

POROUS MEASURES ON \mathbb{R}^n : LOCAL STRUCTURE AND DIMENSIONAL PROPERTIES

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ABSTRACT. We study dimensional properties of porous measures on \mathbb{R}^n . As a corollary of a theorem describing the local structure of nearly uniformly porous measures we prove that the packing dimension of any Radon measure on \mathbb{R}^n has an upper bound depending on porosity. This upper bound tends to $n - 1$ as porosity tends to its maximum value.

1. INTRODUCTION AND PRELIMINARIES

Porosity is a quantity that describes irregularities of fractals. The study of dimensional properties featured by porous sets was pioneered by P. Mattila. In [M1] he verified the existence of a non-increasing function which gives an upper bound for Hausdorff dimension of any set in \mathbb{R}^n as a function of porosity. Furthermore, he showed that this upper bound tends to $n - 1$ as porosity tends to its maximum value. In [S] A. Salli generalized the corresponding results for packing dimension and established the correct asymptotic behaviour for the upper bound.

For measures the following definition of porosity was introduced in [EJJ].

1.1. Definition. The porosity of a Radon measure μ on \mathbb{R}^n at a point $x \in \mathbb{R}^n$ is defined by

$$(1.2) \quad \text{por}(\mu, x) = \lim_{\varepsilon \downarrow 0} \liminf_{r \downarrow 0} \text{por}(\mu, x, r, \varepsilon)$$

where for all $r, \varepsilon > 0$

$$(1.3) \quad \begin{aligned} \text{por}(\mu, x, r, \varepsilon) = \sup \{ p \geq 0 \mid & \text{there is } z \in \mathbb{R}^n \text{ such that } B(z, pr) \subset B(x, r) \\ & \text{and } \mu(B(z, pr)) \leq \varepsilon \mu(B(x, r)) \}. \end{aligned}$$

The porosity of μ is

$$(1.4) \quad \begin{aligned} \text{por}(\mu) &= \text{ess sup}_{x \in \mathbb{R}^n} \text{por}(\mu, x) \\ &= \inf \{ s \geq 0 \mid \text{por}(\mu, x) \leq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n \}. \end{aligned}$$

We will relate porosity of measures to packing dimension defined as follows. (For the definition of the packing dimension of a set see [M2, 5.9 and Theorem 5.11].)

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1.5. Definition. Let μ be a Radon measure on \mathbb{R}^n . The packing dimension of μ is defined in terms of upper local dimensions

$$(1.6) \quad \dim_{\text{p}}(\mu) = \sup\{s \geq 0 \mid \limsup_{i \rightarrow \infty} \frac{\log \mu(D_i(x))}{\log 2^{-i}} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\}$$

where $D_i(x)$ is the closed dyadic cube of side-length 2^{-i} containing x . Equivalently this definition can be given using packing dimensions of Borel sets with positive μ -measure

$$(1.7) \quad \dim_{\text{p}}(\mu) = \inf\{\dim_{\text{p}}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\}.$$

Replacing “liminf” by “limsup” in (1.2) gives the upper porosity of a measure which was studied by M. E. Mera and M. Morán in [MM]. They showed that if μ satisfies the doubling condition, that is,

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty$$

for μ -almost all $x \in \mathbb{R}^n$, then the upper porosity of μ is either 0 or $1/2$. (Above $B(x, r)$ is the closed ball with radius r and centre x .) Furthermore, for any non-doubling measure the upper porosity equals 1. Note that the (lower) porosity may obtain any value between 0 and $1/2$ for both doubling and non-doubling measures (see [JJM, (2.17)] and [EJJ, Example 4]). The upper porosity is too weak for the purpose of obtaining a non-trivial upper bound for dimension; for any $p = 0, 1/2, 1$ and $0 \leq d \leq n$ there exists a Radon measure μ with the upper porosity equal to p and with both Hausdorff and packing dimension equal to d .

In this paper we will establish a connection between porosity and packing dimension for all Radon measures on \mathbb{R}^n . The case $n = 1$ was studied in [JJ]. In [EJJ] the emphasis was given to doubling measures on \mathbb{R}^n . For such measures the porosity can be given in terms of porosities of Borel sets with positive measure:

$$\text{por}(\mu) = \sup\{\text{por}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\}.$$

(The doubling condition is necessary here; see [EJJ] for details.)

