

## ON THE TOTAL CURVATURE OF CONVEX HYPERSURFACES IN HYPERBOLIC SPACES

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ABSTRACT. Let  $C_1 \subset C_2 \subset H^n$  be two convex compact subsets of the hyperbolic space  $H^n$  with smooth boundary. It is shown that the total curvature of the hypersurface  $\partial C_2$  is larger than the total curvature of  $\partial C_1$ .

### 0. INTRODUCTION

Let  $M^n$  be an  $n$ -dimensional Riemannian manifold and let  $F$  be an  $(n - 1)$ -dimensional smooth immersed hypersurface. Denote by  $A_q : T_q F \rightarrow T_q F$  the shape operator of  $F$  at  $q \in F$  with respect to a normal field defined in a neighborhood of  $q$  and set  $K(q) = \det A_q$ . This is well defined up to sign. When  $M$  is the Euclidean space, it is called the Gauss-Kronecker curvature. We adopt the same name for  $K$  in general although it is no longer an intrinsic quantity of the hypersurface.

Let  $H^n$  denote the hyperbolic space and let  $C_0 \subset C_1$  be two convex compact subsets with smooth boundaries. The goal of the paper is to show:

**Theorem 1.** *With the notations introduced above we have*

$$\text{Vol}(S^{n-1}) \leq \int_{\partial C_0} K \leq \int_{\partial C_1} K.$$

Here,  $K$  is computed with respect to the outward normal field of  $\partial C_i$ ,  $i = 0, 1$ , and  $\text{Vol}(S^{n-1})$  denotes the Euclidean  $(n - 1)$ -dimensional volume of the unit sphere  $S^{n-1}$ .

It is well known that if  $C$  is a convex compact subset of the Euclidean space, then

$$\int_F K = \text{Vol}(S^{n-1}).$$

In a general Hadamard manifold  $M^n$ , as a result of the Gauss-Bonnet theorem, we have for  $n = 2$

$$(1) \quad \text{Vol}(S^{n-1}) \leq \int_{\partial C_0} K \leq \int_{\partial C_1} K$$

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and for  $n = 3$

$$(2) \quad \text{Vol}(S^{n-1}) \leq \int_{\partial C_0} K.$$

It seems natural to wonder to what extent the above statements will hold in higher dimensions. Some partial results with respect to (2) were obtained in [1].

Although we have precise results about certain integrals on hypersurfaces due to Chern (the curvature integra [2]), the generalized Gauss-Bonnet-Chern theorem does not seem to help in higher dimension (at least not in an obvious way).

There is another motivation for trying to show that (2) is satisfied for a general nonpositively curved manifold. This is the so-called isoperimetric conjecture (see [3], [4]).

**Isoperimetric Conjecture.** *Let  $M^n$  be a Hadamard manifold and  $D \subset M^n$  be a compact domain with smooth boundary. Then it satisfies the Euclidean isoperimetric inequality:*

$$\text{area}(\partial D) \geq d_n(\text{vol}(D))^{\frac{n-1}{n}},$$

where  $d_n = \text{area}(S^{n-1})/(\text{vol}(B^n))^{\frac{n-1}{n}}$ .

This is now settled in dimension 4 by [3] and in dimension 3 by [4]. In fact, the main part of the proof in [4] is to show how (2) implies the isoperimetric inequality. Although it was carried out in dimension 3 only, it is very likely (and is explicitly mentioned in [4]) that it generalizes to higher dimensions. This means that a possible way of proving the isoperimetric conjecture is to establish (2) for a general Hadamard manifold.

### 1. CONSTRUCTION OF A DIFFERENTIAL FORM

This is a general construction due to Chern [1] which works on any Riemannian manifold  $M^n$ . Our notation follows the notation of the original paper.

Let  $e_n$  be a unit normal field defined on some open subset of  $M^n$ . At each point extend this to an orthonormal frame  $e_1, \dots, e_n$  such that  $e_i$  is a smooth vector field for  $i = 1, \dots, n$ . At least locally it is certainly possible. We now define the connection forms as

$$\omega_j^i(X) = \langle \nabla_X e_j, e_i \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the metric on  $M^n$  and  $X$  is a vector field. The curvature form is defined as

$$(3) \quad \Omega_j^i(X, Y) = -\langle R(X, Y)e_j, e_i \rangle,$$

where  $R(X, Y)$  denotes the curvature tensor defined as:  $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]}Z$ . Then Cartan's second structural equation reads as

$$(4) \quad \Omega_j^i = d\omega_j^i + \omega_k^i \omega_j^k,$$

where we use the usual summation convention, summing over repeated indices.

