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GROWTH OF FUNDAMENTAL GROUPS AND ISOEMBOLIC VOLUME AND DIAMETER

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Abstract. Some properties of fundamental groups of Riemannian manifolds M will be studied without a lower bound assumption on Ricci curvature. The main method is to relate the local packing to global packing instead of using the Bishop-Gromov relative volume comparison. This method allows us to control the volume growth of the universal cover $\tilde{M}$ and yields bounds on the number of generators of $\pi_1(M)$ in terms of some isoembolic geometric invariants of M.

1. Introduction

In this article, we discuss the number of generators and the growth of the fundamental groups of Riemannian manifolds without curvature restrictions.

Let M^n denote a compact n-dimensional Riemannian manifold without boundary, and let $\pi_1(M,p)$, $d(M)$, $v(M)$, and $i(M)$ denote its fundamental group, diameter, volume, and injectivity radius, respectively. For a finitely generated group G, $mg(G)$ denotes the minimal number of elements needed to generate the group. The entropy $h(G)$ of a group is a measurement of the number of distinct words (of a given length or less) in terms of generators and their inverses, and the entropy of a manifold $h(M)$ is a measurement of the volume growth of its universal cover; see Section 2 or [Gr] for their formal definitions.

There are several studies of the fundamental group with lower bounds on Ricci curvature. Bishop(-Gromov) volume comparison is a basic tool for comparing the volumes of metric balls used in these results. In Milnor [Mi], the fundamental groups of manifolds with nonnegative Ricci curvature are shown to have polynomial growth, in contrast to negative sectional curvature cases in which they have exponential growth. The positive and nonnegative Ricci curvature cases are studied further by Anderson [A], and the general case with an arbitrary lower bound on Ricci curvature is studied qualitatively by Gromov [Gr, chap. 6]. The case of negative or nonpositive sectional curvature has been studied extensively; see Preismann [P], Gromoll-Wolf [GW], and Lawson-Yau [LY].

Definition 1.1. The *isoembolic volume* is defined to be $V_e(M^n) = v(M)/i(M)^n$ and *isoembolic diameter* to be $d_e(M^n) = d(M)/i(M)$ for an n-dimensional Riemannian manifold M.

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Let $E^n_N = \{M^n : V_e(M) \le N\}$. The following results show that a bound on the isoembolic volume puts a priori topological and geometric restrictions on the manifolds. In [Be], Berger proved that $V_e(M^n) \ge V_e(S^n(1))$ and equality holds if and only if M^n is isometric to a standard sphere $S^n(r)$ of dimension n with radius r. By the work of Croke [C2], M^n is homeomorphic to a sphere S^n, if $V_e(M^n) \le V_e(S^n(1)) + c(n)$ for a universal constant $c(n)$. In 1988, Yamaguchi proved that E^n_N has finitely many homotopy equivalence types, [Y]. By the author's work [D2], one can estimate a priori upper bound for the number of homotopy types and the Betti numbers $b_k(M, \mathbf{F})$ in terms of $V_e(M^n)$ for any field $\mathbf{F}$. In particular, $b_1(M, \mathbf{F}) \le C(n)V_e(M^n)^2$ for a universal constant $C(n)$, and a linear inequality is not possible, [D2].

Even though $mg(\pi_1(M,p))$ are uniformly bounded on E^n_N, one cannot deduce a priori upper bounds on $mg(\pi_1(M,p))$ from the results above. In particular, the inequality $mg(\pi_1(M,p)) \ge b_1(M, \mathbf{F})$ by the Hurewicz Isomorphism Theorem is in the opposite direction.

Theorem 2. *For any compact Riemannian manifold M^n,*

$$mg(\pi_1(M,p)) \le (cV_e(M))^{14d_e(M)} \le (cV_e(M))^{4cV_e(M)}$$

and

$$h(\pi_1(M,p)) \le 14 d_e(M) \log cV_e(M)$$

where c is a constant depending only on the dimension n.

Theorem 2 is a consequence of Theorem 1 which involves controlling the volume growth of the universal cover by using packing arguments without a lower bound on the Ricci curvature. Here we briefly introduce the notation and definitions needed to state Theorem 1. The formal definitions will be given in Section 2. $B(p,r)$ denotes the metric ball of radius r at p. The packing number $N(a,c,M)$ is the largest number of disjoint open metric balls of radius c that can be fitted into any metric ball of radius a in M. $N(a,c,M)$ obviously bounds $N(b,c,M)$, if $b \le a$. The crucial Proposition 3.1 is about controlling $N(a,c,M)$ in terms of $N(b,c,M)$ for $6c < b \le a$.

