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a-WEYL'S THEOREM FOR OPERATOR MATRICES

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ABSTRACT. If $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a 2×2 upper triangular matrix on the Hilbert space $H \oplus K$, then a-Weyl's theorem for A and B need not imply a-Weyl's theorem for M_C, even when $C = 0$. In this note we explore how a-Weyl's theorem and a-Browder's theorem survive for 2×2 operator matrices on the Hilbert space.

1. INTRODUCTION

Throughout this note let H and K be Hilbert spaces, let $B(H, K)$ denote the set of bounded linear operators from H to K, and abbreviate $B(H, H)$ to $B(H)$ and let $K(H)$ denote the ideal of compact operators acting on H. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of T; $\alpha(T) = \dim N(T)$. An operator $T \in B(H)$ is called upper semi-Fredholm if $R(T)$ is closed with finite dimensional null space and lower semi-Fredholm if $R(T)$ is closed with its range of finite codimension. If T is both upper semi- and lower semi-Fredholm, we call it *Fredholm*. The *index* of a Fredholm operator $T \in B(H)$ is the integer $i(T) = \alpha(T) - \alpha(T^*)$. An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero and is called *Browder* if it is Fredholm of "finite ascent and descent". If $T \in B(H)$ write $\sigma(T)$ for the spectrum of T; $\sigma_a(T)$ for the approximate point spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity; $\pi_{00}^a(T)$ for the isolated points of $\sigma_a(T)$ which are eigenvalues of finite multiplicity. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined by ([10], [11])

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},$$
$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\};$$

evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

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where we write $\operatorname{acc} X$ for the accumulation points of $X \subseteq \mathbb{C}$. We say that *Weyl's theorem holds for* $T \in B(H)$ if there is equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T). \tag{1.1}$$

This obviously implies ([12]) that *Browder's theorem holds for* T:

$$\sigma(T) = \omega(T) \cup \pi_{00}(T). \tag{1.2}$$

By definition, $\sigma_{ea}(T) = \bigcap\{\sigma_a(T+K) : K \in K(H)\}$ is the essential approximate point spectrum and $\sigma_{ab}(T) = \bigcap\{\sigma_a(T+K) : TK = KT \text{ and } K \in K(H)\}$ is the Browder essential approximate point spectrum. We say that *a-Weyl's theorem holds for* $T \in B(H)$ if there is equality

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T), \tag{1.3}$$

and that *a-Browder's theorem holds for* $T \in B(H)$ if there is equality

$$\sigma_{ea}(T) = \sigma_{ab}(T). \tag{1.4}$$

We notice that a-Weyl's theorem and a-Browder's theorem are kinds of "approximate point spectrum" versions of Weyl's theorem and Browder's theorem, respectively. It is known ([5], [7], [18]) that if $T \in B(H)$, then we have

$$a\text{-Weyl's theorem} \Longrightarrow a\text{-Browder's theorem} \Longrightarrow \text{Browder's theorem}. \tag{1.5}$$

V. Rakočević ([18]) has shown that the equality (1.3) holds for cohyponormal operators. Recently S.V. Djordjević and D.S. Djordjević ([5]) showed that if T^* is quasihyponormal, then T obeys a-Weyl's theorem.

2. a-Weyl's theorem for 2×2 diagonal, skew-diagonal operator matrices

a-Weyl's theorem may or may not hold for a direct sum of operators for which a-Weyl's theorem holds. For example, if $U \in B(l_2)$ is the unilateral shift, then a-Weyl's theorem holds for both U and U^*, while it does not hold for $U \oplus U^*$. In fact, $\sigma_a(U \oplus U^*) = \{z \in \mathbb{C} : |z| \le 1\}$ and $\sigma_{ea}(U \oplus U^*) = \{z \in \mathbb{C} : |z| = 1\}$. Therefore a-Weyl's theorem does not hold for $U \oplus U^*$. An operator $T \in B(H)$ is called *approximate-isoloid* (abbrev. a-isoloid) if every isolated point of $\sigma_a(T)$ is an eigenvalue of T and an operator $T \in B(H)$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T . Clearly, if T is a-isoloid, then it is isoloid. However the converse is not true. Consider the following example: let $T = T_1 \oplus T_2$, where T_1 is the unilateral shift on l_2 and T_2 is an injective quasinilpotent on l_2. Then $\sigma(T) = \{z \in \mathbb{C} : |z| \le 1\}$ and $\sigma_a(T) = \{z \in \mathbb{C} : |z| = 1\} \cup \{0\}$. Therefore T is isoloid but is not a-isoloid.

