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A GENERALIZED KOLMOGOROV INEQUALITY FOR THE HILBERT TRANSFORM

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Abstract. If $f \in L^1(\mathbf{R}^1; (1+|x|)^{-1}dx)$ we can define the Hilbert transform Hf almost everywhere (Lebesgue) and obtain an estimate for $\mu\{x : |Hf(x)| \geq \alpha\}$ where μ is a suitable finite measure. The classical Kolmogorov inequality for the Lebesgue measure of $\{x : |Hf(x)| \geq \alpha\}$ is obtained by a scaling argument.

1. Introduction

The purpose of this note is to obtain an extended form of the Kolmogorov inequality for the Hilbert transform

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{f(x-y)}{y} dy. \tag{1}$$

If $f \in L^p(\mathbf{R})$ for some $p \in [1, \infty)$, the almost-everywhere existence of Hf can be obtained from an analysis of the conjugate Poisson kernel in the upper half-plane. The classical Kolmogorov inequality is the statement that

$$\mathrm{Leb}\{x : |Hf(x)| \geq \alpha\} \leq \frac{C}{\alpha} \int_{\mathbf{R}} |f(x)|\, dx, \qquad \alpha > 0, f \in L^1(\mathbf{R}). \tag{2}$$

However, the natural domain of definition of (1) is the Banach space

$$B_1 := \{f : \int_{\mathbf{R}} \frac{|f(x)|}{1+|x|}\, dx < \infty\} \tag{3}$$

which contains all of the Lebesgue spaces $L^p(\mathbf{R})$, $1 \leq p < \infty$, where the operator H has a well-established theory.

In order to study H on B_1, we introduce the norms

$$||f||_{B_1} := \frac{1}{\pi} \int_{\mathbf{R}} \frac{|x\, f(x)|}{1+x^2}\, dx, \qquad ||f||_{B_2} := \frac{1}{\pi} \int_{\mathbf{R}} \frac{|f(x)|}{1+x^2}\, dx. \tag{4}$$

Define a weighted measure by

$$\mu(A) := \frac{1}{\pi} \int_A \frac{dx}{1+x^2}. \tag{5}$$

The main result is the following estimate.

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Theorem. *Suppose that $f \in B_1$. Then $Hf(x)$ exists almost everywhere. If $f \geq 0$, then we have the estimate*

$$\mu\{x : |Hf(x)| \geq \alpha\} \leq \frac{2}{\pi}\left(\frac{||f||_{B_2}}{\alpha - ||f||_{B_1}} + \frac{||f||_{B_2}}{\alpha + ||f||_{B_1}}\right), \qquad \alpha > ||f||_{B_1}. \tag{6}$$

For any complex-valued $f \in B_1$, (6) holds with four terms on the right side, and where α is replaced by $\alpha/4$.

In the final section we show how the classical Kolmogorov inequality (2) can be obtained from (6). We also remark that precise constants for the classical Kolmogorov inequality (2) have been obtained by Davis [D]. Some recent generalizations have been discovered by Choi [C].

2. Related properties of the Poisson kernel

In order to study the conjugate Poisson kernel, we first develop the necessary properties of the Poisson kernel operator, defined by

$$P_y f(x) := \frac{1}{\pi}\int_{\mathbf{R}} \frac{y f(x-t)}{y^2 + t^2}\, dt. \tag{7}$$

This operator is defined on the Banach space

$$B_2 := \{f : ||f||_{B_2} := \frac{1}{\pi}\int_{\mathbf{R}} \frac{|f(x)|}{1+x^2}\, dx < \infty\}. \tag{8}$$

Clearly $L_1(\mathbf{R}) \subset B_1 \subset B_2$. The Poisson kernel has the following properties.

Proposition 2.1. *Suppose that $f \in B_2$. Then $||P_y f||_{B_2} \leq 2||f||_{B_2}$ for $0 < y \leq 1$ and for any $f \in B_2$, $\lim_{y\to 0} ||P_y f - f||_{B_2} = 0$. Furthermore if $f \in B_1$, then $\lim_{y\to 0} P_y f(x) = f(x)$ for almost every $x \in \mathbf{R}$.*