Theorem 1. *Let M^n be a complete Riemannian manifold of dimension n with the Riemannian universal covering map $\Psi : \tilde{M} \to M$. Let R be such that $\forall p \in M$, $B(p,R,M)$ is simply connected in M; that is, every closed curve in $B(p,R,M)$ is contractible in M. Let $N(R,R/7;M) = N_0$. Then, for $\Psi(\tilde{p}) = p$,*

$$card(\Psi^{-1}(p) \cap \bar{B}(\tilde{p}, a, \tilde{M})) \le N_0^{(7a/R)-5}, \quad \forall a > R.$$

Consequently, if M^n is compact with diameter $d(M)$, then

$$mg(\pi_1(M,p)) \le N_0^{14d(M)/R}$$

and the entropies satisfy

$$h(M) \le 7(\log N_0)/R$$

and

$$h(\pi_1(M,p)) \le 14(\log N_0)d(M)/R.$$

2. Basic notation and definitions

We refer to [BC], [CE] and [GKM] for basic Riemannian geometry. Let (M^n, g) denote an n-dimensional compact Riemannian manifold unless otherwise specified. $d_M(.,.)$ is the Riemannian distance function. For $X \subset M$, let $card(X)$, X°, $\bar{X}$ and $v(X)$ denote the cardinality, interior, closure and Riemannian volume of X.

Definition 2.1. (i) $B(p,r) = \{x \in M : d_M(x,\, p) < r\}$ and $\bar{B}(p,r) = \{x \in M : d_M(x,\, p) \le r\}$.

(ii) The packing number $N(a,c,M)$ is defined to be the supremum of the number of disjoint open metric balls of radius c inside each open metric ball of radius a in M.

Definition 2.2 ([Gr, 5B])**.** Let G be a finitely generated group.

(i) The minimal number of generators is defined to be $mg(G) = \inf\{card(H) : H \subset G \text{ and } H \text{ generates } G\}$.

(ii) The entropy of a finite generating set H of G is $h(H) = \liminf\limits_{t\to\infty} \frac{\log N(t,H)}{t}$, where $N(t,H)$ is the number of distinct words of length at most t by using generators from H and their inverses. The entropy of G is $h(G) = \inf\limits_{H} h(H)$, where H runs through the generating subsets of G.

(iii) Let M^n be a complete manifold of dimension n, with the Riemannian universal covering map $\Psi : \tilde{M} \to M$. The entropy of M is defined to be $h(M) = \liminf\limits_{t\to\infty} \frac{\log(v(B(\tilde{p},t,\tilde{M})))}{t}$; that is, a measurement of the volume growth of the universal cover.

Definition 2.3. Let $A \subset B$ be both connected and let i be the inclusion map. A is said to be simply connected in B, if every closed curve in A is contractible in B; that is, $i_*(\pi_1(A,a)) = 0$.

3. Proofs of the theorems

Proposition 3.1. *For any complete Riemannian manifold M and $\forall a \ge b > 6c > 0$, one has $N(a,c,M) \le N(b,c,M)^k$, where k is the smallest integer $\ge \frac{a-6c}{b-6c}$.*

Proof. Let a metric ball $B(p,a)$ and a maximal family $\mathcal{F} = \{B(q,c) : q \in Q\}$ of disjoint open balls inside $B(p,a)$ be given. Define

$$Q(R) = \{q \in Q : B(q,c) \subset B(p,R)\}$$

for $R > 0$. Fix any R with $b \le R \le a$, and choose any $q \in Q(R+b-6c) - Q(R)$. Let p' be a point on any minimal geodesic from p to q such that $d(q,p') = b-3c$, so that $d(p,p') < R-3c$ and $B(p',2c) \subset B(p,R-c)$. There exists a q' in $Q \cap B(p',2c)$, for otherwise, the addition of $B(p',c)$ to $\mathcal{F}$ would contradict the maximality of $\mathcal{F}$. Furthermore,

$$d(p,q') \le d(p,p') + d(p',q') \le R-c, \text{ hence } B(q',c) \subset B(p,R) \text{ and } q' \in Q(R),$$
$$d(q,q') \le d(q,p') + d(p',q') \le b-c, \text{ hence } B(q,c) \subset B(q',b),$$

$$\forall q \in Q(R+b-6c) - Q(R) \exists q' \in Q(R) \text{ such that } B(q,c) \subset B(q',b),$$

$$card(Q(R+b-6c) - Q(R)) \le card(Q(R))(N(b,c)-1),$$
$$card(Q(R+b-6c)) \le card(Q(R))N(b,c).$$