We will generalize the results of [JJ] to higher dimensions by verifying that the packing dimension of any Radon measure on \mathbb{R}^n is bounded above by a function that depends on porosity and by showing that this upper bound goes to $n - 1$ as porosity tends to its maximum value $1/2$ (see Corollary 2.9). In particular, the packing dimension of any Radon measure on \mathbb{R}^n having porosity close to $1/2$ cannot be much larger than $n - 1$.

Our main tools are a dimension estimate obtained from the strong law of large numbers and a description of the local structure of nearly uniformly porous measures. The latter one states that for a given nearly uniformly porous measure any sufficiently small dyadic cube can be divided into three parts, two having small measure and the remaining one being a narrow boundary of a convex set (see Theorem 2.2).

2. LOCAL STRUCTURE AND DIMENSIONAL PROPERTIES

We recall the following lemma from [JJ] according to which we may replace any measure by a nearly uniformly porous measure when estimating packing dimension from above.

2.1. Lemma. *Assume that μ is a Radon measure on \mathbb{R}^n such that $\text{por}(\mu) \geq p$. Let $0 < \delta < 1$. Then there is a Radon measure μ_δ with compact support $\text{spt}(\mu_\delta) \subset \text{spt}(\mu)$ and with $\dim_p(\mu_\delta) \geq \dim_p(\mu)$ such that the following property holds: there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leq \varepsilon_\delta$ there are a Borel set $B_{\delta,\varepsilon}$ and $r_{\delta,\varepsilon} > 0$ with $\mu_\delta(\mathbb{R}^n \setminus B_{\delta,\varepsilon}) \leq \delta\mu_\delta(\mathbb{R}^n)$ and*

$$\text{por}(\mu_\delta, x, r, \varepsilon) > p - \frac{\delta}{2}$$

for all $x \in B_{\delta,\varepsilon}$ and $0 < r \leq r_{\delta,\varepsilon}$.

Proof. See [JJ, Lemma 2.2]. □

For all positive integers i we use the notation \mathcal{D}_i for the family of closed dyadic cubes in \mathbb{R}^n with side-length 2^{-i} , that is, cubes of the form $\{x \in \mathbb{R}^n \mid k_j 2^{-i} \leq x_j \leq (k_j + 1)2^{-i} \text{ for all } j = 1, \dots, n\}$ where $k_j, j = 1, \dots, n$, are integers. For all $Q \in \mathcal{D}_i$ and for all positive integers k , let $\mathcal{N}^k(Q) \subset \mathcal{D}_i$ be the family of the $(2k+1)^n$ neighbouring dyadic cubes of Q with side-length 2^{-i} being located symmetrically around Q (including Q itself).

For all $\delta > 0$, a δ -plate is a $\frac{\delta}{2}$ -neighbourhood of an $(n - 1)$ -dimensional affine subspace of \mathbb{R}^n . An affine δ -boundary of a convex polyhedron P is the union of parts of δ -plates glued on all faces of P such that the union of P and its affine δ -boundary is a polyhedron obtained by magnifying P .

The following theorem describes the local structure of porous measures by stating that in all sufficiently small dyadic cubes such measures are essentially concentrated on a narrow boundary of some convex set.

2.2. Theorem. *Assume that μ is a Radon measure on $[0, 1]^n$ such that $\text{por}(\mu) \geq \frac{1}{2}(1 - \beta)$ for $0 \leq \beta \leq \frac{1}{18}$. Let K be a positive integer. For all $0 < \delta < \frac{1}{18}$ let μ_δ and ε_δ be as in Lemma 2.1. Let $0 < \varepsilon < \varepsilon_\delta$. Then there is a positive integer i_0 depending on K, δ , and ε such that for all $i \geq i_0$ any cube $Q \in \mathcal{D}_i$ can be divided into three disjoint (not necessarily non-empty) parts*

$$(2.3) \quad Q = E \cup P \cup I$$

where

$$\mu_\delta(E) \leq C_Q^K N \varepsilon$$

for an integer N depending on K, δ , and β and for $C_Q^K = \max_{D \in \mathcal{N}^K(Q)} \mu_\delta(D)$, P is an affine $C_{\beta,\delta} 2^{-i}$ -boundary of a convex polyhedron with $C_{\beta,\delta} = 6K(\beta + \delta) + \frac{2n}{K(1-\beta-\delta)}$, and $I \subset Q \setminus B_{\delta,\varepsilon}$. Here $B_{\delta,\varepsilon}$ is as in Lemma 2.1.

