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THE MAGNUS REPRESENTATION OF THE TORELLI GROUP $\mathcal{I}_{g,1}$ IS NOT FAITHFUL FOR $g \geq 2$

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Abstract. In this paper we consider the Magnus representation of the Torelli group. We prove that it is not faithful by showing a non-trivial element in the kernel of this representaion.

1. Introduction

Let Σ_g be a closed oriented surface of genus g and let $\mathcal{M}_g$ be its mapping class group. Namely it is the group of path components of $\mathrm{Diff}_+\Sigma_g$ which is the group of orientation preserving diffeomorphisms of Σ_g. We write $\Sigma_{g,1}$ for an oriented surface obtained from Σ_g by removing an open disk D^2. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$ relative to the boundary, that is, the group of path components of the group of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which restrict to the identity on the boundary. Let $\mathcal{I}_g$ (resp. $\mathcal{I}_{g,1}$) be the Torelli group of Σ_g (resp. $\Sigma_{g,1}$), namely the normal subgroup of $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) consisting of all the elements which act on the homology of Σ_g (resp. $\Sigma_{g,1}$) trivially. Johnson obtained fundamental results concerning the structure of $\mathcal{I}_g$ and $\mathcal{I}_{g,1}$ (e.g. [J]).

As is shown in Birman's book [Bi], Fox's free differential calculus [F] can be used to define a number of interesting matrix representations of free groups of finite ranks and also of various subgroups of the automorphism group of a free group. The first such representation was introduced by Magnus so they are called Magnus representations. For example, the classical Burau and the Gassner representation of the Artin braid group can be obtained in this way. The Magnus representation for the mapping class group $\mathcal{M}_{g,1}$,

$$r : \mathcal{M}_{g,1} \to GL(2g; \mathbf{Z}[\Gamma_0]),$$

has been studied in [Mo1], where $\Gamma_0 = \pi_1(\Sigma_{g,1})$. We restrict this mapping to the Torelli group $\mathcal{I}_{g,1}$ and reduce the coefficients to $\mathbf{Z}[H]$ which is induced by the abelianization $\mathfrak{a} : \Gamma_0 \to H$, where $H = H_1(\Sigma_{g,1}; \mathbf{Z})$. In this way, we obtain a homomorphism

$$\overline{r} : \mathcal{I}_{g,1} \to GL(2g; \mathbf{Z}[H]).$$

We call this mapping $\overline{r}$ the Magnus representation of the Torelli group.

Moody has shown in [M] that the Burau representation τ_n of the braid group B_n is not faithful for $n \geq 9$. His techniques were improved by Long and Paton in [LP],

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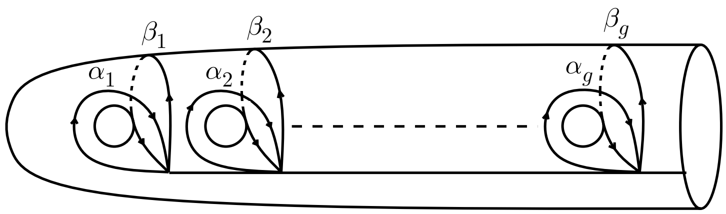

FIGURE 1. Generators of Γ_0

who showed that τ_n is not faithful for $n \geq 6$. Moreover, Bigelow recently proved in [B] that τ_5 is not faithful. On the other hand it seems to still be unknown whether the Gassner representation of the pure braid group is faithful or not. Similarly it has been an open problem to determine whether $\overline{r}$ is injective or not (see [Mo3, Problem 15], [Mo4, Problem 6.23]). In this paper we settle this problem by showing a non-trivial element in the kernel of $\overline{r}$.

2. Definition of the Magnus representation

In this section, we recall the definition of Magnus representation for the mapping class group $\mathcal{M}_{g,1}$ from [Mo1].

