PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 2, Pages 395–404
S 0002-9939(01)06370-5
Article electronically published on September 19, 2001

EXISTENCE AND LIPSCHITZ REGULARITY FOR MINIMA

CARLO MARICONDA AND GIULIA TREU

(Communicated by David S. Tartakoff)

Abstract. We prove the existence, uniqueness and Lipschitz regularity of the minima of the integral functional

$$I(u) = \int_\Omega L(x, u, \nabla u)\, dx$$

on $\bar{u} + W_0^{1,q}(\Omega)$ $(1 \le q \le +\infty)$ for a class of integrands $L(x, z, p) = f(p) + g(x, z)$ that are convex in (z, p) and for boundary data satisfying some barrier conditions. We do not impose regularity or growth assumptions on L.

1. Introduction

We consider the problem of the existence and regularity of the minima of an integral functional of the form

$$I(u) = \int_\Omega L(x, u, \nabla u)\, dx$$

on $\bar{u} + W_0^{1,q}(\Omega)$ $(1 \le q \le +\infty)$ where $L(x, z, p) = f(p) + g(x, z)$ is convex in (z, p).

In the classical approach the Lagrangian L is assumed to be smooth, convex in ∇u and to satisfy, together with its derivatives, some growth conditions. This ensures the existence of a minimum of I that satisfies the integral *Euler equation*. The results on the regularity of the minima then follow from those on the regularity of the solutions to quasilinear elliptic equations. There are some recent results that do not involve conditions on the derivatives of L. We mention the papers [CV] and [AAB] for one–dimensional problems and [GG] for the multi–dimensional case. We point out that all of them assume some growth conditions on L; namely in [GG] it is required that $L(x, z, p)$ behave like a power of p for $|p| \to \infty$.

In this paper we do not assume either regularity or growth conditions on the integrand. In this setting, some conditions that ensure that the minima of I lying between two Lipschitz functions are actually Lipschitz were stated in [TV], [MT1] and [MT3]. In order to study the regularity of the minima we look for conditions on the boundary datum that guarantee the existence of two Lipschitz functions that bound the minima of I. In [MT1] we showed, through a new *Comparison Principle*, that if $\bar{u}$ satisfies a *Generalized Bounded Slope Condition* (GBSC), then every minimum is Lipschitz. Here we consider a wider class of integrands and we

Received by the editors May 20, 2000.
2000 *Mathematics Subject Classification.* Primary 49J52, 49J99, 49K30, 49N60.
Key words and phrases. Barrier, Euler equation, existence of minima, Lavrentiev, Lipschitz regularity.

prove that if the boundary datum fulfills the (GBSC) or a *barrier condition* the minimum for I in $\bar{u} + W_0^{1,q}(\Omega)$ exists for every $1 \leq q \leq +\infty$, it is unique and Lipschitz (and therefore no Lavrentiev phenomenon occurs). As a first step we obtain the existence of a minimum in the class of Lipschitz functions, extending a result by M. Miranda [M] for functionals depending only on the gradient. Then, via Clarke's nonsmooth version of the Euler equation [Cl], we obtain the existence of minima in Sobolev spaces; uniqueness follows from our Comparison Principle for minima.

In the smooth setting the (GBSC) and the barrier condition are well known and are used to give an estimate of the gradient of regular solutions to partial differential equations (see for instance Chapter 14 of [GT]). We point out that we exploit them in a different way since we do not know a priori either whether a minimum exists or if it is a solution to the Euler equation. This approach was inspired by [C] where Cellina obtains a regularity result for functionals of the gradient and boundary data satisfying the Bounded Slope Condition.

In the last section we prove the claim of our main theorem under less restrictive conditions on $\bar{u}$, assuming further that the functional I is strictly convex.

2. Notation and preliminary results

Let A be an open bounded subset of $\mathbb{R}^n$ $(n \geq 1)$; $\bar{A}$ is its closure and ∂A is its boundary. For $0 \leq k \leq +\infty$ we denote by $\mathcal{C}^k(A)$ (resp. $\mathcal{C}^k_c(A)$) the space of the k–times continuously differentiable functions in A (resp. with compact support in A). $\mathrm{Lip}(A)$ is the space of Lipschitz functions in A (that we consider to be extended in $\bar{A}$); we recall that the Lipschitz functions are differentiable almost everywhere. For u in $L^\infty(A)$ we denote by $\|u\|_{L^\infty(A)}$ the usual norm of u in $L^\infty(A)$. If u is in $W^{1,r}(A)$ the weak derivative of u with respect to the i–th variable is denoted by u_{x_i} and its gradient by ∇u; if $u = (u_1, \dots, u_n)$ is in $W^{1,r}(A; \mathbb{R}^n)$ its divergence is denoted by $\mathrm{div}\ u$. For $\bar{u}$ in $\mathrm{Lip}(A)$ and $K > 0$ we set $\mathrm{Lip}(A, \bar{u}) = \{u \in \mathrm{Lip}(A) : u = \bar{u} \text{ on } \partial\Omega\}$, $\mathrm{Lip}_K(A) = \{u \in \mathrm{Lip}(A) : \|\nabla u\|_{L^\infty(A)} \leq K\}$ and $\mathrm{Lip}_K(A, \bar{u}) = \{u \in \mathrm{Lip}_K(A) : u = \bar{u} \text{ on } \partial\Omega\}$. If $L : A \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, $(x, z, p) \longmapsto L(x, z, p)$, is differentiable with respect to z (resp. to $p = (p_1, \dots, p_n)$), then we denote by L_z (resp. L_{p_i}, $i = 1, \dots, n$) the partial derivative of L with respect to z (resp. p_i) and by L_x (resp. L_p) the gradient of L with respect to x (resp. p). In the case where $L(x, z, p)$ is convex in z (resp. p), $\partial_z L(\bar{x}, \bar{z}, \bar{p})$ (resp. $\partial_p L(\bar{x}, \bar{z}, \bar{p})$) is the subdifferential of the map $z \longmapsto L(\bar{x}, z, \bar{p})$ in $\bar{z}$ (resp. $p \longmapsto L(\bar{x}, \bar{z}, p)$ in $\bar{p}$) in the usual sense of convex analysis. For u in $W^{1,1}(\Omega)$ we set $u^+ = \max\{u, 0\}$. Given two vectors a and b in $\mathbb{R}^n$ we denote by $a \cdot b$ their usual scalar product in $\mathbb{R}^n$ and by $|a|$ the euclidean norm of a.