We consider the sets

$$\Phi_+(H) = \{T \in B(H) : R(T) \text{ is closed and } \alpha(T) < \infty\},$$
$$\Phi_-(H) = \{T \in B(H) : R(T) \text{ is closed and } \alpha(T^*) < \infty\},$$
$$\Phi_+^-(H) = \{T \in B(H) : T \in \Phi_+(H) \text{ and } i(T) \le 0\}.$$

It is known ([18]) that if $T \in B(H)$, then $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(H)\}$.

When $A \in B(H)$ and $B \in B(K)$ are given we denote by M_C an operator acting on $H \oplus K$ of the form

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in B(K, H)$.

We begin with:

Lemma 2.1. *For a given pair (A, B) of operators, if $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in \Phi_+^-(H \oplus K)$, then $M_C \in \Phi_+^-(H \oplus K)$ for every $C \in B(K, H)$. Hence, in particular, we have*

$$\sigma_{ea}(M_C) \subset \sigma_{ea} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \subset \sigma_{ea}(A) \cup \sigma_{ea}(B). \tag{2.1.1}$$

Proof. Suppose $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in \Phi_+^-(H \oplus K)$. Then $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in \Phi_+(H \oplus K)$ and $i\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \le 0$. So A and B are upper semi-Fredholm and $i(A) + i(B) \le 0$.

Observe that

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{2.1.2}$$

Since $\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ is invertible for every $C \in B(K, H)$, and since $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ are both upper semi-Fredholm, it follows that M_C is upper semi-Fredholm. But since $i(M_C) = i(A) + i(B) \le 0$, hence $M_C \in \Phi_+^-(H \oplus K)$. The inclusions in (2.1.1) are evident from the first assertion. □

Theorem 2.2. *Suppose a-Weyl's theorem holds for $A \in B(H)$ and $B \in B(K)$.*

(1) *If a-Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, then*

$$\sigma_{ea} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_{ea}(A) \cup \sigma_{ea}(B). \tag{2.2.1}$$

(2) *If A and B are a-isoloid, then the converse of* (1) *is true.*

Proof. (1) By Lemma 2.1, $\sigma_{ea}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \subset \sigma_{ea}(A) \cup \sigma_{ea}(B)$. Since a-Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, a-Browder's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. Therefore $\sigma_{ea}(A) \cup \sigma_{ea}(B) \subset \sigma_{ab}(A) \cup \sigma_{ab}(B) = \sigma_{ab}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \sigma_{ea}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, that is, $\sigma_{ea}(A) \cup \sigma_{ea}(B) \subset \sigma_{ea}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

(2) For the statement (2) observe that if A and B are a-isoloid, then

$$\pi_{00}^a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\pi_{00}^a(A) \setminus \sigma_a(B)) \cup (\pi_{00}^a(B) \setminus \sigma_a(A)) \cup (\pi_{00}^a(A) \cap \pi_{00}^a(B)). \tag{2.2.2}$$

If a-Weyl's theorem holds for A and B, then the right-hand side of (2.2.2) is the set $(\sigma_a(A) \cup \sigma_a(B)) \setminus (\sigma_{ea}(A) \cup \sigma_{ea}(B))$. Thus if (2.2.1) holds, then $\pi_{00}^a\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = (\sigma_a(A) \cup \sigma_a(B)) \setminus (\sigma_{ea}(A) \cup \sigma_{ea}(B)) = \sigma_a\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \setminus \sigma_{ea}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. Thus a-Weyl's holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. □

The assumption "A and B are a-isoloid" is essential in the statement (2) of Theorem 2.2. For example if $A \in B(l_2)$ and $B \in B(l_2)$ are given by

$$A(x_1, x_2, x_3, x_4, \cdots) = (x_1, 0, x_3, x_4, \cdots)$$

and

$$B(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots),$$

then we have that (a) a-Weyl's theorem holds for A and B; (b) $\sigma_{ea}(A) = \{1\}$ and $\sigma_{ea}(B) = \{0\}$; (c) $\sigma_a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_{ea} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \{0, 1\}$; (d) $\pi^a_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \{0\}$; (e) B is not a-isoloid.