Proof. Let $r_{\delta,\varepsilon}$ be as in Lemma 2.1 and let i_0 be the smallest integer such that $(1+K)2^{-i_0} < r_{\delta,\varepsilon}$. Consider an integer $i \geq i_0$. Note that for those $Q \in \mathcal{D}_i$ which do not intersect $B_{\delta,\varepsilon}$ equality (2.3) is trivial. Let $Q \in \mathcal{D}_i$ such that $Q \cap B_{\delta,\varepsilon} \neq \emptyset$. Setting $r_x = \text{dist}(x, \partial Q) + K2^{-i}$ for all $x \in Q \cap B_{\delta,\varepsilon}$, we have $B(x, r_x) \subset \bigcup_{D \in \mathcal{N}^K(Q)} D$. Lemma 2.1 implies that for all $x \in Q \cap B_{\delta,\varepsilon}$ there is a ball B_x with radius $qr_x = \frac{1}{2}(1 - \beta - \delta)r_x$ such that $B_x \subset B(x, r_x)$ and

$$(2.4) \quad \mu_\delta(B_x) \leq (2K + 1)^n C_Q^K \varepsilon.$$

Set $a^{1/n} = 2K(1 - 2q)\alpha^{1/n}$ where $\alpha = \mathcal{L}^n(B(0, 1))$ is the Lebesgue measure of the unit ball. Then there are integers $N(a) \geq N(a, Q) \geq 0$ for which there are

$x_1, \dots, x_{N(a,Q)} \in Q \cap B_{\delta,\varepsilon}$ such that

$$\mathcal{L}^n\left((B_{x_j} \cap Q) \setminus \bigcup_{k=1}^{j-1} B_{x_k}\right) \geq a\mathcal{L}^n(Q)$$

for all $j = 1, \dots, N(a, Q)$ and

$$(2.5) \quad \mathcal{L}^n\left((B_x \cap Q) \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\right) < a\mathcal{L}^n(Q)$$

for all $x \in Q \cap B_{\delta,\varepsilon}$ such that $x \neq x_j$ for all $j = 1, \dots, N(a, Q)$. (We use the interpretation $\bigcup_{k=1}^0 B_{x_k} = \emptyset$. In the case $N(a, Q) = 0$ we have $\mathcal{L}^n(B_x \cap Q) < a\mathcal{L}^n(Q)$ for all $x \in Q \cap B_{\delta,\varepsilon}$.)

Define

$$I_1 = \left\{y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \text{dist}\left(y, \partial\left(Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\right)\right) > 2\left(\frac{a}{\alpha}\right)^{1/n} 2^{-i}\right\}.$$

Then

$$(2.6) \quad B_x \cap I_1 = \emptyset$$

for all $x \in I_1 \cap B_{\delta,\varepsilon}$. In fact, assuming that $B_x = B_{x_j}$ for some $j = 1, \dots, N(a, Q)$, equality (2.6) holds. If $B_x \cap I_1 \neq \emptyset$ for some $x \in I_1 \cap B_{\delta,\varepsilon}$ with $B_x \neq B_{x_j}$ for all $j = 1, \dots, N(a, Q)$, then, as will be indicated shortly, the set $(B_x \cap Q) \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$ contains a ball with radius $(a/\alpha)^{1/n} 2^{-i}$ contradicting (2.5). To find such a ball, take $z \in B_x \cap I_1$. Then $B(z, 2(a/\alpha)^{1/n} 2^{-i}) \subset Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$. Since $(a/\alpha)^{1/n} 2^{-i} \leq \frac{1}{2}qr_x$, the ball B_x contains a ball with radius $(a/\alpha)^{1/n} 2^{-i}$ having z on its boundary such that the centre of the ball belongs to the line going through z and the centre of B_x . Clearly this ball is a subset of $Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$. This completes the proof of (2.6).

