PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 3, Pages 825–831
S 0002-9939(01)06112-3
Article electronically published on June 21, 2001

DOMAIN FUNCTIONALS AND EXIT TIMES FOR BROWNIAN MOTION

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(Communicated by Claudia M. Neuhauser)

ABSTRACT. Two domain functionals describing the averaged expectation of exit times and averaged variance of exit times of Brownian motion from a domain, respectively, are studied. We establish the variational formulas for maximizing the functionals over C^k domains with a volume constraint, and characterize the stationary points and maximizers.

1. INTRODUCTION

Let W_t be a standard n-dimensional Brownian motion satisfying $W_0 = x \in R^n$. For any bounded domain Ω, denote by $\tau_\Omega(\omega)$ the (first) exit time of the Brownian motion from Ω, i.e., $\tau_\Omega(\omega) = \inf\{t \geq 0 : W_t(\omega) \in \partial\Omega\}$, where $\partial\Omega$ is the boundary of Ω. Consider the following domain functionals:

$$\mathfrak{F}_E(\Omega) = \frac{1}{|\Omega|}\int_\Omega E_x(\tau_\Omega)\,dx,$$
$$\mathfrak{F}_V(\Omega) = \frac{1}{|\Omega|}\int_\Omega Var_x(\tau_\Omega)\,dx,$$

where $E_x(\tau_\Omega)$ and $Var_x(\tau_\Omega)$ are the expectation and variance of τ_Ω, respectively. The values $\mathfrak{F}_E(\Omega)$ and $\mathfrak{F}_V(\Omega)$ may be understood as the averaged expected exit time and averaged variance of exit time from Ω for the Brownian motion initiated in Ω.

We are interested in characterizing stationary points for the following optimization problems (P_E) and (P_V):

(P_E) find Ω that maximizes $\mathfrak{F}_E(\Omega)$ over $\mathfrak{D}^k$,

(P_V) find Ω that maximizes $\mathfrak{F}_V(\Omega)$ over $\mathfrak{D}^k$

where $\mathfrak{D}^k$ is the collection of bounded C^k ($k \geq 2$) domains $\Omega \subset R^n$ satisfying the volume constraint

(1.1) volume of $\Omega = A_0$, A_0 is a positive constant.

Received by the editors May 24, 2000 and, in revised form, September 15, 2000.
1991 *Mathematics Subject Classification.* Primary 65J65; Secondary 58G32, 49K20.
Key words and phrases. Boundary value problems, Brownian motion, exit time, domain functionals, domain variations.

A domain Ω in $\mathfrak{D}^k$ is called a stationary point of any domain functional $\mathfrak{F}$ over $\mathfrak{D}^k$ if for any $\{\Omega_\varepsilon\}_{|\varepsilon|\leq\varepsilon_0} \subset \mathfrak{D}^k$ with $\Omega_0 = \Omega$, for some $\varepsilon_0 > 0$, we have $d\mathfrak{F}(\Omega_\varepsilon)/d\varepsilon = 0$ at $\varepsilon = 0$ if it exists. We show the following results.

Theorem 1.1. *Let $\Omega \in \mathfrak{D}^k$ be a stationary point of functional $\mathfrak{F}_E$ over $\mathfrak{D}^k$. Then there exists a solution to the overdetermined boundary value problem*

$$\Delta u(x) = -2 \quad in\ \Omega, \tag{1.2}$$

$$u(x) = 0 \quad on\ \partial\Omega, \tag{1.3}$$

$$\frac{\partial u(x)}{\partial v} \text{ is constant on } \partial\Omega, \tag{1.4}$$

where v is the exterior unit normal vector. In particular, $\Omega \in \mathfrak{D}^k$ is a maximizer of $\mathfrak{F}_E$ over $\mathfrak{D}^k$ if and only if Ω is a ball.

Theorem 1.2. *Let $\Omega \in \mathfrak{D}^k$ be a stationary point of functional $\mathfrak{F}_V$ over $\mathfrak{D}^k$. Then there exists a solution to the overdetermined boundary value problem*

$$\Delta u(x) = -2 \quad in\ \Omega, \tag{1.5}$$

$$\Delta w(x) = -2u(x) \quad in\ \Omega, \tag{1.6}$$

$$u(x) = w(x) = 0 \quad on\ \partial\Omega, \tag{1.7}$$

$$\frac{\partial u(x)}{\partial v}\frac{\partial w(x)}{\partial v} \text{ is constant on } \partial\Omega, \tag{1.8}$$

where v is the exterior unit normal vector. In particular, $\Omega \in \mathfrak{D}^k$ is a maximizer of $\mathfrak{F}_V$ over $\mathfrak{D}^k$ if and only if Ω is a ball.