Next we consider a-Weyl's theorem for 2×2 skew-diagonal operator matrix of the form $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$. a-Weyl's theorem for the skew-diagonal matrices is more intricate in comparison with the diagonal matrices. We begin with:

Lemma 2.3. *If $A \in B(H, K)$ and $B \in B(K, H)$, then*

(1) $\sigma_a(AB) \setminus \{0\} = \sigma_a(BA) \setminus \{0\}$.
(2) $\sigma_{ea}(AB) \setminus \{0\} = \sigma_{ea}(BA) \setminus \{0\}$.

Proof. (1) Remember ([8]) that if $\lambda \neq 0$, then

$$\begin{pmatrix} AB - \lambda & 0 \\ 0 & I \end{pmatrix} = F(\lambda) \begin{pmatrix} BA - \lambda & 0 \\ 0 & I \end{pmatrix} E(\lambda), \tag{2.3.1}$$

where $E(\lambda)$ and $F(\lambda)$ are both invertible for each $\lambda \neq 0$. But if ST is bounded below and T is invertible, then S is bounded below, and so $\sigma_a(AB) \setminus \{0\} \subset \sigma_a(BA) \setminus \{0\}$. Similarly, $\sigma_a(BA) \setminus \{0\} \subset \sigma_a(AB) \setminus \{0\}$.

(2) It follows from (2.3.1) that if $\lambda \neq 0$,

$$N(AB - \lambda) \cong N(BA - \lambda) \text{ and } (R(AB - \lambda))^{\perp} \cong (R(BA - \lambda))^{\perp},$$

which gives the result, where $\cong$ means that there exists an invertible operator between spaces. □

Although $\sigma_a(AB) = \sigma_a(BA)$, we need not expect that $\sigma_{ea}(AB) = \sigma_{ea}(BA)$. Consider the following example: let $\dim\mathcal{H} < \infty$ and let $S, T \in B(l_2)$ be given by

$$S(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, \cdots) \text{ and } T(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots),$$

and let $A = \begin{pmatrix} 0_{\mathcal{H}} & 0 \\ 0 & S \end{pmatrix}$ and $B = \begin{pmatrix} 0_{\mathcal{H}} & 0 \\ 0 & T \end{pmatrix}$. Then $\sigma_a(AB) = \sigma_a(BA) = \{0, 1\}$, while $\sigma_{ea}(AB) = \{0, 1\} \neq \{1\} = \sigma_{ea}(BA)$.

Theorem 2.4. *If $A \in B(H, K)$ and $B \in B(K, H)$, then*

$$\sigma_{ea} \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = \sigma_{ea}(AB) \cup \sigma_{ea}(BA). \tag{2.4.1}$$

Hence, in particular, if AB and BA are a-isoloid and if a-Weyl's throrem holds for AB and BA, then a-Weyl's theorem holds for $\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$.

Proof. By Lemma 2.1, $\sigma_{ea}\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \subset \sigma_{ea}(AB) \cup \sigma_{ea}(BA)$. Conversely, suppose $\lambda \notin \sigma_{ea}\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$. Then $AB - \lambda$ and $BA - \lambda$ are upper semi-Fredholm and $i \begin{pmatrix} AB-\lambda & 0 \\ 0 & BA-\lambda \end{pmatrix} \leq 0$. Now we show that $i(AB - \lambda) = i(BA - \lambda)$. If $\lambda \neq 0$, then by Lemma 2.3 $i(AB - \lambda) = i(BA - \lambda)$. If $\lambda = 0$, then since AB and BA are upper semi-Fredholm, it follows from the continuity of the index that for sufficiently small $|\mu|$ with $\mu \neq 0$, $i(AB) = i(AB-\mu) = i(BA-\mu) = i(BA)$. This proves (2.4.1). The second assertion follows from Theorem 2.2. □

Corollary 2.5. *Let $A \in B(H, K)$ and $B \in B(K, H)$. If AB and BA obeys a-Browder's theorem, then a-Browder's theorem holds for $\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$.*