Set

$$I_2 = \left\{y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \text{dist}\left(y, \partial\left(Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\right)\right) > 3\left(\frac{a}{\alpha}\right)^{1/n} 2^{-i}\right\}.$$

Then $I_2 \subset Q \setminus B_{\delta,\varepsilon}$. To see this, assume that there exists $x \in I_2 \cap B_{\delta,\varepsilon}$. From (2.6) we get

$$\text{dist}(x, \partial B_x) > \left(\frac{a}{\alpha}\right)^{1/n} 2^{-i} = 2K(1-2q)2^{-i}.$$

On the other hand $\text{dist}(x, \partial B_x) \leq r_x(1-2q) \leq (K+1/2)(1-2q)2^{-i}$. Hence $I_2 \subset Q \setminus B_{\delta,\varepsilon}$.

Let $B_{x_k} = B(z_k, qr_{x_k})$. Since $\frac{n}{qK}2^{-i}$ is an upper bound for the height of a segment of any ball with radius qr_x having chord with length at most $\sqrt{n}2^{-i}$, the intersections of each of the annuli $B(z_k, qr_x + 3(a/\alpha)^{1/n} 2^{-i}) \setminus B_{x_k}$ and Q can be included in a $C_{\beta,\delta}2^{-i}$ -plate. Adding parts of the affine $C_{\beta,\delta}$ -boundary of Q if necessary concludes the construction of P . Setting

$$I = I_2 \setminus P$$

and

$$E = Q \setminus (P \cup I) \subset Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k},$$

equality (2.3) follows since

$$\mu_\delta(Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k}) \leq N(a)(2K + 1)^n C_Q^K \varepsilon$$

by (2.4). □

Let μ be a Radon probability measure on $[0, 1]^n$ such that $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. Letting k be a positive integer, set $I = \{1, \dots, 2^{kn}\}$. For all positive integers j , denote by \mathbf{I}^j the set of all j -term sequences of integers belonging to I and by \mathbf{I}^∞ the corresponding set of infinite sequences, that is,

$$\mathbf{I}^j = \{(i_1, \dots, i_j) \mid i_l \in I \text{ for all } l = 1, \dots, j\}$$

and

$$\mathbf{I}^\infty = \{(i_1, i_2, \dots) \mid i_l \in I \text{ for all } l = 1, 2, \dots\}.$$

Divide $[0, 1]^n$ into 2^{kn} dyadic subcubes, enumerate them and denote them by D_j , $j = 1, \dots, 2^{kn}$. Define $f : [0, 1]^n \rightarrow [0, 1]^n$ by setting $f(x) = 2^k x \bmod 1$. For each $x \in [0, 1]^n$ we define a sequence $\mathbf{i}^x = (i_1, i_2, \dots) \in \mathbf{I}^\infty$ such that $f^{l-1}(x) \in D_{i_l}$ for all $l = 1, 2, \dots$. Note that for all $x = (x_1, \dots, x_n)$ the sequence \mathbf{i}^x is unique unless x_j is a dyadic rational for some $j = 1, \dots, n$. (If $f^l(x)_i$ is a dyadic rational for some l and i , one can choose between indices corresponding to “left” and “right” cube. If one chooses left, then $f^{l+1}(x)_i = 1$ and otherwise $f^{l+1}(x)_i = 0$.) For a positive integer l and $\mathbf{i} = (i_1, i_2, \dots) \in \mathbf{I}^\infty$ let $\mathbf{i}|_l = (i_1, \dots, i_l) \in \mathbf{I}^l$ be the sequence of the first l digits of \mathbf{i} and for all $j = 1, \dots, 2^{kn}$ let $n_j(\mathbf{i}|_l)$ be the number of j 's in $\mathbf{i}|_l$.

We can attach a sequence (P_l^μ) of probability measures on I to μ such that for all $j = 1, \dots, 2^{kn}$ $P_l^\mu(\{j\})$ gives the probability that the l th digit (in the above representation) of a random number (with respect to μ) in $[0, 1]^n$ equals j , that is,

$$P_l^\mu(\{j\}) = \sum_{\substack{(i_1, \dots, i_l) \in \mathbf{I}^l \\ i_l = j}} \mu(D_{i_1, \dots, i_l})$$

where D_{i_1, \dots, i_l} is the closed dyadic subcube of $[0, 1]^n$ of side-length 2^{-kl} consisting of points whose expansion begins with (i_1, \dots, i_l) . The measures P_l^μ are well-defined since $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. We use the notation P^μ for the product measure $\prod_{l=1}^\infty P_l^\mu$ on the code space \mathbf{I}^∞ .