In what follows Ω is an open bounded subset of $\mathbb{R}^n$ and L is a function

$$\begin{aligned} L : \Omega \times \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R} \cup \{+\infty\} \\ (x, z, p) &\longmapsto L(x, z, p) \end{aligned}$$

such that $x \longmapsto L(x, z(x), p(x))$ is measurable for every measurable $z : \Omega \to \mathbb{R}$ and $p : \Omega \to \mathbb{R}^n$ (this condition is fulfilled if, for instance, L is a normal integrand; see [ET]). We define the functional I on $W^{1,1}(\Omega)$ by

$$\forall u \in W^{1,1}(\Omega) \qquad I(u) = \int_\Omega L(x, u(x), \nabla u(x))\, dx.$$

We always assume that there exist a in $\mathbb{R}$ and b in $L^1(\Omega)$ such that $L(x,z,p) \geq a|p| + b(x)$ for every (x,z,p); this implies that $I(u) > -\infty$ for every u in $W^{1,1}(\Omega)$.

We recall now a simplified version of a *Comparison Principle* that we proved in [MT1] involving some inequalities on the boundary of Ω for Sobolev functions.

Definition 2.1. For u in $W^{1,1}(\Omega)$ we say that $u \leq 0$ on $\partial\Omega$ if $u^+ \in W_0^{1,1}(\Omega)$. For u, v in $W^{1,1}(\Omega)$ by $u \leq v$ on $\partial\Omega$ we mean that $u - v \leq 0$ on $\partial\Omega$.

Remark 2.2. As we noted in [MT3] the pointwise inequality in $\partial\Omega$ for functions in $\mathcal{C}(\overline{\Omega}) \cap W^{1,1}(\Omega)$ implies the inequality in the sense of $W^{1,1}(\Omega)$.

Theorem 2.3 (Comparison Principle for minima and sub/super–solutions)**.** *Let $L(x,z,p) = f(p) + g(x,z)$ be convex in (z,p) such that either g is strictly convex in z or the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are contained in a vector subspace of dimension less than or equal to $n-1$. Let $1 \leq q \leq \infty$, $\bar{u} \in W^{1,q}(\Omega)$, w be a minimum for I in $\bar{u} + W_0^{1,q}(\Omega)$ and u (resp. v) be a sub–solution (resp. super–solution) of the Euler equation associated to I in $W^{1,q}(\Omega)$. If $u \leq w$ (resp. $w \leq v$) on $\partial\Omega$, then $u \leq w$ (resp. $w \leq v$) a.e. on Ω.*

We will also use the following consequence of Theorem 2.3.

Corollary 2.4. *Assume that L fulfills the assumption of Theorem 2.3. Let $Q > 0$, $\bar{u} \in \mathrm{Lip}_Q(\Omega)$, w be a minimum for I in $\mathrm{Lip}_Q(\Omega, \bar{u})$ and u (resp. v) in $\mathrm{Lip}_Q(\Omega, \bar{u})$ be a sub–solution (resp. super–solution) of the Euler equation associated to I in $W^{1,\infty}(\Omega)$. If $u \leq w$ (resp. $w \leq v$) on $\partial\Omega$, then $u \leq w$ (resp. $w \leq v$) a.e. on Ω.*

Proof. Let j_{B_Q} be the indicator function of the closed ball B_Q of radius Q (i.e. $j_{B_Q}(p) = 0$ if $|p| \leq Q$, $j_{B_Q}(p) = +\infty$ if $|p| > Q$) and $\tilde{L}(x,z,p) = f(p) + j_{B_Q}(p) + g(x,z)$. Since the epigraph of $f + j_Q$ is the epigraph of f restricted to $B_Q \times \mathbb{R}$, then $\tilde{L}$ satisfies the assumption of Theorem 2.3. Moreover the minima of I in $\mathrm{Lip}_Q(\Omega, \bar{u})$ are the minima of the functional

$$J(u) = \int_\Omega \tilde{L}(x, u, \nabla u)\, dx$$

in $\bar{u} + W_0^{1,\infty}(\Omega)$ and the sub/super–solutions for I in $W^{1,\infty}(\Omega)$ are still sub/super–solutions for J in $W^{1,\infty}(\Omega)$. The application of Theorem 2.3 yields the conclusion. □

3. Existence and Lipschitz regularity for minima

We consider the following Assumption (H) on the Lagrangian.