Proof. We have

$$\begin{aligned}
||P_y f||_{B_2} &\leq \frac{1}{\pi^2}\int_{\mathbf{R}}\left(\int_{\mathbf{R}} \frac{y|f(t)|}{y^2 + (x-t)^2}\, dt\right)\frac{dx}{1+x^2} \\
&= \frac{1}{\pi^2}\int_{\mathbf{R}} |f(t)| \left(\int_{\mathbf{R}} \frac{y}{y^2+(x-t)^2}\frac{1}{1+x^2}\, dx\right) dt \\
&= \frac{1}{\pi}\int_{\mathbf{R}} |f(t)| \frac{1+y}{(1+y)^2 + t^2}\, dt \\
&\leq \frac{1}{\pi}\int_{\mathbf{R}} |f(t)| \frac{2}{1+t^2}\, dt, \qquad 0 < y < 1, \\
&= 2||f||_{B_2}
\end{aligned}$$

where we have used the semi-group property of the Poisson kernel in the form $P_y * P_1 = P_{1+y}$. To prove the norm convergence, we first note that if $f = 1_{[a,b]}$, then $\pi P_y f(x) = \arctan[(x-b)/y] - \arctan[(x-a)/y]$ which is bounded by 2, and tends pointwise to $1_{[a,b]}$ except at the endpoints $x = a, b$. By the dominated convergence theorem, we have $||P_y f - f||_{B_2} \to 0$ when $y \to 0$. Similarly for a finite linear combination $f = \sum_{j=1}^N c_j 1_{[a_j,b_j]}$ we have $||P_y f - f||_{B_2} \to 0$. But these functions are dense in the space B_2, and we already have proved that the operator norms $||P_y||_{B_2}$ are uniformly bounded for $0 < y \le 1$, whence the result. To prove the almost-everywhere convergence, we write

$$\begin{aligned} P_y f(x) - f(x) &= \frac{1}{\pi}\int_0^\infty [f(x+t) + f(x-t) - 2f(x)]\frac{y}{t^2+y^2}\,dt, \\ |P_y f(x) - f(x)| &\le \int_0^\infty \frac{y}{t^2+y^2} d\Phi_x(t)\,dt, \\ \Phi_x(t) &:= \frac{1}{\pi}\int_0^t |f(x+u) + f(x-u) - 2f(x)|\,du. \end{aligned}$$

From Lebesgue's theorem, we have for almost every x, $\Phi_x(t)/t \to 0$ when $t \to 0$. On the other hand, $f \in B_1$ implies that $\Phi_x(t) \le Ct$ for all $t \ge 0$. Now we integrate by parts:

$$|P_y f(x) - f(x)| \le \int_0^\infty \frac{2ty}{(t^2+y^2)^2}\Phi_x(t)\,dt,$$

where the estimate $|\Phi_x(t)| \le Ct$ allows one to discard the term at the limits. Setting $t = yz$ in the integration gives

$$|P_y f(x) - f(x)| \le \int_0^\infty \frac{2z}{(1+z^2)^2}\frac{\Phi_x(zy)}{y}\,dz.$$

But the integrand is bounded by an L^1 function and tends to zero pointwise when $y \to 0$, hence $P_y f(x) \to f(x)$ as required.

3. Proof of the Theorem

We follow the method of Carleson and Katznelson, as exposited in [Kz]. Without loss of generality, we assume $f \ge 0$. Indeed, any complex-valued function can be written as $f = f_1 - f_2 + i(f_3 - f_4)$ where $f_j \ge 0$. We begin with the conjugate Poisson kernel operator

$$Q_y f(x) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{tf(x-t)}{t^2+y^2}\,dt, \qquad y > 0, x \in \mathbf{R}. \tag{9}$$

Clearly $|Q_1 f(0)| \le ||f||_{B_1}$. Then

$$P_y f(x) + iQ_y f(x) = \frac{i}{\pi}\int_{\mathbf{R}} \frac{f(t)}{x+iy-t}\,dt. \tag{10}$$

For any $f \in B_1$ (10) defines an analytic function in the upper half-plane $y > 0$. The mapping

$$(x,y) \mapsto \exp[-(P_y f(x) + iQ_y f(x))] \tag{11}$$

is a bounded analytic function in the upper half plane. By the Fatou theorem [G], it possesses a.e. limits when $y \to 0$. But $P_y f(x)$ converges to a finite limit a.e. whenever $f \in B_1 \subset B_2$. Hence we deduce the existence of the a.e. limit of $\exp[-iQ_y f(x)]$ when $y \to 0$. From this it follows that $Q_y f(x)$ can have only one accumulation point when $y \to 0$, hence the existence of $\tilde{H}f(x) := \lim_{y\to 0} Q_y f(x)$.