Since $card(Q(b)) \leq N(b,c)$, it follows that $card(\mathcal{F}) = card(Q(a)) \leq N(b,c)^k$, where k is the smallest integer $\geq \frac{a-b}{b-6c} + 1 = \frac{a-6c}{b-6c}$.

p is arbitrary and $\mathcal{F}$ is maximal, hence $N(a,c,M) \leq N(b,c,M)^k$. $\square$

Proof of Theorem 1. Let $\Psi : \tilde{M} \to M$ be the Riemannian universal covering map and $G = \pi_1(M,p)$. Choose any $\tilde{p} \in \Psi^{-1}(p)$ and let $G = \{g_i : i \in \Lambda\}$ act on $\Psi^{-1}(p) = \{g_i\tilde{p} : i \in \Lambda\}$ by deck transformations.

The following basic facts which were used or proved in [Mi] and [Gr, chap. 5] can also be found in [D1]. $W = M\text{-}cutlocus(p)$ is an open, dense and contractible subset of M, [CE]. $\forall g_i \in G, \exists$ an open contractible subset W_i of $\tilde{M}$ such that $g_i\tilde{p} \in W_i$, $\Psi \mid W_i$ is an isometry of W_i onto W, $W_i \cap W_j = \emptyset$ if $g_i \neq g_j$, $v(W_i) = v(M)$, and $\bigcup_{i\in\Lambda} \bar{W}_i = \tilde{M}$. If M is compact, then $H = \{g_i \in G : \bar{W}_i \cap \bar{W}_0 \neq \emptyset\}$ is a generating set for G, where $\tilde{p} \in W_0$. For compact M, $\forall g_i \in H, g_i\tilde{p} \in \bar{B}(\tilde{p}, 2d(M); \tilde{M})$.

If A is a path-connected subset of $\tilde{M}$ and $\Psi \mid A$ is not one-to-one, then $\Psi(A)$ contains a closed curve γ which has a non-closed lift to $\tilde{M}$ and $0 \neq [\gamma] \in \pi_1(M,q)$. By the hypothesis, $\forall r \leq R$ and $\tilde{q} \in \tilde{M}$, $\Psi \mid B(\tilde{q}, r; \tilde{M})$ is one-to-one and an isometry, and $v(B(\tilde{q}, r; \tilde{M})) = v(B(\Psi(\tilde{q}), r; M)$.

$card(\Psi^{-1}(p) \cap \bar{B}(\tilde{p}, a; \tilde{M})) = 1$, if $0 < a < R$.

Let $a \geq R$ be fixed, set $c = \frac{R}{7}$ and choose a maximal family $\mathcal{F} = \{B(q,c) : q \in Q\}$ of disjoint open balls inside $B(\tilde{p}, a+c; \tilde{M})$. From the maximality of $\mathcal{F}$, it follows that $\bar{B}(\tilde{p}, a; \tilde{M}) \subset \bigcup_{q\in Q} B(q, 2c, \tilde{M})$. Since the restrictions of Ψ to metric balls of radius R are isometries, $N(R,c;\tilde{M}) = N(R,c;M) = N_0$. Given two distinct elements $g_1\tilde{p}$ and $g_2\tilde{p}$ of $\Psi^{-1}(p) \cap \bar{B}(\tilde{p}, a; \tilde{M})$, one has $g_1\tilde{p} \in B(q_{i_1}, 2c; \tilde{M})$ and $g_2\tilde{p} \in B(q_{i_2}, 2c; \tilde{M})$ for $q_{i_1} \neq q_{i_2}$ of Q, otherwise, $\Psi \mid B(q_{i_1}, 2c; \tilde{M})$ will not be one-to-one. Hence, $\forall q \in Q$, $B(q, 2c, \tilde{M})$ contains at most one element of $\Psi^{-1}(p) \cap \bar{B}(\tilde{p}, a; \tilde{M})$. By Proposition 3.1,

$$\begin{aligned} card(\Psi^{-1}(p) \cap \bar{B}(\tilde{p}, a; \tilde{M})) &\leq card(Q) \leq N(a+c, c, \tilde{M}) \\ &\leq N(R, c, \tilde{M})^{\frac{a+c-6c}{R-6c}} \leq N_0^{\frac{7a}{R}-5}. \end{aligned}$$