In the case $k = \infty$, the optimization problems (P_E) and (P_V) have been studied in [4, 5], where the same results with a different definition of stationary points (called critical points there) were established. See also [1, 2, 6, 8] for more general domain functionals. However, the notion of "critical point" adopted in these papers is with respect to infinitesimal volume preservation. More precisely, a domain Ω is called a critical point in [4, 5, 6] if $d\mathfrak{F}_E(\Omega_\varepsilon)/d\varepsilon = 0$ at $\varepsilon = 0$ for any normal domain variations Ω_ε ($\Omega_0 = \Omega$) satisfying $d|\Omega_\varepsilon|/d\varepsilon = 0$ at $\varepsilon = 0$. This same definition was also employed in [1, 2] for interior domain variations. In general, this domain variation Ω_ε may not satisfy constraint (1.1), i.e., $\Omega_\varepsilon \notin \mathfrak{D}^k$. Strictly speaking, a maximizer of $\mathfrak{F}_E$ or $\mathfrak{F}_V$ over $\mathfrak{D}^k$ may not be a critical point in this sense.

The novelty of this note is the introduction of a family of volume-preserving domain variations Ω_ε, i.e., $\Omega_\varepsilon \in \mathfrak{D}^k$ for all ε. This guarantees that a maximizer of $\mathfrak{F}_E$ or $\mathfrak{F}_V$ over $\mathfrak{D}^k$ is necessarily a stationary point. In addition, we weaken the regularity requirement on Ω which cannot be achieved by using the method of normal domain variations.

2. Volume-preserving domain variations

For simplicity, throughout the paper, we assume that in (1.1), $A_0 = 1$. Let Ω be a bounded C^k domain. For any $V \in C^k(\Omega', R^n)$, where Ω' is a neighborhood of $\bar{\Omega}$, we denote by $\Phi(x,\varepsilon)$ the solution of the initial value problem, for fixed $x \in \bar{\Omega}$,

$$\frac{d\Phi(x,\varepsilon)}{d\varepsilon} = V(\Phi(x,\varepsilon)), \quad \Phi(x,0) = x. \tag{2.1}$$

A standard ODE theorem shows that $\Phi(x,\varepsilon)$ is a C^k diffeomorphism for $|\varepsilon| < \varepsilon_0$ for some $\varepsilon_0 > 0$. We define a variation $\Omega_{\varepsilon,V}$ of the domain Ω by the vector field V, by

$$\Omega_{\varepsilon,V} = \Phi(\Omega,\varepsilon) = \{\Phi(x,\varepsilon) : x \in \Omega\}.$$

When no confusion arises, we shall omit the subscript $V : \Omega_\varepsilon = \Omega_{\varepsilon,V}$. Obviously, $\Omega_0 = \Omega$, and $\Omega_\varepsilon \subset \Omega'$ (for small ε_0) is a C^k domain. Also, $\Omega_\varepsilon \to \Omega$ as $\varepsilon \to 0$, in the sense of the C^k topology.

Lemma 2.1. *Suppose $\Omega \in \mathfrak{D}^k$. Let $\Omega' \supset \bar{\Omega}$ be a neighborhood of $\bar{\Omega}$, and let $V \in C^k(\Omega', R^n)$ be a divergence free vector field, i.e., it satisfies $\nabla \cdot V(x) = 0$ for $x \in \Omega'$. Let $\Omega_\varepsilon = \Phi(\Omega,\varepsilon)$ be the variation of Ω by the vector field V. Then $\Omega_\varepsilon \in \mathfrak{D}^k$.*

Proof. By the ODE theory and the fact that $\nabla \cdot V(x) = 0$, its Jacobian $J(\Phi(x,\varepsilon))$ satisfies

$$\frac{dJ(\Phi(x,\varepsilon))}{d\varepsilon} = (\nabla \cdot V)(\Phi(x,\varepsilon))\, J(\Phi(x,\varepsilon)) = 0.$$