To prove (6), we consider the harmonic function $J_\alpha(w)$ defined for $\mathrm{Re}(w) > 0$ as the harmonic measure of the two rays $\{w = iv, v \ge \alpha\}$ and $\{w = iv, v \le -\alpha\}$. This is the harmonic function which takes the value 1 on these rays and takes the value zero on the segment $\{w = iv : -\alpha \le v \le \alpha\}$. Equivalently it can be obtained as the imaginary part of $(1/\pi)\,\mathrm{Log}[(w - i\alpha)/(w + i\alpha)]$ for a suitable branch of the logarithm. The set $\{w : J_\alpha(w) \geq 1/2\}$ is the exterior of the semi-circle described by $\{w : \text{Re } w > 0, |w| = \alpha\}$. On the strip $|\text{Im}(w)| < \alpha$ we have

$$J_\alpha(u+iv) = \frac{1}{\pi}[\arctan(\frac{u}{\alpha - v}) + \arctan(\frac{u}{\alpha+v})], \quad u > 0, |v| < \alpha\,. \tag{12}$$

We now consider the harmonic function

$$U_\alpha(x,y) = J_\alpha[P_y f(x) + iQ_y f(x)].$$

We first recall a basic fact from ([G], p. 17, Lemma 3.4).

Lemma 3.1. *If $U(x,y)$ is any bounded harmonic function in $y > 0$, then for any $y_1, y_2 > 0$, we have*

$$U(x, y_1 + y_2) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{y_2 U(t, y_1)}{(x-t)^2 + y_2^2} dt. \tag{13}$$

Applying (13) with $x = 0, y_2 = 1, U = U_\alpha$, we have

$$J_\alpha\left(P_{1+y}f(0) + iQ_{1+y}f(0)\right) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{J_\alpha(P_y f(t) + iQ_y f(t))}{1+t^2}\, dt. \tag{14}$$

The right side of (14) is underestimated by

$$\frac{1}{\pi}\int_{\mathbf{R}} \frac{J_\alpha(P_y f(t) + iQ_y f(t))}{1+t^2}\, dt \geq \frac{1}{2\pi}\int_{\{t:|Q_y f(t)| \geq \alpha\}} \frac{dt}{1+t^2}.$$

Using the inequality $|\arctan(x)| \leq |x|$, applied to (12), we can overestimate the left side of (14) by writing

$$J_\alpha(P_{1+y}f(0) + iQ_{1+y}f(0)) \leq \frac{1}{\pi}\left(\frac{P_{1+y}f(0)}{\alpha - |Q_{1+y}f(0)|} + \frac{P_{1+y}f(0)}{\alpha + |Q_{1+y}f(0)|}\right).$$

Therefore we have

$$\begin{aligned} \mu\{x : |Q_y f(x)| \geq \alpha\} &= \frac{1}{\pi}\int_{\{x:|Q_y f(x)| \geq \alpha} \frac{dx}{1+x^2} \\ &\leq \frac{2}{\pi}\left(\frac{P_{1+y}f(0)}{\alpha - |Q_{1+y}f(0)|} + \frac{P_{1+y}f(0)}{\alpha + |Q_{1+y}f(0)|}\right). \end{aligned} \tag{15}$$

In Section 4 (below) we prove that $Hf(x) = \tilde{H}f(x) := \lim_{y\to 0} Q_y f(x)$ a.e., in particular we have convergence in measure. Now from (4), $|Q_1 f(0)| \leq ||f||_{B_1}$, $P_1 f(0) = ||f||_{B_2}$ and the right side of (15) is only increased when we replace $Q_1 f(0)$ by its upper bound $||f||_{B_1}$. Hence

$$\mu\{x : |Hf(x)| \geq \alpha\} \leq \frac{2}{\pi}\left(\frac{||f||_{B_2}}{\alpha - ||f||_{B_1}} + \frac{||f||_{B_2}}{\alpha + ||f||_{B_1}}\right),$$

which proves the result in case $f \geq 0$. In the general case, we write

$$\mu\{x : |Hf(x)| \geq \alpha\} \leq \sum_{j=1}^{4} \mu\{x : |Hf_j(x)| \geq \alpha/4\}$$

and apply the result for non-negative functions to each of the terms on the right.

The upper bound assumes a more familiar form in case f is even, as follows.

Corollary 3.1. *Suppose that $0 \leq f \in B_1$ is even: $f(-x) = f(x)$, $\forall x \in \mathbf{R}$. Then for any $\alpha > 0$ we have*

$$\mu\{x : |Hf(x)| \geq \alpha\} \leq \frac{4||f||_{B_2}}{\pi\alpha}.$$

Proof. In this case we have $Q_y f(0) = 0$ for all $y > 0$. Thus the right side of (15) becomes $\frac{4}{\pi} P_{1+y} f(0) \to \frac{4\|f\|_{B_2}}{\pi}$ when $y \to 0$.