Let $\mathbf{Z}[\Gamma_0]$ be the integral group ring of $\Gamma_0 = \pi_1(\Sigma_{g,1})$. Then the Magnus representation for the mapping class group is a mapping

$$r : \mathcal{M}_{g,1} \to GL(2g; \mathbf{Z}[\Gamma_0])$$

defined as follows. We fix a system of generators $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g$ of the free group Γ_0 as shown in Figure 1. Let us simply write $\gamma_1, \dots, \gamma_{2g}$ for them. Then we have the free differential calculus $\frac{\partial}{\partial \gamma_i} : \mathbf{Z}[\Gamma_0] \to \mathbf{Z}[\Gamma_0]$. Let $^{-} : \mathbf{Z}[\Gamma_0] \to \mathbf{Z}[\Gamma_0]$ be the antiautomorphism induced by the mapping $\gamma \mapsto \gamma^{-1}$.

Definition 2.1. We call the mapping

$$\begin{array}{cccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL(2g; \mathbf{Z}[\Gamma_0]) \\ & \varphi & \longmapsto & \left(\overline{\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i}} \right)_{i,j} \end{array}$$

the Magnus representation for the mapping class group $\mathcal{M}_{g,1}$.

This mapping is not a homomorphism in the usual sense but is rather a crossed homomorphism. That is to say,

Proposition 2.2 ([Mo1]). *For any two elements $\varphi, \psi \in \mathcal{M}_{g,1}$, we have*

$$r(\varphi\psi) = r(\varphi) \cdot {}^{\varphi}r(\psi)$$

where ${}^{\varphi}r(\psi)$ denotes the matrix obtained from $r(\psi)$ by applying the automorphism $\varphi : \mathbf{Z}[\Gamma_0] \to \mathbf{Z}[\Gamma_0]$ on each entry.

To obtain a genuine homomorphism, we have to restrict this mapping to the Torelli group $\mathcal{I}_{g,1}$ and reduce the coefficients to $\mathbf{Z}[H]$ which is induced by the abelianization $\mathfrak{a} : \Gamma_0 \to H$, where $H = H_1(\Sigma_{g,1}; \mathbf{Z})$. Since the Torelli group $\mathcal{I}_{g,1}$ acts trivially on H, we obtain a homomorphism

$$\overline{r} : \mathcal{I}_{g,1} \to GL(2g; \mathbf{Z}[H]).$$

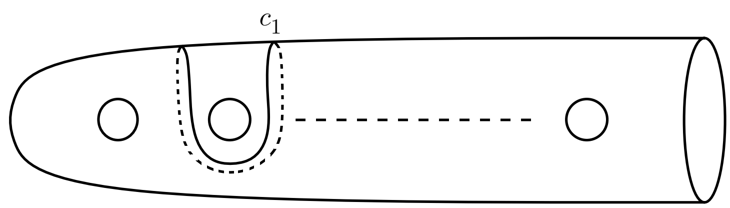

FIGURE 2. Simple closed curve c_1

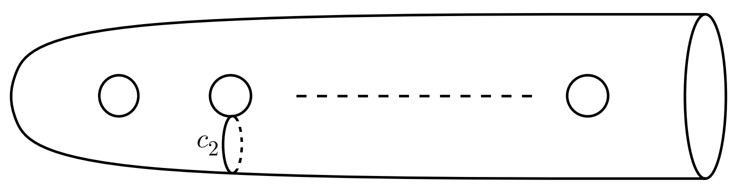

FIGURE 3. Simple closed curve c_2

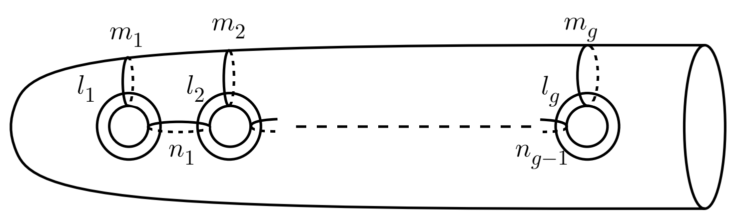

FIGURE 4. Lickorish generators

Thus we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_{g,1} \arrow[r, "r"] \arrow[d, phantom, "\bigtriangledown"] & GL(2g; \mathbf{Z}[\Gamma_0]) \arrow[d, "\mathfrak{a}"] \\
\mathcal{I}_{g,1} \arrow[r, "\overline{r}"'] & GL(2g; \mathbf{Z}[H])
\end{tikzcd}$$

3. MAIN THEOREM

We can easily show that the Magnus representation of the Torelli group is faithful for $g = 1$. However, for higher genera, we obtain the following.