Proof. Since $\sigma_{ea} \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = \sigma_{ea}(AB) \cup \sigma_{ea}(BA)$, the result follows from [7, Theorem 3.11]. □

Example 2.6. (1) If $H = K$ in Theorem 2.4, one might expect to replace the condition "a-Weyl's theorem holds for AB and BA" by the condition "a-Weyl's theorem holds for A and B". However, this is not true. Consider the following example as follows: let T_1 and T_2 be defined on l_2 by

$$T_1(x_1, x_2, x_3, \cdots) = (x_1, 0, \frac{1}{2}x_2, \frac{1}{2}x_3, \cdots),$$
$$T_2(x_1, x_2, x_3, \cdots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \cdots).$$

Let $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 - I \end{pmatrix}$ on $l_2 \bigoplus l_2$. Then

$$\sigma_a(T) = \{-1\} \cup \{z \in \mathbb{C} : |z| = \frac{1}{2}\} \cup \{1\},$$
$$\sigma_{ea}(T) = \{-1\} \cup \{z \in \mathbb{C} : |z| = \frac{1}{2}\},$$
$$\pi_{00}^a(T) = \{1\}.$$

Therefore a-Weyl's theorem holds for T. However,

$$\sigma_a(T^2) = \{z \in \mathbb{C} : |z| = \frac{1}{4}\} \cup \{1\},$$
$$\sigma_{ea}(T^2) = \{z \in \mathbb{C} : |z| = \frac{1}{4}\} \cup \{1\},$$
$$\pi_{00}^a(T^2) = \{1\},$$

and hence a-Weyl's theorem doesn't hold for T^2.

(2) Since

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix},$$

we might expect that "a-Weyl's theorem for $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$" is inherited from "$a$-Weyl's theorem for $\begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$". However, in general, a-Weyl's theorem need not be trasmitted from T^2 to T. Consider the following example as follows: let T_1 and T_2 be defined on l_2 by

$$T_1(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, x_4, \cdots),$$
$$T_2(x_1, x_2, x_3, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots).$$

Let $T = \begin{pmatrix} T_1 + I & 0 \\ 0 & T_2 - I \end{pmatrix}$ on $l_2 \bigoplus l_2$. Then

$$\sigma_a(T) = \{-1\} \cup \{z \in \mathbb{C} : |z - 1| = 1\},$$
$$\sigma_{ea}(T) = \{-1\} \cup \{z \in \mathbb{C} : |z| = 1\},$$
$$\pi_{00}^a(T) = \{-1\}.$$

Therefore a-Weyl's theorem does not hold for T. However,

$$\sigma_a(T^2) = \{re^{i\theta} : r = 2(1 + \cos\theta)\},$$
$$\sigma_{ea}(T^2) = \{re^{i\theta} : r = 2(1 + \cos\theta)\},$$
$$\pi_{00}^a(T^2) = \phi,$$

and hence a-Weyl's theorem holds for T^2.

(3) Generally, "a-Weyl's theorem holds for AB" does not imply "a-Weyl's theorem holds for BA". For example suppose the operators $T_1, T_2, T_3 \in B(l_2)$ are given by

$$\begin{aligned} T_1(x_1, x_2, x_3, \cdots) &= (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots), \\ T_2(x_1, x_2, x_3, \cdots) &= (0, x_1, 0, x_2, 0, x_3, 0, \cdots), \\ T_3(x_1, x_2, x_3, \cdots) &= (x_2, x_4, x_6, x_8, \cdots). \end{aligned}$$

Let $A = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & T_2 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} I & 0 \\ 0 & T_3 \end{smallmatrix}\right)$. Then

$$\begin{aligned} &\sigma_a(AB) = \sigma_{ea}(AB) = \sigma_a(BA) = \sigma_{ea}(BA) = \{0, 1\}, \\ &\pi_{00}^a(AB) = \phi, \\ &\pi_{00}^a(BA) = \{0\}. \end{aligned}$$

Therefore a-Weyl's theorem holds for AB but fails for BA.

Theorem 2.7. *Let $A \in B(K, H)$ and $B \in B(H, K)$ be such that AB and BA are a-isoloid. If a-Weyl's theorem holds for AB and BA, then the following statements are equivalent:*

(1) *a-Weyl's theorem holds for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$;*
(2) *$\sigma_{ea}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)}$;*
(3) *a-Browder's theorem holds for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$,*

where $\sqrt{X}$ denotes the set of square roots of complex numbers in $X \subset \mathbb{C}$.