Assumption (H) (Structure conditions on $L(x,z,p)$)**.** $L(x,z,p) = f(p) + g(x,z)$ is convex in (z,p) and one of the following conditions holds:

(H1) The function g is of class $\mathcal{C}^2$ in $\overline{\Omega} \times \mathbb{R}$ and $\inf\{g_{zz}(x,z) : (x,z) \in \Omega \times \mathbb{R}\} > 0$; in this case we set $\iota = \inf\{g_{zz}(x,z) : (x,z) \in \Omega \times \mathbb{R}\}$ and $\sigma = \sup\{|g_{zx}(x,z)| : (x,z) \in \Omega \times \mathbb{R}\}$.

(H2) The function g does not depend on x and either g is strictly convex or the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are contained in a vector subspace of dimension less than or equal to $n-1$.

In what follows we set

$$M(g) = \begin{cases} \sigma/\iota & \text{if (H1) holds;} \\ 0 & \text{if (H2) holds.} \end{cases}$$

The next result gives some conditions under which a minimum of I in a Sobolev space lying between two Lipschitz functions is actually Lipschitz, together with a bound of the Lipschitz constant. The two cases have been obtained in some recent papers by using different methods; more precisely case i) was studied in [TV], [MT1] and case ii) was established in [MT3].

Theorem 3.1 (Regularity for constrained minima)**.** *Let* $\bar{u} \in \mathrm{Lip}(\Omega)$ *and* ℓ^1, ℓ^2 *belong to* $\mathrm{Lip}(\Omega, \bar{u})$. *Assume that one of the following conditions holds:*

i) L *satisfies Assumption* (H1)*; or*

ii) $L(x, z, p) = f(p) + g(z)$ *is convex and either* f *or* g *is strictly convex.*

Let w *be a minimum for* I *in* $\bar{u} + W_0^{1,q}(\Omega)$ $(1 \le q \le \infty)$ *such that* $\ell^1 \le w \le \ell^2$ *a.e. on* Ω. *Then* w *is Lipschitz, a Lipschitz constant being*

$$\max\{M(g), \|\nabla \ell^1\|_{L^\infty(\Omega)}, \|\nabla \ell^2\|_{L^\infty(\Omega)}\}.$$

The main results of the paper are based on the following consequence of Theorem 3.1.

Theorem 3.2 (A Lipschitz constant for constrained minima)**.** *Assume that* L *satisfies Assumption* (H1) *or that* $L(x, z, p) = f(p) + g(z)$ *is convex. Let* $Q > 0$, $\bar{u}$ *in* $\mathrm{Lip}_Q(\Omega)$ *and* ℓ^1, ℓ^2 *belong to* $\mathrm{Lip}(\Omega, \bar{u})$. *Then there exists a minimum for* I *in* $\mathcal{K}_Q = \{u \in \mathrm{Lip}_Q(\Omega, \bar{u}) : \ell^1 \le u \le \ell^2\}$ *whose Lipschitz constant is bounded by* $\max\{M(g), \|\nabla \ell^1\|_{L^\infty(\Omega)}, \|\nabla \ell^2\|_{L^\infty(\Omega)}\}$.

Proof. Notice first that I has a minimum in $\mathcal{K}_Q$ since $\mathcal{K}_Q$ is compact and I is lower semicontinuous in the weak* topology of $\bar{u} + W_0^{1,\infty}(\Omega)$. Moreover, denoting by j_{B_Q} the indicator function of the closed ball of radius Q (i.e. $j_{B_Q}(p) = 0$ if $|p| \le Q$, $j_{B_Q}(p) = +\infty$ if $|p| > Q$), the minima of I in $\mathcal{K}_Q$ are the minima of the functional

$$J(u) = \int_\Omega f(\nabla u) + g(x, u) + j_{B_Q}(\nabla u)\, dx$$

in $\{u \in \bar{u} + W_0^{1,\infty}(\Omega) : \ell^1 \le u \le \ell^2\}$. If L fulfills (H1), then so does the function $\tilde{L}(x, z, p) = f(p) + j_{B_Q}(p) + g(x, z)$; the application of Theorem 3.1 i) with $\tilde{L}$ instead of L yields $\|\nabla w\|_{L^\infty(\Omega} \le \max\{M(g), \|\nabla \ell^1\|_{L^\infty(\Omega)}, \|\nabla \ell^2\|_{L^\infty(\Omega)}\}$. Assume now that $L(x, z, p) = f(p) + g(z)$ is convex: if f or g are strictly convex, the claim follows directly from Theorem 3.1 ii); otherwise for every $\varepsilon > 0$ we consider the functional $I^\varepsilon(u) = \int_\Omega f(\nabla u) + g_\varepsilon(u)\, dx$ where $g_\varepsilon(z) = g(z) + \varepsilon z^2$ and we let w_ε be a minimum of I^ε in $\mathcal{K}_Q$. By Theorem 3.1 ii) applied to $L_\varepsilon(x, z, p) = f(p) + g_\varepsilon(z)$ a Lipschitz constant of w_ε is $\max\{\|\nabla \ell^1\|_{L^\infty(\Omega)}, \|\nabla \ell^2\|_{L^\infty(\Omega)}\}$. We may therefore assume that $(w_\varepsilon)_{\varepsilon > 0}$ converges to a function w in the weak* topology of $\bar{u} + W_0^{1,\infty}(\Omega)$; clearly $\|\nabla w\|_{L^\infty(\Omega)} \le \max\{\|\nabla \ell^1\|_{L^\infty(\Omega)}, \|\nabla \ell^2\|_{L^\infty(\Omega)}\}$. Since $\mathcal{K}_Q$ is weakly* closed, then w belongs to $\mathcal{K}_Q$. We claim that w is a minimum for I in $\mathcal{K}_Q$: in fact let $v \in \mathcal{K}_Q$; since $I^\varepsilon(w_\varepsilon) \le I^\varepsilon(v)$, then the following inequalities hold:

$$I(w_\varepsilon) \le I^\varepsilon(w_\varepsilon) \le I^\varepsilon(v) \le I(v) + \varepsilon C$$

where C is a constant depending only on Ω, ℓ^1 and ℓ^2. By the lower semi–continuity of I we obtain $I(w) \le \liminf_{\varepsilon \to 0} I(w_\varepsilon) \le I(v)$, proving the claim. □

We recall now some definitions given in [MT1] that will be useful in the sequel.

Definition 3.3. Let $K > 0$, $1 \le q \le +\infty$ and $q' = q/(q-1)$ be the conjugate of q. We say that u in $W^{1,q}(\Omega)$ is a *sub–solution* for the *weak Euler equation* associated to I in $W^{1,q}(\Omega)$ if there exist k in $L^{q'}(\Omega;\mathbb{R}^n)$ and h in $L^{q'}(\Omega)$ such that $k(x) \in \partial_p L(x,u(x),\nabla u(x))$ and $h(x) \in \partial_z L(x,u(x),\nabla u(x))$ a.e. on Ω satisfying

$$\forall \eta \in W_0^{1,q}(\Omega), \qquad \eta \geq 0 \text{ a.e.}, \qquad \int_\Omega k \cdot \nabla\eta + h\eta \, dx \leq 0. \tag{3.1}$$

We say that u in $W^{1,q}(\Omega)$ is a *super–solution* for the *weak Euler equation* associated to I in $W^{1,q}(\Omega)$ if the inequality (3.1) holds with the opposite sign for every positive η.

Definition 3.4 (Generalized Bounded Slope Condition)**.** Let $K > 0$. We say that the pair $(I, \bar{u})$ satisfies the *Generalized Bounded Slope Condition* $(GBSC)_K$ if $\bar{u} \in \mathrm{Lip}_K(\Omega)$ and for every x_0 in $\partial\Omega$ there exist a sub–solution $\ell^1_{x_0} \in \mathrm{Lip}_K(\Omega)$ and a super–solution $\ell^2_{x_0} \in \mathrm{Lip}_K(\Omega)$ for I in $W^{1,\infty}(\Omega)$ such that

$$\forall x \in \partial\Omega \qquad \ell^1_{x_0}(x) \leq \bar{u}(x) \leq \ell^2_{x_0}(x) \qquad \text{and} \qquad \ell^1_{x_0}(x_0) = \bar{u}(x_0) = \ell^2_{x_0}(x_0).$$

In the case where the functions $\ell^i_{x_0}$ are affine the above definition turns out to be the *Bounded Slope Condition.*

Let us denote by $\mathrm{dist}(x, \partial\Omega)$ the distance from x to $\partial\Omega$. For every $t > 0$ we set

$$\Sigma_t = \{x \in \Omega : \, \mathrm{dist}(x,\partial\Omega) < t\}, \qquad \Gamma_t = \{x \in \Omega : \, \mathrm{dist}(x,\partial\Omega) = t\}$$

and we define the functional I_{Σ_t} by

$$\forall u \in W^{1,1}(\Sigma_t) \qquad I_{\Sigma_t}(u) = \int_{\Sigma_t} L(x,u,\nabla u) \, dx.$$

Definition 3.5 (Barriers)**.** An *upper barrier* for $(I, \bar{u})$ is a super–solution v for the functional I_{Σ_t} in $W^{1,\infty}(\Sigma_t)$ (for some $t > 0$) satisfying

$$v = \bar{u} \text{ on } \partial\Omega \quad \text{and} \quad v \geq \sup_{\partial\Omega} \bar{u} \text{ on } \Gamma_t.$$

Analogously a *lower barrier* for $(I, \bar{u})$ is a sub–solution u for the functional I_{Σ_t} in $W^{1,\infty}(\Sigma_t)$ (for some $t > 0$) satisfying

$$u = \bar{u} \text{ on } \partial\Omega \quad \text{and} \quad u \leq \inf_{\partial\Omega} \bar{u} \text{ on } \Gamma_t.$$

The above definitions generalize the classical notions of *Bounded Slope Condition* and *barriers* that are widely used in partial differential equations to give an estimate of the Lipschitz constant of Lipschitz solutions.

Example 3.6. Let $L(x, z, p) = f(p)$ be elliptic of class $\mathcal{C}^2$ and let Ω be of class $\mathcal{C}^2$. We denote by $\Lambda(p)$ the maximum eigenvalue of the Hessian $\big(f_{p_i p_j}(p)\big)_{i,j}$ of f at p. We set $\mathcal{E}(p) = \sum_{i,j} f_{p_i p_j}(p) p_i p_j$ to be the *Bernstein function.* Then, for every smooth boundary data $\bar{u}$, the pair $(I, \bar{u})$ admits lower and upper barriers if either Ω is convex and $\displaystyle\lim_{|p|\to\infty} \frac{\Lambda(p)}{\mathcal{E}(p)} = 0$ or if $\displaystyle\limsup_{|p|\to\infty} \frac{|p|\Lambda(p)}{\mathcal{E}(p)} < \infty$. Sufficient conditions for the existence of barriers for more general integrands are given in Chapter 14 of [GT].