Theorem 3.1. *The Magnus representation of the Torelli group $\mathcal{I}_{g,1}$ is not faithful for $g \geq 2$.*

Proof. We consider two elements φ_1, $\varphi_2\varphi_1{\varphi_2}^{-1}$ of $\mathcal{I}_{g,1}$, where φ_1 and φ_2 are the Dehn twists about simple closed curves c_1 and c_2 as depicted in Figure 2 and Figure 3 respectively. We can write $\varphi_1 = (\mu_2\lambda_2)^6$, $\varphi_2 = \delta\mu_2\delta^{-1}$, $\delta = \lambda_2\nu_1\lambda_1{\mu_1}^2\lambda_1\nu_1\lambda_2$, where λ_i, μ_i, ν_i are the Lickorish generators which are the Dehn twists along simple closed curves l_i, m_i, n_i as shown in Figure 4. We can see the formulas for the action of λ_i, μ_i and ν_i on the free group generators α_i and β_j in [Mo2, Lemma 4.4].

Since

$$\varphi_1(\gamma_j) = \begin{cases} \beta_2\alpha_2{\beta_2}^{-1}\alpha_2\beta_2{\alpha_2}^{-1}{\beta_2}^{-1}, & j=3,\\ \beta_2\alpha_2{\beta_2}^{-1}{\alpha_2}^{-1}\beta_2\alpha_2\beta_2{\alpha_2}^{-1}{\beta_2}^{-1}, & j=4,\\ \gamma_j, & \text{otherwise,} \end{cases}$$

$$\varphi_2(\gamma_j) = \begin{cases} \alpha_2{\beta_2}^{-1}{\alpha_2}^{-1}\beta_1\alpha_1{\beta_1}^{-1}\alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\alpha_2\beta_2{\alpha_2}^{-1}, & j=1,\\ \alpha_2{\beta_2}^{-1}{\alpha_2}^{-1}\beta_1\alpha_1{\beta_1}^{-1}{\alpha_1}^{-1}\beta_1\alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\alpha_2\beta_2{\alpha_2}^{-1}, & j=2,\\ \alpha_2{\beta_2}^{-1}{\alpha_2}^{-1}\beta_1\alpha_1{\beta_1}^{-1}{\alpha_1}^{-1}\alpha_2, & j=3,\\ \gamma_j, & \text{otherwise,} \end{cases}$$

$${\varphi_2}^{-1}(\gamma_j) = \begin{cases} \alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\alpha_2\beta_2{\alpha_2}^{-1}\alpha_1\alpha_2{\beta_2}^{-1}{\alpha_2}^{-1}\beta_1\alpha_1{\beta_1}^{-1}{\alpha_1}^{-1}, & j=1,\\ \alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\alpha_2\beta_2{\alpha_2}^{-1}\beta_1\alpha_2{\beta_2}^{-1}{\alpha_2}^{-1}\beta_1\alpha_1{\beta_1}^{-1}{\alpha_1}^{-1}, & j=2,\\ \alpha_1\beta_1{\alpha_1}^{-1}{\beta_1}^{-1}\alpha_2\beta_2, & j=3,\\ \gamma_j, & \text{otherwise,} \end{cases}$$

we obtain

$$\frac{\partial\varphi_1(\gamma_3)}{\partial\gamma_i} = \begin{cases} \beta_2+\beta_2\alpha_2{\beta_2}^{-1}-\beta_2\alpha_2{\beta_2}^{-1}\alpha_2\beta_2{\alpha_2}^{-1}, & i=3,\\ 1-\beta_2\alpha_2{\beta_2}^{-1}+\beta_2\alpha_2{\beta_2}^{-1}\alpha_2 \\ \quad -\beta_2\alpha_2{\beta_2}^{-1}\alpha_2\beta_2{\alpha_2}^{-1}{\beta_2}^{-1}, & i=4,\\ 0, & \text{otherwise,} \end{cases}$$

$$\mathfrak{a}\left(\frac{\partial\varphi_1(\gamma_3)}{\partial\gamma_i}\right) = \begin{cases} y_2+x_2-x_2y_2, & i=3,\\ 1-x_2+{x_2}^2-x_2, & i=4,\\ 0, & \text{otherwise.} \end{cases}$$