Proof. (1) $\Leftrightarrow$ (2) Suppose $\sigma_{ea}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)}$. Then it follows from a similar argument of ([13], (3.10); (4.3)) that

$$\begin{aligned} \sigma_a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \setminus \sigma_{ea} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} &= \sqrt{\sigma_a(AB) \cup \sigma_a(BA)} \setminus \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)} \\ &= \sqrt{(\sigma_a(AB) \cup \sigma_a(BA)) \setminus (\sigma_{ea}(AB) \cup \sigma_{ea}(BA))} \\ &= \sqrt{\sigma_a \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \setminus \sigma_{ea} \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}} \\ &= \sqrt{\pi_{00}^a \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}}. \end{aligned}$$

Now we show that

$$\sqrt{\pi_{00}^a \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}} = \pi_{00}^a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{2.7.1}$$

But since $\sigma_a\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ is symmetric with respect to the origin, it follows from the approximate point spectral mapping theorem that

$$\sqrt{\text{iso } \sigma_a \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}} = \sqrt{\text{iso } \left(\sigma_a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right)^2} = \text{iso } \sigma_a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

where we write iso$X$ for the isolated points of $X \subset \mathbb{C}$. Thus for (2.7.1) it suffices to show that for any $\lambda \in \mathbb{C}$,

$$0 < \alpha \begin{pmatrix} AB - \lambda & 0 \\ 0 & BA - \lambda \end{pmatrix} < \infty \Longleftrightarrow 0 < \alpha \begin{pmatrix} -\sqrt{\lambda} & A \\ B & -\sqrt{\lambda} \end{pmatrix} < \infty. \tag{2.7.2}$$

If $\lambda = 0$, then (2.7.2) follows from the observation

$$0 < \alpha(N(A) \oplus N(B)) < \infty \Longleftrightarrow 0 < \alpha(N(AB) \oplus N(BA)) < \infty. \tag{2.7.3}$$

If $\lambda \neq 0$, then (2.7.2) follows from the observation

$$\bigvee\{(x, \frac{1}{\sqrt{\lambda}}Bx) : x \in N(AB - \lambda)\} \cup \bigvee\{(\frac{1}{\sqrt{\lambda}}Ay, y) : y \in N(BA - \lambda)\}$$

$$\subset N \begin{pmatrix} -\sqrt{\lambda} & A \\ B & -\sqrt{\lambda} \end{pmatrix} \subset N(AB - \lambda) \oplus N(BA - \lambda),$$

where $\bigvee F$ denotes the closed linear span of F. This proves (2.7.1).

Suppose a-Weyl's theorem holds for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$.

Then $\sigma_{ea}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sigma_a\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \setminus \pi_{00}^a\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$. It will be shown that

$$\sigma_a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \setminus \pi_{00}^a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)}.$$

Indeed,

$$\begin{aligned} \sigma_a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \setminus \pi_{00}^a \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} &= \sqrt{\sigma_a(AB) \cup \sigma_a(BA)} \setminus \sqrt{\pi_{00}^a \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}} \\ &= \sqrt{\sigma_a \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \setminus \pi_{00}^a \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}} \\ &= \sqrt{\sigma_{ea} \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}} \\ &= \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)}. \end{aligned}$$

Therefore $\sigma_{ea}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)}$.

(2) $\Leftrightarrow$ (3) Suppose $\sigma_{ea}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)}$. Then a-Weyl's theorem holds for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$, and so a-Browder's theorem holds for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$. Conversely, suppose a-Browder's theorem holds for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$. Since AB and BA obeys a-Weyl's theorem, $\sigma_{ea}(AB) = \sigma_{ab}(AB)$ and $\sigma_{ea}(BA) = \sigma_{ab}(BA)$, respectively.

But since $\sigma_{ab}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sqrt{\sigma_{ab}(AB) \cup \sigma_{ab}(BA)}$, $\sigma_{ea}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sigma_{ab}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sqrt{\sigma_{ab}(AB) \cup \sigma_{ab}(BA)} = \sqrt{\sigma_{ea}(AB) \cup \sigma_{ea}(BA)}$ and completes the proof. □

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