2.7. Proposition. *Let μ be a Radon probability measure on $[0, 1]^n$ such that $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. Let $p \leq 2^{-kn}$ and $L \in I$. Assume that $\limsup_{l \rightarrow \infty} \frac{1}{l} \sum_{i=1}^l P_i^\mu(\{j\}) \leq p$ for all $j = 1, \dots, L$. Then*

$$\dim_p \mu \leq -\frac{1}{\log 2^k} (Lp \log p + (1 - Lp) \log(\frac{1 - Lp}{2^{kn} - L})) =: \alpha(p, L).$$

Proof. The strong law of large numbers [Fe, X.7.1] gives for all $j = 1, \dots, L$ that

$$\limsup_{l \rightarrow \infty} \frac{1}{l} n_j(\mathbf{i}|_l) \leq p$$

for \mathbb{P}^μ -almost all $\mathbf{i} \in \mathbf{I}^\infty$. Defining

$$E_{p,L} = \{x \in [0, 1]^n \mid \limsup_{l \rightarrow \infty} \frac{1}{l} n_j(\mathbf{i}^x|_l) \leq p \text{ for all } j = 1, \dots, L\},$$

this implies that $\mu(E_{p,L}) = 1$. Since $E_{p,L}$ is a Borel set it is enough to prove that $\dim_p(E_{p,L}) \leq \alpha(p, L)$.

Let ρ be a probability measure on I such that $\rho(\{j\}) = p$ for all $j = 1, \dots, L$ and $\rho(\{j\}) = (1 - Lp)/(2^{kn} - L)$ for all $j = L + 1, \dots, 2^{kn}$. Let ν be the image of the infinite product of the measures ρ under the map $\pi : \mathbf{I}^\infty \rightarrow [0, 1]^n$. Note that since $p \leq 2^{-kn}$ we have

$$-u \log p - (1 - u) \log \left(\frac{1 - Lp}{2^{kn} - L} \right) \leq \alpha(p, L) \log 2^k$$

for all $u \leq Lp$. Let $x \in E_{p,L}$. The equality

$$\log \nu(D_{kl}(x)) = \log p \sum_{j=1}^L n_j(\mathbf{i}^x|_l) + \log \left(\frac{1 - Lp}{2^{kn} - L} \right) \sum_{j=L+1}^{2^{kn}} n_j(\mathbf{i}^x|_l)$$

gives

$$\liminf_{l \rightarrow \infty} \frac{1}{l} \log \left(\frac{\nu(D_{kl}(x))}{2^{-klt}} \right) \geq -\log 2^k \alpha(p, L) + t \log 2^k$$

where $D_{kl}(x)$ is the dyadic cube of side-length 2^{-kl} containing x . Thus if $t > \alpha(p, L)$, then $\liminf_{l \rightarrow \infty} \frac{\nu(D_{kl}(x))}{2^{-klt}} = \infty$, implying

$$\limsup_{l \rightarrow \infty} \frac{\log(\nu(D_{kl}(x)))}{\log 2^{-kl}} \leq t.$$

By [Fa, Proposition 2.3 (d)] we get $\dim_p(E_{p,L}) \leq \alpha(p, L)$. \square

Let k and i be positive integers. Dyadic cubes in \mathcal{D}_{ki} form a brood if they belong to the same dyadic cube belonging to $\mathcal{D}_{k(i-1)}$. Note that each brood consists of 2^{kn} dyadic cubes. Given a measure μ on \mathbb{R}^n , order the cubes of every brood such that $\mu(D_j) \leq \mu(D_{j+1})$ for all $j = 1, \dots, 2^{kn}$. Let $\mathcal{D}_{ki}^j(\mu)$ be the set of the j th cubes of all broods.

2.8. Theorem. *Let μ be a Radon probability measure on $[0, 1]^n$ such that $\text{por}(\mu) \geq \frac{1}{2}(1 - \beta)$ for $0 \leq \beta \leq \frac{1}{18}$ and $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. For all $0 < \delta < \frac{1}{18}$ let μ_δ be as in Lemma 2.1. Then there is an integer L with $2^{k(\beta, \delta)n} \geq L \geq 2^{k(\beta, \delta)n} - c2^{k(\beta, \delta)(n-1)}$ where c is a constant depending only on n and $k(\beta, \delta) \rightarrow \infty$ as $\delta \rightarrow 0$ and $\beta \rightarrow 0$ such that the following inequality is valid: for all $j = 1, \dots, L$ we have*

$$\limsup_{l \rightarrow \infty} \frac{1}{l} \sum_{i=1}^l \sum_{D \in \mathcal{D}_{ki}^j(\mu_\delta)} \mu_\delta(D) \leq \delta \mu_\delta([0, 1]^n).$$

