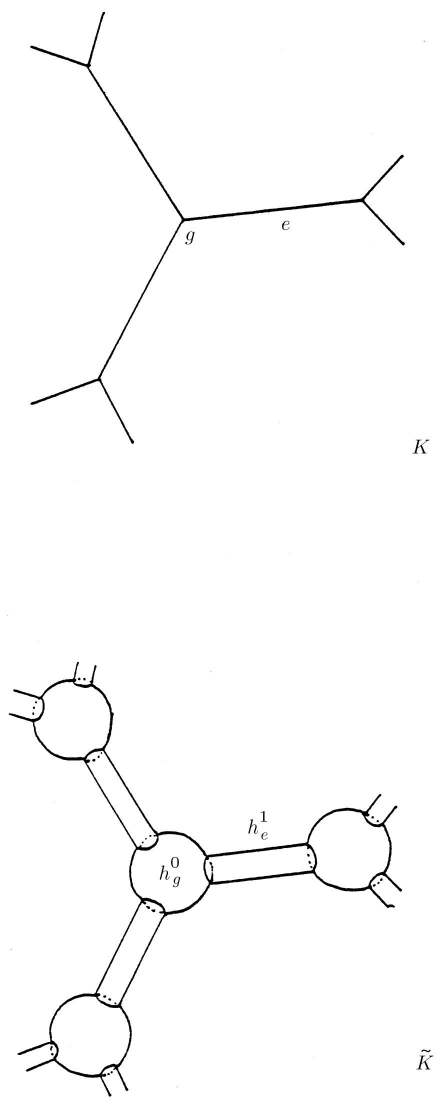

FIGURE 1.

It is obvious that $\widetilde{K}$ can be constructed in such a way that

(3) there exists a triangulation L of $\widetilde{K}$ such that G acts effectively on $\widetilde{K}$ by simplicial isomorphisms with respect to L (which is "induced" by the action of G on K).

Start the Lefschetz construction of a connected μ^1-manifold X with $M_0 = \widetilde{K}$ and $L_0 = L$. Then G naturally acts effectively on X and the action extends to the one on the Freudenthal compactification $\overline{X}$. By the Proposition, $\overline{X}$ is homeomorphic to μ^1. This completes the proof.

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