We give some conditions on $(I, \bar{u})$ that will ensure that the minima of I are constrained.

Assumption ($\mathbf{A}_K$) (Barrier conditions on the boundary datum)**.** Let $K > 0$ and $\bar{u} \in \mathrm{Lip}(\Omega)$. The pair $(I, \bar{u})$ satisfies Assumption (A_K) if one of the following conditions holds:

A_Ki) The pair $(I, \bar{u})$ satisfies the $(GBSC)_K$.

$A_K$ii) The pair $(I, \bar{u})$ admits upper and lower barriers of Lipschitz constant K. Moreover we assume that if (H1) holds, then $g_z(x, \inf_{\partial\Omega} \bar{u}) \le 0$ and $g_z(x, \sup_{\partial\Omega} \bar{u}) \ge 0$ on Ω, whereas if (H2) holds, then there exist $\alpha \in \partial g(\inf_{\partial\Omega} \bar{u})$ and $\beta \in \partial g(\sup_{\partial\Omega} \bar{u})$ such that $\alpha \le 0$ and $\beta \ge 0$.

Theorem 3.7 (Sufficient conditions for the existence of constraints)**.** *Let L satisfy Assumption* (H) *and $Q \ge K > 0$. Let $\bar{u} \in \mathrm{Lip}_K(\Omega)$ and assume that the pair $(I, \bar{u})$ fulfills Assumption (A_K) for some $K > 0$. Then there exist two functions ℓ^1 and ℓ^2 in $\mathrm{Lip}_K(\Omega)$ such that $\ell^1 \le w \le \ell^2$ a.e. on Ω for every minimum w of I in $\mathrm{Lip}_Q(\Omega, \bar{u})$.*

Proof of Theorem 3.7. We first point out that Corollary 2.4 can be applied to every Lagrangian satisfying Assumption (H). If A_Ki) holds, then for every x_0 in $\partial\Omega$ there exists a sub–solution $\ell^1_{x_0}$ and a super–solution $\ell^2_{x_0}$ (both belonging to $\mathrm{Lip}_K(\Omega)$) of I in $W^{1,\infty}(\Omega)$ satisfying $\ell^1_{x_0} \le \bar{u} \le \ell^2_{x_0}$ on $\partial\Omega$ and $\ell^1_{x_0}(x_0) = \bar{u}(x_0) = \ell^2_{x_0}(x_0)$. Corollary 2.4 then implies that $\ell^1_{x_0} \le w \le \ell^2_{x_0}$ on Ω. Therefore if we set $\ell^1(x) = \sup\{\ell^1_{x_0}(x) : x_0 \in \partial\Omega\}$, $\ell^2(x) = \inf\{\ell^2_{x_0}(x) : x_0 \in \partial\Omega\}$ we obtain $\ell^1 \le w \le \ell^2$ on Ω. Clearly ℓ^1 and ℓ^2 are Lipschitz (with Lipschitz constant less than K) and $\ell^1 = \ell^2 = \bar{u}$ on $\partial\Omega$, proving the claim under A_Ki). Assume now that v is an upper barrier for $(I, \bar{u})$ of Lipschitz constant K: let Σ_t and Γ_t be as in Definition 3.5; we set

$$s = \sup_{\partial\Omega} \bar{u}, \qquad \ell^2 = \begin{cases} \min\{v, s\} \text{ on } \Sigma_t; \\ s \text{ on } \Omega \setminus \Sigma_t. \end{cases}$$

Since $v \ge s$ on Γ_t, then ℓ^2 is Lipschitz and $\|\nabla \ell^2\|_{L^\infty(\Omega)} \le \|\nabla v\|_{L^\infty(\Omega)}$. The assumption on the sub–differential of g yields, both under (H1) and (H2), the existence of a positive function $\beta(x)$ satisfying $\beta(x) \in \partial_z g(x, s)$ a.e. More precisely under (H1) we set $\beta(x) = g_z(x, s)$ whereas under (H2) we set $\beta(x) = \beta$. It follows that the constant function equal to s is a super–solution for I in $W^{1,\infty}(\Omega)$. In fact $\partial f(\nabla s) = \partial f(0)$ and for every k in $\partial f(0)$ we have

$$\forall \eta \in W^{1,\infty}_0(\Omega) \qquad \int_\Omega k \cdot \nabla \eta + \beta \eta \, dx = \int_\Omega \beta \eta \, dx \ge 0.$$