Here we denote by x_i, y_i the abelianization of α_i, β_i respectively. By similar calculation, we have $\overline{r}(\varphi_1) = I_{2g}+M_1$ and $\overline{r}(\varphi_2\varphi_1{\varphi_2}^{-1}) = I_{2g}+M_2$, where

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & -a_2b_2 & -{b_2}^2 & 0 & \cdots & 0\\ 0 & 0 & {a_2}^2 & a_2b_2 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -a_1b_1b_2b_2' & -{b_1}^2b_2b_2' & -a_2b_1b_2' & -b_1b_2b_2' & 0 & \cdots & 0\\ {a_1}^2b_2b_2' & a_1b_1b_2b_2' & a_1a_2b_2' & a_1b_2b_2' & 0 & \cdots & 0\\ -a_1{b_2}^2 & -b_1{b_2}^2 & -a_2b_2 & -{b_2}^2 & 0 & \cdots & 0\\ a_1a_2b_2 & a_2b_1b_2 & {a_2}^2 & a_2b_2 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Here $a_i = 1-{x_i}^{-1}$, $b_i = 1-{y_i}^{-1}$, $b_2' = 1-y_2$. Because of the fact that $M_1M_2 = M_2M_1$ is the zero matrix, we have

$$\overline{r}(\varphi_1)\overline{r}(\varphi_2\varphi_1{\varphi_2}^{-1}) = \overline{r}(\varphi_2\varphi_1{\varphi_2}^{-1})\overline{r}(\varphi_1).$$

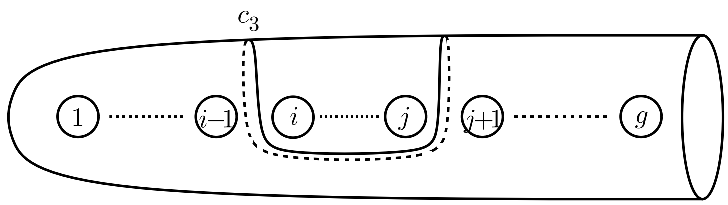

FIGURE 5. Simple closed curve c_3

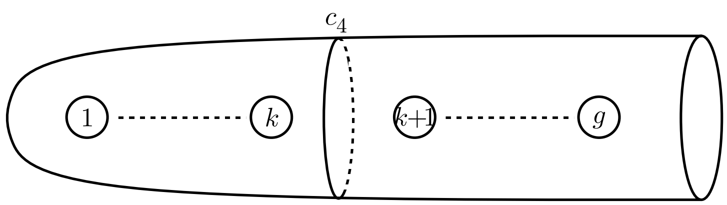

FIGURE 6. Simple closed curve c_4

It follows that

$$\overline{r}([\varphi_1, \varphi_2\varphi_1{\varphi_2}^{-1}]) = I_{2g}.$$

It is easy to see that $[\varphi_1, \varphi_2\varphi_1\varphi_2^{-1}]$ is a non-trivial element. Hence $\overline{r}$ is not injective and this completes the proof. □

Remark 3.2. We set $\psi = [\varphi_1, \varphi_2\varphi_1{\varphi_2}^{-1}]$. Then ψ can be expressed as a word of length 116 in the Lickorish generators. The words $\psi(\gamma_i)$ $(1 \leq i \leq 4)$ are so large that we cannot write them down here. However, it can be written as

$$\psi(\gamma_i) = l_i\gamma_i l_i'$$

for some $l_i, l_i' \in [[\Gamma_0, \Gamma_0], [\Gamma_0, \Gamma_0]]$. This equation suggests $\psi \in \mathrm{Ker}\ \overline{r}$ (cf. [Bi, Theorem 3.5, Theorem 3.16]).

Remark 3.3. Here is another element in the kernel of $\overline{r}$. Let φ' and φ'' be the Dehn twists about simple closed curves c_3 and c_4 as depicted in Figure 5 and Figure 6 respectively. Here we can take i, j, k arbitrarily. Again explicit computation shows that

$$\overline{r}(\varphi')\overline{r}(\varphi'') = \overline{r}(\varphi'')\overline{r}(\varphi').$$

Hence $[\varphi', \varphi'']$ is also an element in the kernel of $\overline{r}$.

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