Proof. Let k_0 be the largest integer such that $\beta + \delta \leq 2^{-2k_0}$. Set $K = 2^{k_0}$. Let $0 < \varepsilon < \varepsilon_\delta$ and let $i \geq i_0$ where ε_δ and i_0 are as in Theorem 2.2. Let k be the largest integer such that $2^{-k} > 4(6 + 4n)2^{-k_0}$. Then $2^{-k} > 4C_{\beta, \delta}$ where $C_{\beta, \delta}$ is as in Theorem 2.2. Consider $Q \in \mathcal{D}_{ki}$. Let $Q = E_Q \cup P_Q \cup I_Q$ be the partition of Q given in Theorem 2.2. Take any $x \in P_Q$. Then $D_{k(i+1)}(x)$ and its neighbouring cubes in $\mathcal{D}_{k(i+1)}$ cover a part of P_Q such that the \mathcal{L}^{n-1} -measure of the covered part of both inner and outer boundary of P_Q is at least $2^{-(n-1)}2^{-k(i+1)(n-1)}$. By

the convexity of P_Q the \mathcal{L}^{n-1} -measure of the outer boundary of P_Q is less than $2n2^{-ki(n-1)}$. Hence we need at most $3^{2n}2n2^{n-1}2^{k(n-1)}$ cubes from $\mathcal{D}_{k(i+1)}$ to cover P_Q . Thus there are $L \geq 2^{kn} - 3^{2n}2n2^{n-1}2^{k(n-1)}$ cubes in $\mathcal{D}_{k(i+1)}$ which belong to $E_Q \cup I_Q$. Clearly $\mu_\delta(D_j) \leq \mu_\delta(E_Q \cup I_Q)$ for all $j = 1, \dots, L$. Theorem 2.2 and Lemma 2.1 give

$$\begin{aligned} \limsup_{l \rightarrow \infty} \frac{1}{l} \sum_{i=1}^l \sum_{D \in \mathcal{D}_{k_i}^j(\mu_\delta)} \mu_\delta(D) &\leq \limsup_{l \rightarrow \infty} \frac{1}{l} \sum_{i=1}^l \sum_{Q \in \mathcal{D}_{k_i}} \mu_\delta(E_Q \cup I_Q) \\ &\leq \limsup_{l \rightarrow \infty} \frac{1}{l} \sum_{i=1}^l ((2K+1)^n N\varepsilon + \delta) \mu_\delta([0, 1]^n) \xrightarrow{\varepsilon \rightarrow 0} \delta \mu_\delta([0, 1]^n). \end{aligned}$$

Since $k_0 \rightarrow \infty$ as δ and β tend to zero we may let k tend to infinity when $\delta \rightarrow 0$ and $\beta \rightarrow 0$. □

2.9. Corollary. *Assume that μ is a Radon measure on \mathbb{R}^n . If $0 < \beta \leq 1$ such that $\text{por}(\mu) \geq \frac{1}{2}(1 - \beta)$, then $\dim_p(\mu) \leq d(\beta)$ where $d(\beta) \rightarrow n - 1$ as $\beta \rightarrow 0$.*

Proof. The claim follows from Theorem 2.8 and from the obvious generalization of [JJ, Lemma 3.3] (see [JJ, Corollary 3.4]) with $d(\beta) = \lim_{\delta \rightarrow 0} \alpha(\delta, L)$ where L is as in Theorem 2.8. Note that by the choices of k and k_0 in Theorem 2.8 we have $C_n(\beta + \delta)^{-1/2} \leq 2^k \leq \tilde{C}_n(\beta + \delta)^{-1/2}$ for constants C_n and \tilde{C}_n depending only on n . By the lower and upper bounds given in Theorem 2.8 for L we obtain that $d(\beta) \rightarrow n - 1$ when $\beta \rightarrow 0$. □

2.10. *Remark.* After finishing this paper we obtained the preprint [BS] from D. B. Beliaev and S. K. Smirnov where similar dimension results have been proved using different methods.

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