Since $w \le \bar{u} \le s$ on $\partial\Omega$, Corollary 2.4 yields $w \le s$ on Ω. By Remark 2.2 we deduce that $w \le s$ on $\partial\Sigma_t$ and thus $w \le s \le v$ on Γ_t. Now $\bar{u} = v = w$ on $\partial\Omega$ so that $w \le v$ on $\partial\Sigma_t$. Moreover the restriction $w_{|\Sigma_t}$ of w to Σ_t is a minimum for I_{Σ_t} in $\mathrm{Lip}_Q(\Sigma_t, w_{|\Sigma_t})$ and v is a super–solution for the functional I_{Σ_t} in $W^{1,\infty}(\Sigma_t)$ of Lipschitz constant $K \le Q$: the Comparison Principle applies again yielding $w \le v$ on Σ_t which, together with $w \le s$ on Ω, gives $w \le \ell^2$ on Ω. A similar argument shows that $w \ge \ell^1$on Ω, where ℓ^1 is the Lipschitz function defined by

$$\ell^1 = \begin{cases} \max\{u, \inf_{\partial\Omega} \bar{u}\} \text{ on } \Sigma_t; \\ \inf_{\partial\Omega} \bar{u} \text{ on } \Omega \setminus \Sigma_t. \end{cases} \qquad \square$$

A first consequence of Theorem 3.2 and Theorem 3.7 is the existence of a minimum for I among Lipschitz functions with a suitable prescribed datum. The case where L does not depends on (x, z) and $\bar{u}$ satisfies the Bounded Slope Condition was established by M. Miranda in [M].

Theorem 3.8 (Existence of minima in $\mathrm{Lip}(\Omega, \bar{u})$)**.** *Let L satisfy Assumption* (H). *Let $\bar{u}$ be Lipschitz and assume that the pair $(I, \bar{u})$ fulfills Assumption (A_K) for some $K > 0$. Then I has a minimum in* $\mathrm{Lip}(\Omega, \bar{u})$*, a Lipschitz constant being* $\max\{M(g), K\}$.

We postpone the proof of this result until after the following simple lemma.

Lemma 3.9. *Let I be convex, $Q > 0$ and $\bar{u} \in \mathrm{Lip}_Q(\Omega)$. Assume that w is a minimum for the functional I in $\mathrm{Lip}_Q(\Omega, \bar{u})$ and that $\|\nabla w\|_{L^\infty(\Omega)} < Q$. Then w is a minimum for I in $\mathrm{Lip}(\Omega, \bar{u})$.*

Proof of Lemma 3.9. Let $u \in \mathrm{Lip}(\Omega, \bar{u})$ and $0 < \varepsilon < 1$ be such that the Lipschitz constant of $w+t(u-w)$ is less than Q for $0 < t < \varepsilon$. Then $I(w) \leq I(w+t(u-w)) \leq tI(u) + (1-t)I(w)$ and thus $I(w) \leq I(u)$. □

Proof of Theorem 3.8. Fix $Q > \max\{M(g), K\}$; by Theorem 3.7 there exist ℓ^1 and ℓ^2 in $\mathrm{Lip}_K(\Omega)$ such that $\ell^1 \leq w \leq \ell^2$ a.e. on Ω for every minimum w of I in $\mathrm{Lip}_Q(\Omega, \bar{u})$. Therefore, if we set $\mathcal{K}_Q = \{u \in \mathrm{Lip}_Q(\Omega, \bar{u}) : \ell^1 \leq u \leq \ell^2\}$, every minimum of I in $\mathrm{Lip}_Q(\Omega, \bar{u})$ belongs to $\mathcal{K}_Q$. It follows that the minima for I in $\mathcal{K}_Q$ are the minima for I in $\mathrm{Lip}_Q(\Omega, \bar{u})$. By Theorem 3.2 there exists a minimum of I in $\mathcal{K}_Q$ whose Lipschitz constant is bounded by $\max\{M(g), K\}$. Lemma 3.9 then shows that it is a minimum of I in $\mathrm{Lip}(\Omega, \bar{u})$. □

We are now in the position to formulate the following result stating the existence, uniqueness and Lipschitz regularity of the minimum of I in a Sobolev space, with no growth assumption on the Lagrangian. We will use a result by Clarke generalizing, in a nonsmooth setting, the well–known fact that the Lipschitz minimima satisfy the Euler equation [GT, §11.5].

Theorem 3.10 (Existence and Lipschitz regularity for minima)**.** *Let L be real valued and satisfy Assumption* (H). *Let $\bar{u} \in \mathrm{Lip}(\Omega)$ and assume that the pair $(I, \bar{u})$ fulfills Assumption (A_K) for some $K > 0$. Then, for every $q \in [1, +\infty]$, the functional I has a unique minimum w in $\bar{u} + W_0^{1,q}(\Omega)$. Moreover w is Lipschitz with Lipschitz constant bounded by* $\max\{M(g), K\}$.

Proof. By Theorem 3.8 the functional I has a minimum w in $\mathrm{Lip}(\Omega, \bar{u})$ with Lipschitz constant less than $\max\{M(g), K\}$. The main Theorem in [Cl] then yields the validity of the Euler equation for w; more precisely there exist two essentially bounded functions k and h satisfying $k(x) \in \partial f(\nabla w(x))$ and $h(x) \in \partial_z g(x, w(x))$ a.e. on Ω such that

$$\forall \eta \in W_0^{1,1}(\Omega) \qquad \int_\Omega k \cdot \nabla \eta + h\eta \, dx = 0.$$

This in particular shows that w is both a sub–solution and a super–solution for I in $W^{1,q}(\Omega)$ for every $1 \leq q \leq \infty$. Let $v \in \bar{u} + W_0^{1,q}(\Omega)$; by convexity we have

$$L(x, v, \nabla v) \geq L(x, w, \nabla w) + k \cdot \nabla(v - w) + h(v - w) \quad \text{a.e. on } \Omega$$

so that, since $v - w \in W_0^{1,1}(\Omega)$, by integration we obtain $I(v) \geq I(w)$ showing that w is a minimum for I in $\bar{u} + W_0^{1,q}(\Omega)$. Let ζ be another minimum for I in $\bar{u} + W_0^{1,q}(\Omega)$: then clearly $w \leq \zeta \leq w$ on $\partial\Omega$. The Comparison Principle for sub/super–solutions (Theorem 2.3) yields $\zeta = w$. □

Example 3.11. As an application of Theorem 3.10 we obtain existence and Lipschitz regularity for the minima of functionals whose Lagrangian L satisfies Assumption (H) together with the structure conditions stated in Example 3.6 or more generally in [GT, Ch. 14].

