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UNIVERSAL PERTURBATIONS OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $X : [0, \infty) \to L(\mathbb{R}^n)$ be a fundamental solution of $x' = A(t)x$ with X and X^{-1} bounded on $[0, \infty)$. We prove that there exist arbitrary small matrix functions $B : [0, \infty) \to L(\mathbb{R}^n)$ with limit 0 as $t \to \infty$ such that $y' = (A(t) + B(t))y$ has solutions with $y([0, \infty))$ dense in $\mathbb{R}^n$.

1. INTRODUCTION

Let $\mathbb{R}^n$ be endowed with a norm $||\cdot||$, and let $L(\mathbb{R}^n)$ be the space of all linear endomorphisms endowed with the corresponding operator norm $|||\cdot|||$. For a given function $A \in C([0, \infty), L(\mathbb{R}^n))$ we fix a fundamental system $X : [0, \infty) \to L(\mathbb{R}^n)$ for $x'(t) = A(t)x(t)$. We assume that A is such that X and X^{-1} are bounded on $[0, \infty)$, that is, $|||X(t)||| \leq \alpha$, $|||X^{-1}(t)||| \leq \alpha$ $(t \geq 0)$ for some $\alpha > 0$. Note that a fundamental system (or its inverse) is bounded if and only if each fundamental system has this property.

The following perturbation theorem is well known; see for example [2], pp. 98-99.

Theorem 1. *Let A be as above, and let $B \in C([0, \infty), L(\mathbb{R}^n))$ be such that*

$$\int_0^\infty |||B(t)||| \, dt < \infty.$$

If $Y : [0, \infty) \to L(\mathbb{R}^n)$ is a fundamental system for $y'(t) = (A(t) + B(t))y(t)$, then Y is bounded on $[0, \infty)$.

Remark. The boundedness of X together with the conditions $|||B(t)||| \to 0$ $(t \to \infty)$, $\int_0^\infty |||B(t)|||dt < \infty$ is not sufficient to ensure the boundedness of solutions of $y'(t) = (A(t) + B(t))y(t)$; see for example [1], 1.135.

Now let F denote the Banach space of all $B \in C([0, \infty), L(\mathbb{R}^n))$ with $B(t) \to 0$ $(t \to \infty)$ endowed with the maximum norm

$$|||B|||_\infty = \max\{|||B(t)||| : t \geq 0\}.$$

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We will prove

Theorem 2. *Let $n \geq 2$ and let A be as above. Let $y_0 \in \mathbb{R}^n$, $y_0 \neq 0$ and let $(t_k)_{k=1}^{\infty}$ be a strictly increasing sequence in $[0, \infty)$ with $t_k \to \infty$ $(k \to \infty)$. Let U denote the set of all $B \in F$ with the following property: The solution $y : [0, \infty) \to \mathbb{R}^n$ of*

$$y'(t) = (A(t) + B(t))y(t), \qquad y(0) = y_0,$$

satisfies

$$\overline{\{y(t_k) : k \in \mathbb{N}\}} = \mathbb{R}^n.$$

Then U is a dense G_δ-subset of F.

Remarks. 1) In particular, given $\varepsilon > 0$ there exist $B \in U$ such that $|||B|||_\infty < \varepsilon$. The perturbation of A with such a function B turns the stable system $x'(t) = A(t)x(t)$ into an unstable system. For a related destabilization result for Hamiltonian linear systems see [5].

2) Of course, our result cannot hold for $n = 1$ since in this case each solution $y : [0, \infty) \to \mathbb{R}$ has constant sign.

3) We do not know if Theorem 2 holds for infinite dimensional Banach spaces E (instead of $\mathbb{R}^n$). It is easy to see that the proof of Theorem 2 works for an infinite dimensional separable Banach space if it is assumed, in addition, that $(X(t_k))_{k=1}^{\infty}$ has a subsequence which is convergent in $(L(E), |||\cdot|||)$. This is the case, for example, if $A(t) = 0$ $(t \geq 0)$. Hence, in this case there are functions $B \in F$ such that for a solution y of $y'(t) = B(t)y(t)$ the sequence $(y(t_k))_{k=1}^{\infty}$ is dense in E. In the infinite dimensional case $(y(t_k))_{k=1}^{\infty}$, for a solution y of $y'(t) = B_0 y(t)$, can be dense in E (for certain $B_0 \in L(E)$). The case that $\exp(B_0)$ is a so-called hypercyclic operator refers to $(t_k)_{k=1}^{\infty} = (k)_{k=1}^{\infty}$; see for example [4]. In the finite dimensional case a solution y of $y'(t) = B_0 y(t)$ $(B_0 \in L(\mathbb{R}^n))$ never has the property that $y([0, \infty))$ is dense in $\mathbb{R}^n$. Here $||y(t)||$ either tends to 0 or tends to ∞ or is bounded and bounded away from 0.