4. Identification of H

It remains to identify the Hilbert transform as defined in (1), with the boundary values of $Q_y f$, namely to show that for a.e. $x \in \mathbf{R}$,

$$Q_y f(x) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{t f(x-t)}{t^2 + y^2}\, dt \to H f(x), \qquad y \downarrow 0. \tag{16}$$

Proposition 4.1. *Suppose that $f \in B_1$. Then*

$$\lim_{y \to 0} \left(\int_{\mathbf{R}} \frac{t f(x-t)}{t^2+y^2}\, dt - \int_{|t|>y} \frac{f(x-t)}{t}\, dt \right) = 0$$

for almost every $x \in \mathbf{R}$.

Proof. We write the above difference as $I_1 + I_2$ where

$$I_1 = \int_{|t|<y} \frac{t f(x-t)}{t^2+y^2}\, dt,$$

$$I_2 = \int_{|t| \geq y} \left(\frac{t}{t^2+y^2} - \frac{1}{t} \right) f(x-t)\, dt.$$

The function $t \to t/(t^2+y^2)$ is odd and increasing for $|t| < y$, so that we can write

$$\begin{aligned} I_1 &= \int_{|t|<y} \frac{t}{t^2+y^2} [f(x-t) - f(x)]\, dt, \\ |I_1| &\leq \frac{1}{2y} \int_{|t|<y} |f(x-t) - f(x)|\, dt \to 0 \end{aligned}$$

at every Lebesgue point of f, especially almost everywhere.

To estimate I_2 we note that its kernel is odd, hence for any $\delta > 0$ and $y < \delta$,

$$\begin{aligned} -I_2 &= \int_{|t|>y} \frac{y^2}{t(t^2+y^2)} [f(x-t) - f(x)]\, dt, \\ |I_2| &\leq \int_{|t|>y} \frac{y^2}{|t|^3} |f(x-t) - f(x)|\, dt \\ &= y^2 \int_{|t|>y} \frac{dF(t)}{|t|^3} \end{aligned}$$

where $F(t) := \int_0^t |f(x-s) - f(x)|\, ds$. Clearly $F(t)/t \to 0$ at every Lebesgue point when $t \to 0$, whereas $F(t) \leq Ct$ when for all t. Therefore we can integrate-by-parts to obtain

$$\int_{|t|>y} \frac{dF(t)}{|t|^3} = \frac{F(y)}{|y|^3} + 3 \int_{|t|>y} \frac{F(t)}{t^4}\, dt.$$

The term at the limits is clearly $o(y^{-2})$ when $y \to 0$. To analyse the new integral, write $F(t)/t = \eta(t)$, $v = 1/t$ to obtain

$$\int_{|t|>y} \frac{F(t)}{t^4}\, dt = \int_0^{1/y} v \eta(1/v)\, dv = o(y^{-2}), \quad y \to 0,$$

which completes the proof that $I_2 \to 0$ when $y \to 0$ for almost every $x \in \mathbf{R}$.

5. Retrieval of the classical Kolmogorov inequality

Our result (6) contains the classical Kolmogorov inequality (2) as a special case, when we introduce a scaling parameter Y. In detail, define

$$\mu_Y(A) = \frac{1}{\pi}\int_A \frac{Y}{t^2+Y^2}\,dt. \tag{17}$$

Then we have the following scaled replacement for (6) when $\alpha > |Q_Y f(0)|$:

$$\mu_Y\{x : |Hf(x)| \geq \alpha\} \leq \frac{2}{\pi}\left(\frac{P_Y f(0)}{\alpha - Q_Y f(0)} + \frac{P_Y f(0)}{\alpha + Q_Y f(0)}\right). \tag{18}$$

Now multiply (18) by Y and take $Y \to \infty$. For the left side, we note that for any Borel set of finite Lebesgue measure we have from the dominated convergence theorem

$$\lim_{Y\to\infty} Y\mu_Y(A) = \frac{1}{\pi}\,\mathrm{Leb}(A).$$

For the right side, we see that when $Y \to \infty$, the dominated convergence theorem shows that for any $f \in L^1(\mathbf{R})$

$$\lim_{Y\to\infty} YP_Y f(0) = \frac{1}{\pi}\int_{\mathbf{R}} f(x)\,dx$$

whereas

$$|Q_Y f(0)| \leq ||f||_{L^1(\mathbf{R})} \times \sup_{t\in\mathbf{R}} \frac{|t|}{t^2+Y^2} \to 0, \quad Y \to \infty.$$

Hence when we multiply (18) by Y and take $Y \to \infty$, we obtain the original form (2) of Kolmogorov's inequality.

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Added in proof

We have recently learned that a different approach to this problem has appeared in the book of Paul Koosis, *The Logarithmic Integral.* I, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, 1998, pp. 59–65.

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