The differential form which is of interest to us is defined as:

$$\Phi = \omega_n^1 \omega_n^2 \dots \omega_n^{n-1}.$$

This is an  $(n-1)$ -form on  $M^n$  which is essentially the same as Chern's form denoted by  $\Phi_0$  in [2]. More precisely,  $\Phi_0$  is an  $(n-1)$ -form on the unit tangent bundle and

$\Phi$  is its pull back via the map  $E_n : M^n \rightarrow TM^n$  defined as  $E_n(p) = e_n(p)$ . As a consequence we have the following important observation:

**Fact.** *The differential form  $\Phi$  depends only on the vector field  $e_n$ . It does not depend on how  $e_n$  is extended to an orthonormal frame  $e_1, \dots, e_n$ .*

From the second structural equation (4) one can derive that

$$(5) \quad d\Phi = \sum \epsilon_{i_1 \dots i_{n-1}} \Omega_n^{i_1} \omega_n^{i_2} \dots \omega_n^{i_{n-1}},$$

where  $\epsilon_{i_1 \dots i_{n-1}}$  is the Kronecker index which is equal to  $+1$  or  $-1$  according to whether the permutation  $i_1 \dots i_{n-1}$  of the numbers  $1, 2, \dots, n-1$  is even or odd and the summation is extended over all the indices  $i_1 \dots i_{n-1}$  subject to the condition  $i_2 < i_3 < \dots < i_{n-1}$ . It is essentially the same as the  $n$ -form  $\Psi_0$  in [2].

## 2. CONVEX EXHAUSTION

The other important ingredient is a lemma about convex exhaustion. Although we only need this in the hyperbolic space we state it for Hadamard manifolds. We say that a convex set  $C$  with smooth boundary is strictly convex if the second fundamental form of the boundary (with respect to the outward normal) is positive definite everywhere.

**Lemma 1.** *Let  $M^n$  be a Hadamard manifold and  $C_0 \subset \text{int}(C_1)$  be two compact strictly convex subsets with smooth boundary. Then there exists a continuous function  $F : C_1 - \text{int}(C_0) \rightarrow [0, 1]$  which is smooth in the interior such that:  $\nabla F \neq 0$ ,  $\partial C_i = \{p \in C_2 - \text{int}(C_1) : F(p) = i\}$  for  $i = 0, 1$  and the sublevel sets  $C_b = \{p \in C_2 - \text{int}(C_1) : F(p) \leq b\}$  for  $0 \leq b \leq 1$  are convex.*

*Proof.* The statement is clear intuitively. Denote by  $\varrho_i$  the distance function from the set  $\partial C_i$  for  $i = 0, 1$ .

For  $\delta > 0$  we set  $N_{2\delta} = \{p \in C_1 : \varrho_1(p) < 2\delta\}$ . Since  $\partial C_1$  has a positive definite second fundamental form, we can choose  $\delta > 0$  small enough such that:  $3\delta < \text{dist}(C_0, \partial C_1)$ , the function  $\varrho_1$  is smooth on  $N_{2\delta}$  (there are no focal points of  $\partial C_1$  inside  $N_{2\delta}$ ) and

$$(6) \quad -D^2\varrho_1 > c_1 > 0,$$

when restricted to  $\nabla\varrho_1^\perp$  for some positive constant  $c_1$ . Here,  $D^2\varrho_1$  denotes the Hessian and we adapted the notation that  $-D^2\varrho_1 > c_1$  on  $\nabla\varrho_1^\perp$  if  $-D^2\varrho_1(X, X) > c_1$  for every unit tangent vector  $X \in \nabla\varrho_1^\perp$ . We observe also that on  $N_{2\delta}$  the following inequality holds for the angle between the gradients:

$$(7) \quad \angle(\nabla\varrho_0, -\nabla\varrho_1) < \pi/2 - \alpha,$$

for some  $\alpha > 0$  depending on  $\delta$ .

We are going to construct  $F$  in the form

$$(8) \quad F = 1 - e^{-a\varrho_0} f^\epsilon.$$

Here  $f = h(\varrho_1)$  is the reparametrized distance function from  $\partial C_1$  and  $h$  is a fixed smooth increasing real function  $h : [0, \infty) \rightarrow [0, 1]$  such that  $h(t) = \frac{1}{2\delta}t$  for  $0 \leq t \leq \delta$  and  $h(t) = 1$  on  $[2\delta, \infty)$ . For the derivative and the Hessian of  $f$  we have

$$df = h'd\varrho_1 \quad \text{and} \quad D^2f = h''d\varrho_1 \otimes \varrho_1 + h'D^2\varrho_1.$$

The choice of  $a, \epsilon$  will be discussed later.