4) In the proof of Theorem 2 only bounded solutions of linear systems are used, although finally unbounded solutions are produced. This is typical if Baire's Theorem is involved and illustrates its power for handling "explosions".

2. Proof of Theorem 2

First note that the space F_0 of all $B \in F$ with $B(t) = 0$ $(t \geq T)$ for some $T > 0$ is dense in F. We consider the continuous operators $L_k : F \to \mathbb{R}^n$ $(k \in \mathbb{N})$ defined in the following way. Let $y : [0, \infty) \to \mathbb{R}^n$ be the solution of the initial value problem

$$y'(t) = (A(t) + B(t))y(t), \qquad y(0) = y_0.$$

We set $L_k(B) = y(t_k)$. Since F is a Baire space and since $\mathbb{R}^n$ is separable, we are done, according to a result of Grosse-Erdmann [3], Theorem 1, if we can show the following condition:

To each $B \in F_0$, $u \in \mathbb{R}^n$ and $\varepsilon > 0$ there exist $C \in F$ and $k_0 \in \mathbb{N}$ such that

$$|||B - C|||_\infty < \varepsilon \text{ and } ||L_{k_0}(C) - u|| < \varepsilon.$$

We fix $B \in F_0$, $u \in \mathbb{R}^n$ and $\varepsilon > 0$. Let $Y : [0, \infty) \to \mathbb{R}^n$ be the fundamental system of $y'(t) = (A(t) + B(t))y(t)$ with $Y(0) = I := id_{\mathbb{R}^n}$. Let $T > 0$ be such that $|||B(t)||| = 0$ $(t \geq T)$. Since X is bounded, the sequence $(X(t_k))_{k=1}^{\infty}$ has a convergent subsequence, and we assume without loss of generality that $(X(t_k))_{k=1}^{\infty}$

is convergent, to $R \in L(\mathbb{R}^n)$, say. Since X^{-1} is bounded, R is invertible and $X^{-1}(t_k) \to R^{-1}$ $(k \to \infty)$.

The solution of our initial value problem is $y(t) = Y(t)y_0$ $(t \geq 0)$, and we set $v = y(T)$. Since $n \geq 2$ and $v \neq 0$, we can choose $w \in \mathbb{R}^n$ such that $||w - u|| < \varepsilon/2$, and such that $\xi := X^{-1}(T)v$ and $\eta := R^{-1}w$ are linear independent. Hence, there exists a linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(\xi) = \varphi(\eta) = 1$. We define $Q \in L(\mathbb{R}^n)$ by $Qx = \varphi(x)(\eta - \xi)$. Then $Q^2x = \varphi(x)\varphi(\eta - \xi)(\eta - \xi) = 0$; hence $\exp(Q) = I + Q$ and we have

$$\exp(Q)\xi = (I + Q)\xi = \xi + \varphi(\xi)(\eta - \xi) = \eta.$$

Obviously there exists a continuous function $g : [T, \infty) \to [0, \infty)$ with the following properties:

1. $g(T) = 0$ and $\lim_{t\to\infty} g(t) = 0$.
2. $\int_T^\infty g(t)dt = 1$.
3. $|g(t)| \leq \varepsilon/(2\alpha^2|||Q|||)$ $(t \geq T)$.

We set

$$C(t) = B(t) \quad (t \in [0, T]), \quad C(t) = g(t)X(t)QX^{-1}(t) \quad (t > T).$$

Then $C \in F$ and

$$|||B - C|||_\infty \leq (\sup_{t\geq T} |g(t)|)\alpha^2|||Q||| < \varepsilon.$$

Let $Z : [0, \infty) \to L(\mathbb{R}^n)$ be the fundamental system of $z'(t) = (A(t) + C(t))z(t)$ with $Z(0) = I$. We have $Z(t) = Y(t)$ for $0 \leq t \leq T$, hence $Z(T)y_0 = v$. Moreover,

$$Z(t)Z^{-1}(T)X(T) = X(t) \exp\left(\int_T^t g(s)ds\ Q\right) \quad (t \geq T),$$

as can be verified by differentiation. For $t_k \geq T$ we have

$$Z(t_k)y_0 = X(t_k) \exp\left(\int_T^{t_k} g(s)ds\ Q\right) X^{-1}(T)Z(T)y_0$$

$$= X(t_k) \exp\left(\int_T^{t_k} g(s)ds\ Q\right) \xi \to R \exp(Q)\xi = R\eta = w$$

as $k \to \infty$. Thus, there exists $k_0 \in \mathbb{N}$ such that $||Z(t_{k_0})y_0 - w|| < \varepsilon/2$. Then

$$||L_{k_0}(C) - u|| = ||Z(t_{k_0})y_0 - u|| \leq ||Z(t_{k_0})y_0 - w|| + ||w - u|| < \varepsilon. \quad \square$$

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