Hence

$$J(\Phi(x,\varepsilon)) = J(\Phi(x,0)) = 1. \tag{2.2}$$

This immediately leads to $|\Omega_\varepsilon| = |\Omega| = 1$. □

Let V be such a divergence free C^k vector field and let Ω_ε be the variation of Ω by V. Consider the Dirichlet problem

$$\Delta u_\varepsilon(x) = -2 \text{ in } \Omega_\varepsilon,\ u_\varepsilon(x) = 0 \text{ on } \partial\Omega_\varepsilon.$$

Set $u = u_0$. We define the first variation $\delta_V(u)$ of u along the vector field V by

$$\delta_V(u)(x_0) = \left.\frac{d}{d\varepsilon} u_\varepsilon(\Phi(x_0,\varepsilon))\right|_{\varepsilon=0}, \tag{2.3}$$

for $x_0 \in \bar{\Omega}$.

Lemma 2.2. *Let u be the solution of (1.2) and (1.3) and $G_0(x,y)$ be the Green's function for the Laplacian Δ for Ω. Then for any divergence free C^k vector field V, defined in a neighborhood of $\bar{\Omega}$, and $x_0 \in \bar{\Omega}$, we have*

$$\begin{aligned}\delta_V(u)(x_0) = &-\int_{\partial\Omega} (V(\sigma)\cdot v(\sigma)) \frac{\partial u(\sigma)}{\partial v} \frac{\partial G_0(x_0,\sigma)}{\partial v} d\sigma \\ &+ V(x_0)\cdot\nabla u(x_0),\ \textit{for } x_0 \textit{ in } \bar{\Omega}.\end{aligned} \tag{2.4}$$

Proof. Let $G_\varepsilon(x,y)$ be the Green's function for the Laplacian Δ for Ω_ε. By Green's formula and (2.2), we have

$$\begin{aligned} u_\varepsilon(\Phi(x_0,\varepsilon)) &= -2\int_{\Omega_\varepsilon} G_\varepsilon(\Phi(x_0,\varepsilon), x)\, dx \\ &= -2\int_{\Omega} G_\varepsilon(\Phi(x_0,\varepsilon), \Phi(x,\varepsilon))\, dx. \end{aligned}$$

By [3, 2.4.28] (an early argument of this type was given by Hadamard) and direct computations, we have the formula

$$\frac{\partial}{\partial\varepsilon}G_{\varepsilon}\left(\Phi\left(x_{0},\varepsilon\right),\Phi\left(x,\varepsilon\right)\right)\Big|_{\varepsilon=0}$$
$$=-\int_{\Omega}\nabla_{y}G_{0}\left(x_{0},y\right)\cdot\left(\nabla V(y)+\left(\nabla V(y)\right)^{T}\right)\nabla_{y}G_{0}\left(x,y\right)dy.$$

Using the fact that

$$\nabla u\left(y\right)=-2\int_{\Omega}\nabla_{y}G_{0}\left(x,y\right)dx,$$

it follows from (2.3) that

$$\delta_{V}\left(u\right)\left(x_{0}\right)=-2\int_{\Omega}\frac{\partial}{\partial\varepsilon}G_{\varepsilon}\left(\Phi\left(x_{0},\varepsilon\right),\Phi\left(x,\varepsilon\right)\right)\Big|_{\varepsilon=0}dx$$
$$=-\int_{\Omega}\left(\nabla_{y}G_{0}\left(x_{0},y\right)\right)\cdot\left(\nabla V(y)+\left(\nabla V(y)\right)^{T}\right)\nabla u\left(y\right)dy.$$

Since $\nabla^2 u$ and $\nabla^2 G_0$ are symmetric matrices and since V is divergence free, a direct computation leads to

$$\nabla_{y}G_{0}\left(x_{0},y\right)\cdot\left(\nabla V(y)+\left(\nabla V(y)\right)^{T}\right)\nabla u\left(y\right)$$
$$=\nabla G_{0}\cdot\nabla\left(V\cdot\nabla u\right)+\nabla u\cdot\nabla\left(V\cdot\nabla G_{0}\right)-\nabla\cdot\left(\left(\nabla u\cdot\nabla G_{0}\right)V\right),$$

for $y\neq x_0$. Since u solves (1.2) and (1.3), $\Delta G_0(x_0,y)$ is the Dirac measure at $y=x_0$, $\nabla\cdot V(x)=0$, and $G_0(x_0,\sigma)=0$ for $\sigma\in\partial\Omega$, the divergence theorem yields

