PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 2, Pages 529–531
S 0002-9939(01)06109-3
Article electronically published on June 6, 2001

ASYMPTOTIC PROPERTIES OF THE VECTOR CARLESON EMBEDDING THEOREM

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(Communicated by Christopher D. Sogge)

ABSTRACT. The dyadic Carleson embedding operator acting on $\mathbb{C}^n$-valued functions has norm at least $C\log n$. Thus the Carleson Embedding Theorem fails for Hilbert space valued functions.

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and $\{I\}_{I\in D}$ its collection of dyadic arcs. Let w_I be nonnegative real numbers indexed by $I \in D$. For integrable functions f on $\mathbb{T}$, denote by $\langle f\rangle_I$ the average $|I|^{-1}\int_I f(y)dy$. The classical Carleson embedding theorem [1] is equivalent to the following dyadic result:

Theorem. *If $\sum_{I\subset K} w_I \le |K|$ for all $K \in D$, then $\sum_{I\in D} w_I\langle f\rangle_I^2 \le C\|f\|^2$ for all $f \in L^2(\mathbb{T})$.*

The converse is also true (up to the placement of constants) and is verified by considering functions of the form $f = \chi_J$, $J \in D$.

An analogous statement may be made for functions taking values in $\mathbb{C}^n$ with matrix-valued weights $W_I \ge 0$ in the sense of quadratic forms. We wish to consider the following n-dimensional embedding theorem:

Proposition. *If $\left\|\sum_{I\subset K} W_I\right\| \le |K|$ for all $K \in D$, then*

$$\sum_{I\in D}(W_I\langle f\rangle_I, \langle f\rangle_I) \le C_n\|f\|^2$$

for all $f \in L^2(\mathbb{T};\mathbb{C}^n)$.

The space $\mathbb{C}^n$ here is viewed as a finite-dimensional Hilbert space. One might ask whether a similar result still holds when f takes values in a general Hilbert space $\mathbb{H}$ and W_I are positive selfadjoint operators. This is answered in the negative by [4], which proves that C_n must be bounded from below by $c\log n$. In the current paper we will use the construction in [4] to verify the stronger bound $C_n \ge c(\log n)^2$, which is also proved in [5]. A precise statement is as follows:

Theorem 1. *There exist a function $f \in L^2(\mathbb{T};\mathbb{C}^n)$ and matrix weights $W_I \ge 0$ such that $\left\|\sum_{I\subset K} W_I\right\| \le |K|$ and $\sum_{I\in D}(W_I\langle f\rangle_I, \langle f\rangle_I) \ge c(\log n)^2\|f\|^2$, where $c > 0$ is independent of n.*

Received by the editors July 5, 2000.
2000 *Mathematics Subject Classification.* Primary 42B20, 42A50.
Key words and phrases. Carleson embedding theorem, vector valued functions, operator valued measures, weights.

Remarks. The example presented here is due to Nazarov, Treil, and Volberg [4]. It is further shown in [3] and [4] that the best possible C_n is bounded above by $C(\log n)^2$, making these results sharp up to a constant factor.

Proof of Theorem 1. Let $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis for $\mathbb{C}^{n+1}$. Define the Rademacher functions $r_j(e^{2\pi it}) = (-1)^{\lfloor 2^j t\rfloor}$. For a dyadic interval $I, |I| \le 2^{-j}$, r_j is seen to be constant along I. Its value throughout the interval will be called $r_j(I)$.

Let $f(x) = \sum_{j=0}^n r_j(x)\mathbf{e}_j$. Clearly $\|f\|^2 = n+1$. The averages of f over dyadic intervals are also easy to compute. When $|I| = 2^{-i}, i \le n$, $\langle f\rangle_I = \sum_{j=0}^i r_j(I)\mathbf{e}_j$.

Let $W_I, |I| \ge 2^{-n}$, be the rank-one operator satisfying $W_I\mathbf{v} = |I|(\mathbf{v}, \phi_I)\phi_I$, where $\phi_I = \sum_{j=0}^i \frac{1}{i+1-j} r_j(I)\mathbf{e}_j$. Define ϕ_I to be 0 when $|I| < 2^{-n}$. Already we can estimate the sum

$$\sum_{I\in D}(W_I\langle f\rangle_I, \langle f\rangle_I) = \sum_{I\in D}|I|(\langle f\rangle_I, \phi_I)^2 = \sum_{i=0}^n\Big(\sum_{j=0}^i \frac{1}{i+1-j}\Big)^2 \ge cn(\log n)^2.$$

The only task remaining is to show that $\|\sum_{I\subset K} W_I\|$ is controlled by $|K|$. We will prove the estimate $\sum_{I\subset K}(W_I\mathbf{v},\mathbf{v}) = \sum_{I\subset K}|I|(\mathbf{v},\phi_I)^2 \le C|K||\mathbf{v}|^2$ for all $\mathbf{v}\in\mathbb{C}^{n+1}$.

For each interval I with $|I| = 2^{-i}$, split the vector ϕ_I into the sum of two parts, $\phi_I = \sum_{j=0}^k \frac{1}{i+1-j} r_j(K)\mathbf{e}_j + \sum_{j=k+1}^i \frac{1}{i+1-j} r_j(I)\mathbf{e}_j$. Denote the first sum, which depends only on the length of $I \subset K$, by $\mathbf{g}_i$. Summing over all I with $|I| = 2^{-i}$, and exploiting the orthogonality of the Rademacher functions,

$$\sum_{\substack{I\subset K\\ |I|=2^{-i}}} |I|(\mathbf{v},\phi_I)^2 = |K|\Big((\mathbf{v},\mathbf{g}_i)^2 + \sum_{j=k+1}^i \frac{1}{(i+1-j)^2}|\mathbf{v}_j|^2\Big).$$

Thus

$$\sum_{I\subset K}(W_I\mathbf{v},\mathbf{v}) = |K|\Big(\sum_{i=k}^n(\mathbf{v},\mathbf{g}_i)^2 + \sum_{j=k+1}^n |\mathbf{v}_j|^2\sum_{i=j}^n \frac{1}{(i+1-j)^2}\Big).$$

The second sum is less than $C|K|\sum_{j=0}^n|\mathbf{v}_j|^2 = C|K|\,|\mathbf{v}|^2$. To estimate the first sum, let $\mathbf{G}$ represent the $(n-k+1)\times(k+1)$ matrix whose ij^{th} entry is the coefficient of $\mathbf{e}_{j-1}$ in $\mathbf{g}_{i+k-1}$. Then $\sum_{i=k}^n(\mathbf{v},\mathbf{g}_i)^2 \le \|\mathbf{G}\|^2|\mathbf{v}|^2$. Here $\|\mathbf{G}\|$ is taken as an operator from $\mathbb{C}^{k+1}$ to $\mathbb{C}^{n-k+1}$. Under a suitable permutation of indices, however, $\mathbf{G}$ is seen to be a restriction of the Hilbert matrix $\mathbf{A}, (\mathbf{A}_{ij} = \frac{1}{i+j-1})\big|_{i,j=1}^\infty$, to finite-dimensional subspaces. It is well known [2] that $\mathbf{A}$ is bounded from $\ell^2(\mathbb{N})$ to itself. Thus the first sum is less than $|K|\,\|\mathbf{A}\|^2|\mathbf{v}|^2 = C|K|\,|\mathbf{v}|^2$. Dividing all weights W_I by an appropriate constant proves the theorem.

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