If M is compact, then $\pi_1(M,p)$ can be generated by loops of length at most $2d(M)$ by above. By taking $a = 2d(M)$ in the above inequality yields

$$mg(\pi_1(M,p)) \leq N_0^{14d(M)/R}.$$

Any word of length at most k constructed by using loops of length at most $2d(M)$ has a lift to the universal cover starting at $\tilde{p}$ and with length at most $2kd(M)$. So, the end point is in $B(\tilde{p}, 2kd(M); \tilde{M})$. This proves that

$$h(\pi_1(M,p)) \leq 14d(M)(\log N_0)/R.$$

For the entropy of M, $h(M) \leq 7(\log N_0)/R$ follows

$$\begin{aligned} v(B(\tilde{p}, a; \tilde{M})) &\leq card(\Psi^{-1}(p) \cap \bar{B}(\tilde{p}, a + d(M); \tilde{M}))v(M) \\ &\leq N_0^{7(a+d(M))/R} v(M). \end{aligned}$$

$\square$

Proposition 3.2. *Let $R, V, v > 0$ be given. For any compact Riemannian manifold M, if $v(M) \leq V$, $\forall p \in M$, $B(p,R)$ is contractible in M, and $v(B(p, R/7)) \geq v$, then $mg(\pi_1(M,p)) \leq L^{14d(M)/R} \leq L^{4L}$, where $L = V/v$.*

Proof. Clearly, $N_0 = N(R, R/7; M) \leq \frac{V}{v} = L$. Let p and q be a pair of furthest points of M, that is, $d(p, q) = d(M)$. Choose any normal geodesic γ from p to q, with $\gamma(0) = p$. Let $p_i = \gamma((2i-1)R/7)$ for $1 \leq i \leq s$ so that $d(p_s, q) < R/7$ and hence $\frac{2}{7}sR \geq d(M)$. The balls $B(p_i, R/7)$ are disjoint, since a nonempty intersection of $B(p_i, R/7)$ and $B(p_j, R/7)$ contradicts the minimality of γ between p_i and p_j. This implies $sv \leq V$. By Theorem 1,

$$mg(\pi_1(M,p)) \leq \left(\frac{V}{v}\right)^{14d(M)/R} \leq \left(\frac{V}{v}\right)^{4s} \leq \left(\frac{V}{v}\right)^{4\frac{V}{v}} = L^{4L}.$$

□

Proof of Theorem 2. If $r \leq i(M)/2$, then $v(B(p, r; M^n)) \geq r^n 2^{n-1} \alpha_{n-1}^n n^{-n} \alpha_n^{1-n}$ by Croke [Cr], where $\alpha_n = v(S^n(1))$. Hence, by taking $v(M) = V$ and $R = i(M)$, one uses $\frac{d(M)}{R} = d_e(M)$ and $N_0 \leq \frac{V}{v} = L \leq V_e(M)c$ in Proposition 3.2 to conclude the proof, where $c = 2(7/2)^n \alpha_{n-1}^{-n} n^n \alpha_n^{n-1}$. □

The method we developed above yields the following for noncompact manifolds.

Theorem 3. *Let M be a noncompact Riemannian manifold and G_A be any subgroup of $\pi_1(M, p)$ generated by loops of length at most A. Let R be such that $\forall p \in B(p, A, M)$, $B(p, R, M)$ is simply connected in M. Let $N_0 = N(R, R/7; B(p, A, M))$. Then*

$$mg(G_A) \leq N_0^{7A/R} \text{ and } h(G_A) \leq 7A(\log N_0)/R.$$

The proof of Theorem 3 follows the proof of Theorem 1, with the fact that $\Psi(B(\tilde{p}, A, \tilde{M})) \subset (B(p, A, M)$.

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