Example 3.12. Let $\Omega =]0,1[$, $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $L(z,p) = e^p + g(z)$ where g is a smooth Lipschitz convex function. We consider the problem (P) of minimizing the functional $I(u) = \int_0^1 e^{u'} + g(u)\, dx$ among the absolutely continuous functions u such that $u(0) = \alpha$, $u(1) = \beta$. Here the Lagrangian does not satisfy the classical Tonelli's growth conditions for the existence of a solution; it clearly does satisfy Assumption (H2). Assuming that $m \leq g'(z) \leq M$ for every z the application of Theorem 3.10 yields the following result. Let α, β, m, M satisfy one of the following compatibility conditions:

i) $m \leq M \leq 0$ and $\beta - \alpha \leq \log(-m/2) + 1$;
ii) $m < 0 < M$ and $\log(M/2) + 1 \leq \beta - \alpha \leq \log(-m/2) + 1$;
iii) $0 \leq m \leq M$ and $\log(M/2) + 1 \leq \beta - \alpha$.

Then problem (P) has a unique solution which turns out to be Lipschitz.

To prove the claim we set $\bar{u}(0) = \alpha, \bar{u}(1) = \beta$ and we show that the pair $(I, \bar{u})$ satisfies Assumption A_Ki). A sub–solution of the Euler equation is a function u satisfying $-u''e^{u'} + g'(u) \leq 0$: it is therefore enough to look for functions u satisfying $-u''e^{u'} + M \leq 0$. If $M \leq 0$, then every concave function fulfills this requirement: to ensure the validity of Assumption $\mathrm{A}_K i)$ it is enough to choose a concave function coinciding with $\bar{u}$ at 0 and 1. Otherwise, if $M > 0$ we look for functions of the form $u(x) = ax^2 + bx + c$. In this case $-u''e^{u'} + M \leq 0$ if, for instance, $b = \log t$ and $a = M/(2t)$ for some $t > 0$. The conditions $[u(0) = \alpha$ and $u(1) \leq \beta]$ or $[u(0) \leq \alpha$ and $u(1) = \beta]$ are fulfilled if there exists t such that $M/(2t) + \log t = \beta - \alpha$ and this occurs if $\beta - \alpha$ is greater than $\log(M/2) + 1$, the minimum of $M/(2t) + \log t$. Similarly, a super–solution v of $-u''e^{u'} + g'(u) = 0$ satisfying $[v(0) = \alpha$ and $v(1) \geq \beta]$ or $[v(0) \geq \alpha$ and $v(1) = \beta]$ exists if either $m \geq 0$ or $\beta - \alpha \leq \log(-m/2) + 1$. We note that the Lipschitz regularity of the solution is also a consequence of the results stated in [CV].

Remark 3.13. Theorem 3.10 extends Theorem 5.8 of [MT1] where we established that just the regularity part of the claim under the assumption that L is regular satisfies (H) and $(I, \bar{u})$ fulfills $(\mathrm{A}_K\mathrm{i})$.

4. On a barrier condition involving sub/super–minima

In this section we obtain an existence and regularity result similar to Theorem 3.10 in a slightly different setting. More precisely we relax Assumption (A_K) by using the notion of sub/super–minima instead of that of sub/super–solution and we strengthen Assumption (H) by imposing the strict convexity of the functional I.

Definition 4.1. A convex subset X of $W^{1,1}(\Omega)$ is said to be a *convex sublattice* if

$$\forall u, v \in X \quad \max\{u, v\} \in X, \quad \min\{u, v\} \in X.$$

Definition 4.2. Let X be a convex sublattice of $W^{1,1}(\Omega)$. A function u in $W^{1,1}(\Omega)$ is said to be a *sub–minimum* (resp. *super–minimum*) for I in X if u belongs to X, $I(u)$ is finite, and $I(u) \leq I(v)$ for every v in $X \cap (u + W_0^{1,1}(\Omega))$ such that $v \leq u$

(resp. $v \geq u$) a.e. on Ω. Moreover the function u is a *minimum* for I in X whenever $I(u) \leq I(v)$ for every v in $X \cap (u + W_0^{1,1}(\Omega))$.

Remark 4.3. In [MT3] we showed the notion of sub/super–minima generalizes that of sub/super–solution in the sense that every sub–solution (resp. super–solution) to the Euler equation associated to I in $W^{1,q}(\Omega)$ is a sub–minimum (resp. superminimum) for I in $W^{1,q}(\Omega)$. The converse is not true since there are integral functionals whose minima are not solutions of the Euler equation [BM].

