

ASYMPTOTIC PROPERTIES OF THE VECTOR CARLESON EMBEDDING THEOREM

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ABSTRACT. The dyadic Carleson embedding operator acting on \mathbb{C}^n -valued functions has norm at least $C \log n$. Thus the Carleson Embedding Theorem fails for Hilbert space valued functions.

Let \mathbb{T} be the unit circle in \mathbb{C} , and $\{I\}_{I \in D}$ its collection of dyadic arcs. Let w_I be nonnegative real numbers indexed by $I \in D$. For integrable functions f on \mathbb{T} , denote by $\langle f \rangle_I$ the average $|I|^{-1} \int_I f(y) dy$. The classical Carleson embedding theorem [1] is equivalent to the following dyadic result:

Theorem. *If $\sum_{I \subset K} w_I \leq |K|$ for all $K \in D$, then $\sum_{I \in D} w_I \langle f \rangle_I^2 \leq C \|f\|^2$ for all $f \in L^2(\mathbb{T})$.*

The converse is also true (up to the placement of constants) and is verified by considering functions of the form $f = \chi_J$, $J \in D$.

An analogous statement may be made for functions taking values in \mathbb{C}^n with matrix-valued weights $W_I \geq 0$ in the sense of quadratic forms. We wish to consider the following n -dimensional embedding theorem:

Proposition. *If $\|\sum_{I \subset K} W_I\| \leq |K|$ for all $K \in D$, then*

$$\sum_{I \in D} (W_I \langle f \rangle_I, \langle f \rangle_I) \leq C_n \|f\|^2$$

for all $f \in L^2(\mathbb{T}; \mathbb{C}^n)$.

The space \mathbb{C}^n here is viewed as a finite-dimensional Hilbert space. One might ask whether a similar result still holds when f takes values in a general Hilbert space \mathbb{H} and W_I are positive selfadjoint operators. This is answered in the negative by [4], which proves that C_n must be bounded from below by $c \log n$. In the current paper we will use the construction in [4] to verify the stronger bound $C_n \geq c(\log n)^2$, which is also proved in [5]. A precise statement is as follows:

Theorem 1. *There exist a function $f \in L^2(\mathbb{T}; \mathbb{C}^n)$ and matrix weights $W_I \geq 0$ such that $\|\sum_{I \subset K} W_I\| \leq |K|$ and $\sum_{I \in D} (W_I \langle f \rangle_I, \langle f \rangle_I) \geq c(\log n)^2 \|f\|^2$, where $c > 0$ is independent of n .*

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Remarks. The example presented here is due to Nazarov, Treil, and Volberg [4]. It is further shown in [3] and [4] that the best possible C_n is bounded above by $C(\log n)^2$, making these results sharp up to a constant factor.

Proof of Theorem 1. Let $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis for \mathbb{C}^{n+1} . Define the Rademacher functions $r_j(e^{2\pi it}) = (-1)^{\lfloor 2^j t \rfloor}$. For a dyadic interval $I, |I| \leq 2^{-j}$, r_j is seen to be constant along I . Its value throughout the interval will be called $r_j(I)$.

Let $f(x) = \sum_{j=0}^n r_j(x)\mathbf{e}_j$. Clearly $\|f\|^2 = n+1$. The averages of f over dyadic intervals are also easy to compute. When $|I| = 2^{-i}, i \leq n$, $\langle f \rangle_I = \sum_{j=0}^i r_j(I)\mathbf{e}_j$.

Let $W_I, |I| \geq 2^{-n}$, be the rank-one operator satisfying $W_I \mathbf{v} = |I|(\mathbf{v}, \phi_I)\phi_I$, where $\phi_I = \sum_{j=0}^i \frac{1}{i+1-j} r_j(I)\mathbf{e}_j$. Define ϕ_I to be 0 when $|I| < 2^{-n}$. Already we can estimate the sum

$$\sum_{I \in \mathcal{D}} (W_I \langle f \rangle_I, \langle f \rangle_I) = \sum_{I \in \mathcal{D}} |I| (\langle f \rangle_I, \phi_I)^2 = \sum_{i=0}^n \left(\sum_{j=0}^i \frac{1}{i+1-j} \right)^2 \geq cn(\log n)^2.$$

The only task remaining is to show that $\|\sum_{I \subset K} W_I\|$ is controlled by $|K|$. We will prove the estimate $\sum_{I \subset K} (W_I \mathbf{v}, \mathbf{v}) = \sum_{I \subset K} |I|(\mathbf{v}, \phi_I)^2 \leq C|K|\|\mathbf{v}\|^2$ for all $\mathbf{v} \in \mathbb{C}^{n+1}$.

For each interval I with $|I| = 2^{-i}$, split the vector ϕ_I into the sum of two parts, $\phi_I = \sum_{j=0}^k \frac{1}{i+1-j} r_j(K)\mathbf{e}_j + \sum_{j=k+1}^i \frac{1}{i+1-j} r_j(I)\mathbf{e}_j$. Denote the first sum, which depends only on the length of $I \subset K$, by \mathbf{g}_i . Summing over all I with $|I| = 2^{-i}$, and exploiting the orthogonality of the Rademacher functions,

$$\sum_{\substack{I \subset K \\ |I|=2^{-i}}} |I|(\mathbf{v}, \phi_I)^2 = |K| \left((\mathbf{v}, \mathbf{g}_i)^2 + \sum_{j=k+1}^i \frac{1}{(i+1-j)^2} |\mathbf{v}_j|^2 \right).$$

Thus

$$\sum_{I \subset K} (W_I \mathbf{v}, \mathbf{v}) = |K| \left(\sum_{i=k}^n (\mathbf{v}, \mathbf{g}_i)^2 + \sum_{j=k+1}^n |\mathbf{v}_j|^2 \sum_{i=j}^n \frac{1}{(i+1-j)^2} \right).$$

The second sum is less than $C|K| \sum_{j=0}^n |\mathbf{v}_j|^2 = C|K|\|\mathbf{v}\|^2$. To estimate the first sum, let \mathbf{G} represent the $(n-k+1) \times (k+1)$ matrix whose ij^{th} entry is the coefficient of \mathbf{e}_{j-1} in \mathbf{g}_{i+k-1} . Then $\sum_{i=k}^n (\mathbf{v}, \mathbf{g}_i)^2 \leq \|\mathbf{G}\|^2 \|\mathbf{v}\|^2$. Here $\|\mathbf{G}\|$ is taken as an operator from \mathbb{C}^{k+1} to \mathbb{C}^{n-k+1} . Under a suitable permutation of indices, however, \mathbf{G} is seen to be a restriction of the Hilbert matrix $\mathbf{A}, (\mathbf{A}_{ij} = \frac{1}{i+j-1})_{i,j=1}^{\infty}$, to finite-dimensional subspaces. It is well known [2] that \mathbf{A} is bounded from $\ell^2(\mathbb{N})$ to itself. Thus the first sum is less than $|K|\|\mathbf{A}\|^2 \|\mathbf{v}\|^2 = C|K|\|\mathbf{v}\|^2$. Dividing all weights W_I by an appropriate constant proves the theorem.

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