$$\delta_{V}\left(u\right)\left(x_{0}\right)=-\int_{\partial\Omega}\left(V\left(\sigma\right)\cdot\nabla u\right)\frac{\partial G_{0}\left(x_{0},\sigma\right)}{\partial v}d\sigma+V\left(x_{0}\right)\cdot\nabla u\left(x_{0}\right)$$
$$-\int_{\partial\Omega}\left(V\left(\sigma\right)\cdot\nabla G_{0}\left(x_{0},\sigma\right)\right)\frac{\partial u\left(\sigma\right)}{\partial v}d\sigma-2\int_{\Omega}V\left(y\right)\cdot\nabla G_{0}\left(x_{0},y\right)dy$$
$$+\int_{\partial\Omega}\left(\nabla u\left(\sigma\right)\cdot\nabla G_{0}\left(x_{0},\sigma\right)\right)V\left(\sigma\right)\cdot v\left(\sigma\right)d\sigma$$
$$=-\int_{\partial\Omega}\left(V\left(\sigma\right)\cdot v\left(\sigma\right)\right)\frac{\partial u\left(\sigma\right)}{\partial v}\frac{\partial G_{0}\left(x_{0},\sigma\right)}{\partial v}d\sigma+V\left(x_{0}\right)\cdot\nabla u\left(x_{0}\right).$$

In deriving the last equality, we have used the fact that for $\sigma\in\partial\Omega$,

$$\nabla u\left(\sigma\right)=\frac{\partial u\left(\sigma\right)}{\partial v}v\left(\sigma\right),\ \nabla G_{0}\left(x_{0},\sigma\right)=\frac{\partial G_{0}\left(x_{0},\sigma\right)}{\partial v}v\left(\sigma\right),$$

and the fact that V is divergence free. □

We point out that the mappings defining interior variations used in [2, 3] are actually the first order approximation to $\Phi(x,\varepsilon)$.

3. Variational formulations for functionals

In this section we develop variational formulations for problems (P_E) and (P_V), utilizing the results established in the previous section. Define the first variation of any domain functional $\mathfrak{F}$ along a divergence free vector field V by

$$\delta_{V}\left(\mathfrak{F}\right)\left(\Omega\right)=\frac{d\mathfrak{F}\left(\Omega_{\varepsilon}\right)}{d\varepsilon}\Big|_{\varepsilon=0}=\lim_{\varepsilon\to0}\frac{\mathfrak{F}\left(\Omega_{\varepsilon}\right)-\mathfrak{F}\left(\Omega\right)}{\varepsilon},$$

where Ω_ε is the variation of Ω along V.

Lemma 3.1. *Let $\Omega \in \mathfrak{D}^k$ and V be a divergence free C^k vector field defined in a neighborhood of $\bar{\Omega}$. Then*

$$\delta_V(\mathfrak{F}_E)(\Omega) = \frac{1}{2}\int_{\partial\Omega} (V(\sigma)\cdot v)\left(\frac{\partial u(\sigma)}{\partial v}\right)^2 d\sigma \tag{3.1}$$

where u is the solution of (1.2) *and* (1.3), *and*

$$\delta_V(\mathfrak{F}_V)(\Omega) = \int_{\partial\Omega} (V(\sigma)\cdot v)\frac{\partial u(\sigma)}{\partial v}\frac{\partial w(\sigma)}{\partial v} d\sigma \tag{3.2}$$

where u and w satisfy (1.5)-(1.7).

The formulas (3.1) and (3.2) were previously derived in [6] for normal variations of smooth domains, and in [2] for interior variations of Lipschitz domains. By the same techniques used in [6], together with Lemma 2.2, one can show that (3.1) and (3.2) remain true for our domain variations by any divergence free vector field V. We omit the details of the proof to Lemma 3.1.