Assumption (H′) (Structure conditions on $L(x, z, p)$)**.** $L(x, z, p) = f(p) + g(x, z)$ is convex in (z, p) and one of the following conditions holds:

(H1) As stated in Assumption (H).

(H′2) The function g does not depend on x and either g or f is strictly convex.

Assumption ($\mathbf{A}'_K$). Let $K > 0$ and $\bar{u} \in \mathrm{Lip}(\Omega)$. We say that the pair $(I, \bar{u})$ satisfies Assumption (A'_K) if it fulfills Assumption (A_K) where by $(GBSC)_K$ or *barriers* we mean the conditions stated in Definition 3.4 and Definition 3.5 in which the words *sub–solutions* (resp. *super–solutions*) are replaced by *sub–minima* (resp. *super–minima*).

Remark 4.4. It is clear that Assumption (H′) implies (H) and (A_K) implies (A'_K).

We need here a Comparison Principle that does not involve the Euler equation at all; we give here a version of Theorem 4.1 in [MT3] whose hypotheses are fulfilled under Assumption (H′).

Theorem 4.5 (Comparison Principle for sub/super–minima)**.** *Let X be a convex sublattice of $W^{1,1}(\Omega)$ and let $L(x, z, p) = f(p) + g(x, z)$ be convex in (z, p) where either f or the map $z \mapsto g(x, z)$ is strictly convex for almost every x in Ω. Let u be a sub–minimum and v a super–minimum for I in X such that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ a.e. on Ω.*

If we follow the steps of section 3 using Theorem 4.5 for $X = \mathrm{Lip}_Q(\Omega, \bar{u})$ instead of Corollary 2.4 and for $X = W^{1,q}(\Omega)$ instead of Theorem 2.3 we obtain the following result.

Theorem 4.6 (Existence and Lipschitz regularity for minima)**.** *Let L be real valued and satisfy Assumption* (H′)*. Let $\bar{u} \in \mathrm{Lip}(\Omega)$ and assume that the pair $(I, \bar{u})$ fulfills Assumption (A'_K) for some $K > 0$. Then, for every $q \in [1, +\infty]$, the functional I has a unique minimum w in $\bar{u} + W_0^{1,q}(\Omega)$. Moreover w is Lipschitz with Lipschitz constant bounded by* $\max\{M(g), K\}$.

References

[AAB] L. Ambrosio, O. Ascenzi, G. Buttazzo, *Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands*, J. Math. Anal. Appl. **142** (1989), 301–316. MR **91c:**49060

[BM] J.M. Ball, V.J. Mizel, *One-dimensional variational problems whose minimizers do not satisfy the Euler–Lagrange equation*, Arch. Rational Mech. Anal. **90** (1985), 325–388. MR **86k:**49002

[C] A. Cellina, *On the Bounded Slope Condition and the validity of the Euler Lagrange equation*, SIAM J. Control Optim. (2001) (to appear).

[Cl] F.H. Clarke, *Multiple integrals of Lipschitz functions in the calculus of variations*, Proc. Amer. Math. Soc. **64** (1977), 260–264. MR **56:**9443

[CV] F. H. Clarke, R. B. Vinter, *Regularity properties of solutions to the basic problem in the calculus of variations*, Trans. Amer. Math. Soc. **289** (1985), 73–98. MR **86h:**49020

[ET] I. Ekeland, R. Temam, *Convex analysis and variational problems*, vol. I, North–Holland Publishing Company, Amsterdam–Oxford; American Elsevier Publishing Co., Inc., New York, 1976. MR **57:**3931b

[E] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, 1998. MR **99e:**35001

[GG] M. Giaquinta, E. Giusti, *Differentiability of minima of nondifferentiable functionals*, Invent. Math. **72** (1983), 285–298. MR **84f:**58037

[GT] D.Gilbarg, N.S. Trudinger, *Elliptic Partial Differential equations of second order*, Third Edition; Grundenlehren der Matematischen Wissenschaften, vol. 224, Springer–Verlag, Berlin– Heidelberg–New York, 1998. MR **86c:**35035 (review of 2nd edition)

[G] E. Giusti, *Metodi diretti nel Calcolo delle Variazioni*, U.M.I., Bologna, 1994. MR **2000f:**49001

[MT1] C. Mariconda, G. Treu, *A Comparison Principle and the Lipschitz continuity for minimizers*, preprint, Dipartimento di Matematica pura e applicata **27** (1999).

[MT2] C. Mariconda, G. Treu, *A Comparison Principle for minimizers*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 681–686. CMP 2000:14

[MT3] C. Mariconda, G. Treu, *A gradient Maximum Principle for minima*, preprint, Dipartimento di Matematica pura e applicata (2000).

[M] M. Miranda, *Un teorema di esistenza e unicità per il problema dell'area minima in n variabili*, Ann. Sc. Normale di Pisa **19** (1965), 233–249. MR **31:**6144

[R] R.T. Rockafeller, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[TV] G. Treu, M. Vornicescu, *On the equivalence of two variational problems*, Calculus of Variations and Partial Differential Equations **11** (2000), 307–319. MR **2001h:**49007

Dipartimento di Matematica pura e applicata, Università di Padova, 7 via Belzoni, I-35131 Padova, Italy

E-mail address: `maricond@math.unipd.it`

Dipartimento di Matematica pura e applicata, Università di Padova, 7 via Belzoni, I-35131 Padova, Italy

E-mail address: `treu@math.unipd.it`