First, we show that the derivative of  $F$  is never zero. From the definition we obtain

$$(9) \quad dF = e^{-a\varrho_0} f^\epsilon (ad\varrho_0 - \frac{\epsilon}{f}df).$$

Then, taking into consideration that  $\nabla f \parallel \nabla\varrho_1$ , the statement follows from (7).

To show that the sublevel sets are convex we need to show that the Hessian  $D^2F$  is positive definite on  $\nabla F^\perp$ . We have

$$(10) \quad \begin{aligned} D^2F = & e^{-a\varrho_0} (-a^2d\varrho_0 \otimes d\varrho_0 + aD^2\varrho_0) f^\epsilon \\ & + \epsilon e^{-a\varrho_0} f^{\epsilon-1} (\frac{1-\epsilon}{f}df \otimes df - D^2f) \\ & + \epsilon a e^{-a\varrho_0} f^{\epsilon-1} (d\varrho_0 \otimes df + df \otimes d\varrho_0). \end{aligned}$$

The argument depends on certain estimates of the Hessians  $D^2\varrho_i$  for  $i = 0, 1$ , on various subspaces. It will be useful to keep in mind that  $\varrho_i$  (for  $i = 0, 1$ ) are distance functions; therefore the gradients  $\nabla\varrho_i$  are eigenvectors of  $D^2\varrho_i$  with eigenvalue zero.

First, we consider the region where  $f \equiv 1$ . On this region  $df, D^2f = 0$  and  $\nabla F \parallel \nabla d\varrho_0$ . Therefore  $D^2F = a f^\epsilon e^{-a\varrho_0} D^2\varrho_0$ , when it is restricted to vectors in  $\nabla F^\perp$ . But on this subspace  $D^2\varrho_0$  is positive definite since it is the distance function from a convex set in a Hadamard manifold.

Next, we consider the region where  $1/2 < f < 1$ . Since  $C_0$  is strictly convex and compact, we know that  $D^2\varrho_0 > c_2 > 0$  for some  $c_2 > 0$ , when restricted to  $(\nabla\varrho_0)^\perp$ . Therefore, taking (6) and (9) into account, we conclude that  $D^2\varrho_0 > c_3 > 0$  for some positive constant  $c_3$ , when restricted to  $\nabla F^\perp$ . So, for a small enough  $a$ , where the choice of  $a$  depends only on  $c_3$ , the term  $aD^2\varrho_0$  will dominate  $a^2d\varrho_0 \otimes d\varrho_0$ . This will remain true throughout the whole region  $C_1 - C_0$ . The derivatives of  $h$  are bounded and so are the terms  $df$  and  $D^2f$ . Therefore, for a small enough  $\epsilon$ , the term  $aD^2\varrho_0$  will dominate all the other terms as well. The choice of  $\epsilon$  depends on  $a$  and on the bounds for  $df$  and  $D^2f$ .

At last, we consider the region where  $0 < f < 1/2$ . It is clear from the definition of the function  $f$  that this region is a subset of  $N_{2\delta}$ . As before,  $aD^2\varrho_0$  will dominate  $a^2d\varrho_0 \otimes d\varrho_0$ . The term involving  $df \otimes df$  is positive semi-definite. As for the rest of the terms, we will show that the term involving  $-D^2f$  will dominate, for a small enough  $a$ , the term involving  $d\varrho_0 \otimes df + df \otimes d\varrho_0$ . Since  $h$  is linear on this region, from (6) we obtain  $-D^2f = -h'D^2\varrho_1 > c_1/2\delta > 0$ , when restricted to  $(\nabla\varrho_1)^\perp$ . Taking into consideration (7) and (9) we conclude that  $-D^2f > c_4 > 0$  for some sufficiently small  $c_4 > 0$ , when restricted to  $\nabla F^\perp$ . The constant  $c_4$  depends on  $c_1, \delta$  and the angle  $\alpha$ . So, if the constant  $a > 0$  was chosen small enough, where the choice of  $a$  depended only on  $c_2$  and  $c_4$ , then  $-D^2f$  dominates the term involving  $d\varrho_0 \otimes df + df \otimes d\varrho_0$ . This concludes the proof of the lemma.  $\square$

### 3. PROOF OF THEOREM 1

With the preparation done in the previous sections, the proof of the theorem is simple. Let us return to the hyperbolic space  $H^n$ .

First, we prove the inequality between the two integrals. Assume that  $C_0 \subset \text{int}(C_1)$  and both sets are strictly convex. We are going to show that

$$(11) \quad \int_{\partial C_0} K \leq \int_{\partial C_1} K.$$

The general case will follow by a trivial limiting procedure.