We are now in a position to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Suppose that the functional $\mathfrak{F}_E$ has a stationary point $\Omega \in \mathfrak{D}^k$. Then for any divergence free C^k vector field V, $\delta_V(\mathfrak{F}_E)(\Omega) = 0$. Let u solve (1.2), (1.3). By Lemma 3.1, it follows that

$$\int_{\partial\Omega} (V(\sigma)\cdot v(\sigma))\left(\frac{\partial u(\sigma)}{\partial v}\right)^2 d\sigma = 0. \tag{3.3}$$

To show (1.4), it suffices to show that for any smooth function φ defined in a neighborhood of $\partial\Omega$ and satisfying

$$\int_{\partial\Omega} \varphi(\sigma)\, d\sigma = 0, \tag{3.4}$$

the following equality holds:

$$\int_{\partial\Omega} \varphi(\sigma)\left(\frac{\partial u(\sigma)}{\partial v}\right)^2 d\sigma = 0. \tag{3.5}$$

To verify (3.5), we select a sequence of decreasing smooth domains $\left\{\Omega^{(m)}\right\}_{m=0}^{\infty}$ with $\Omega^{(0)} = \Omega$ such that $\Omega^{(m)} \supset \bar{\Omega}^{(m+1)}$, $\Omega^{(m)} \supset \bar{\Omega}$ for $m \geq 1$, and as $m \to \infty$, $\Omega^{(m)} \to \Omega$ in the C^k topology. Let φ be a fixed smooth function satisfying (3.4). We extend φ smoothly to R^n. Define a sequence of constants θ_m by

$$\theta_m = -\frac{1}{\int_{\partial\Omega^{(m)}} d\sigma}\int_{\partial\Omega^{(m)}} \varphi(\sigma)\, d\sigma.$$

It is easy to see that $\theta_m \longrightarrow 0$ (by (3.4)) as $m \longrightarrow \infty$ and that

$$\int_{\partial\Omega^{(m)}} (\varphi(\sigma) + \theta_m)\, d\sigma = 0. \tag{3.6}$$

Consider the Neumann boundary value problem

$$\Delta g_m = 0 \text{ in } \Omega^{(m)},$$
$$\frac{\partial g_m}{\partial v} = \varphi + \theta_m \text{ on } \partial\Omega^{(m)}.$$

Note that (3.6) is the solvability condition for the Neumann problem. Hence, the above boundary value problem admits a smooth solution g_m (which is unique up to a constant). Furthermore, the $H^1(\Omega)$ norm of the function g_m is bounded uniformly in m. Set $V_m = \nabla g_m$. Obviously V_m is a divergence free vector field. Let $\eta(x)$ be a C^k extension of $\partial u(\sigma)/\partial v$. Then, by (3.3) and the divergence theorem (in $\Omega_m \backslash \Omega$), we have

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial g_m(\sigma)}{\partial v} \left(\frac{\partial u(\sigma)}{\partial v}\right)^2 d\sigma \\ &= \int_{\Omega_m \backslash \Omega} \nabla \cdot \left(\eta^2(x) \nabla g_m(x)\right) dx + \int_{\partial\Omega_m} \frac{\partial g_m(\sigma)}{\partial v} \eta^2(\sigma)\, d\sigma \\ &= \int_{\Omega_m \backslash \Omega} \nabla \eta^2(x) \cdot \nabla g_m(x)\, dx + \int_{\partial\Omega_m} (\varphi(\sigma) + \theta_m)\, \eta^2(\sigma)\, d\sigma \\ &\to \int_{\partial\Omega} \varphi(\sigma) \left(\frac{\partial u(\sigma)}{\partial v}\right)^2 d\sigma \quad (\text{as } m \to \infty). \end{aligned}$$

Our previous assertion (3.5) follows.

The rest of the assertions follow from the fact that any solution of (1.2)-(1.4) is radially symmetric [6] (see [7] for the original results) and that any ball is a maximizer [4]. □

Proof of Theorem 1.2. Suppose that the functional $\mathfrak{F}_V$ has a stationary point $\Omega \in \mathfrak{D}^k$. Then for any divergence free C^k vector field V, $\delta_V(\mathfrak{F}_V)(\Omega) = 0$. Let u and w be the solutions of (1.5)-(1.7). By Lemma 3.1, it follows that

$$\int_{\partial\Omega} (V(\sigma) \cdot v(\sigma)) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma = 0.$$

Using exactly the same argument as in the proof of Theorem 1.1, we may show that for any smooth function φ defined in a neighborhood of $\bar{\Omega}$ and satisfying $\int_{\partial\Omega} \varphi(\sigma)\, d\sigma = 0$, we have

$$\int_{\partial\Omega} \varphi(\sigma) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma = 0.$$

This immediately leads to (1.8). The rest of the assertions follow from [6]. □

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