Let  $F$  be the smooth function of Lemma 1 and define the unit vector field  $e_n$  by  $e_n = \nabla F/|\nabla F|$ . This is defined on  $\text{int}(C_1) - C_0$  but it extends continuously to the boundary. To the vector field  $e_n$  we construct the form  $\Phi$  as in the previous section and by Stokes's theorem we have

$$(12) \quad \int_{\partial C_0 \cup \partial C_1} \Phi = \int_{C_1 - C_0} d\Phi.$$

To evaluate the integrals in (12) we are going to compute the forms  $\Phi$  and  $d\Phi$ . Let  $q \in C_1 - \text{int}(C_0)$  be an arbitrarily chosen point.

Since  $\Phi$  depends only on the vector field  $e_n$ , we can express  $\Phi$  in a special frame. Let us choose the frame  $e_1, \dots, e_n$  such that at the point  $q \in C_1 - \text{int}(C_0)$  the vectors  $e_1, \dots, e_{n-1}$  are the principal directions for the hypersurface  $\{F = F(q)\}$ . For the other points of the hypersurface the vectors  $e_1, \dots, e_{n-1}$  may no longer be principal directions. Then, from the definition of the  $\omega_n^i$ 's we have

$$(13) \quad \omega_n^i(e_j) = \delta_j^i \lambda_j, \quad \text{for } 1 \leq i, j \leq n - 1$$

at  $q \in C_1 - \text{int}(C_0)$ , where  $\lambda_j$  denotes the principal curvature at  $q$  of the hypersurface  $\{F = F(q)\}$  in the direction of  $e_j$  and  $\delta_j^i$  is the Kronecker symbol. Therefore

$$\Phi(e_1, \dots, e_{n-1}) = \lambda_1 \cdot \dots \cdot \lambda_{n-1} = K$$

at  $q \in C_1 - \text{int}(C_0)$ , where  $K$  denotes the Gauss-Kronecker curvature of the hypersurface  $\{F = F(q)\}$  with respect to the normal field  $e_n$ . Since  $q \in C_1 - \text{int}(C_0)$  was chosen arbitrarily, the left-hand side of (12) reads as follows:

$$\int_{\partial C_0 \cup \partial C_1} \Phi = \epsilon_{n12\dots n-1} \left( \int_{\partial C_1} K - \int_{\partial C_0} K \right) = (-1)^{n-1} \left( \int_{\partial C_1} K - \int_{\partial C_0} K \right).$$

The curvature tensor has the form  $R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$ ; therefore

$$\Omega_n^i(e_j, e_k) = 0$$

if  $\{j, k\} \neq \{i, n\}$  as sets. From this and (13) we have

$$\Omega_n^{i_1} \omega_n^{i_2} \dots \omega_n^{i_{n-1}}(e_1, \dots, e_n) = \epsilon_{i_1 n i_2 \dots i_{n-1}} K_{i_1 n} \lambda_{i_2} \cdot \dots \cdot \lambda_{i_{n-1}}$$

at  $q \in C_1 - \text{int}(C_0)$ , where the indices satisfy the condition  $i_2 < i_3 < \dots < i_{n-1}$  and  $K_{in} \equiv -1$  denotes the sectional curvature of the two-plane determined by  $e_i, e_n$  and  $\lambda_i$  is the principal curvature at the point  $q \in C_1 - \text{int}(C_0)$  in the direction of  $e_i$ . Then (12) reads as follows:

$$\int_{\partial C_1} K - \int_{\partial C_0} K = \int_{C_1 - C_0} \sum -K_{i_1 n} \lambda_{i_2} \cdot \dots \cdot \lambda_{i_{n-1}},$$

where the summation is extended over all the indices  $i_1 \dots i_{n-1}$  subject to the condition  $i_2 < i_3 < \dots < i_{n-1}$ .

Since all sublevel sets are convex, all the principal curvatures are positive. Therefore, the integral on the right-hand side is positive. This completes the proof of the theorem, when  $C_0 \subset \text{int}(C_1)$  and both sets are strictly convex. The general case follows by slightly "blowing up" the sets; that is, instead of  $C_0$  we consider an  $\eta$ -neighborhood  $C_\eta$  and instead of  $C_1$  we take a  $2\eta$ -neighborhood  $C_{1+2\eta}$ . These are now strictly convex sets satisfying the conditions set forth at the beginning of the proof. Then letting  $\eta$  go to 0 will yield the general case.

All that remains is to prove the inequality

$$\text{Vol}(S^{n-1}) \leq \int_{\partial C_0} K.$$

This is a simple consequence of (11). Choose a ball  $B_\epsilon$  inside  $C_0$ . Applying (11), we obtain

$$\int_{\partial B_\epsilon} K < \int_{\partial C_0} K.$$

Letting  $\epsilon$  go to 0 will yield the desired inequality. This completes the proof of the theorem.

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