PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 3, Pages 903–907
S 0002-9939(01)06131-7

LOCAL CONNECTEDNESS IN TRANSFORMATION GROUPS

KEITH WHITTINGTON

(Communicated by Alan Dow)

Abstract. This note shows that under very general conditions, in a topological transformation group, the natural map from the group onto an orbit is almost open. The implications for local connectedness are investigated. In particular, if the image of a path component of the group is sufficiently "robust", the orbit will be locally connected.

1. Introduction

Throughout our discussion (G, X) will be a topological transformation group, p a fixed element of X, T the function from G to X that carries g to gp, and $H = T^{-1}(p)$.

A popular result of Effros [2, Theorem 2.1, p. 39] states that if G and X are Polish spaces, each such T will be open as a map from G to its image Gp if and only if each orbit of G is second category in itself. This note shows T will be *almost open* under greatly relaxed conditions. For instance, if G is either separable or Lindelöf and $T(G)$ is second category in X (resp., in itself), T will be almost open from G to X (resp., to $T(G)$).

The implications for local connectedness are investigated. For instance, in [4, Theorem 3.7, p. 396], Ungar shows that every path connected, complete, separable metric group is locally path connected. We show that completeness and path connectedness can be relaxed simply to the group having a second category path component, if the conclusion is relaxed to the group being locally connected. There are broader implications for topological transformation groups. It is a long standing question whether every arc-connected homogeneous continuum is locally connected. We give an affirmative answer in the case that enough paths in the space can be "raised" to paths in the group.

I am indebted to Stephen Watson for his contributions, both material and intellectual, and to York University for hosting me on my sabbatical.

2. Preliminaries

A topological transformation group is a pair (G, X) where G is a topological group that acts continuously on the topological space X.

Received by the editors April 12, 2000 and, in revised form, August 28, 2000.
2000 *Mathematics Subject Classification.* Primary 54H15; Secondary 54D05.

If $a \in G$, the function $x \to ax$ is a homeomorphism of X. Thus for every $B \subseteq X$ and $C \subseteq G$ we have

$$(aB)^\circ = aB^\circ, \tag{1}$$

$$\overline{aB} = a\overline{B}, \tag{2}$$

$$(\overline{T(aC)})^\circ = a(\overline{T(C)})^\circ. \tag{3}$$

Notation. If $\emptyset \neq S \subseteq G$, $x \in X$, and $w \in W \subseteq$ some space, then:

1. $[w]_W =$ the path component of w in W.
2. $(w)_W =$ the connected component of w in W.
3. $P_x = \{y \in X : \text{there is a path } \phi \text{ in } G \text{ with } x, y \in T(\phi)\}$.
4. $\mathrm{w}(S) = \min\{|\mathfrak{B}| : \mathfrak{B} \text{ is a base for } S\} + \omega$.
5. $d(S) = \min\{|D| : D \text{ is a dense subset of } S\}$.
6. $L(S) =$ the smallest cardinal κ such that every open cover of S has a subcover of cardinal number $\leq \kappa$.
7. $\mu(S) =$ the smallest cardinal κ such that for each open neighborhood U of e in G, there is a set S' in S such that $|S'| \leq \kappa$ and $S \subseteq S'U$.

If $\kappa > 0$ is a cardinal, we will say that a subset A of a space Y is *second κ-category* in Y if A is not contained in the union of $\leq \kappa$ nowhere dense subsets of Y. Thus second ω-category means the same as second category, and second 1-category means that the closure has nonempty interior.

By [1, Theorem 12, p. 492], under the compact-open topology, the space G^I of all paths in G is a topological group under the operation $\alpha\beta(t) = \alpha(t)\beta(t)$. It easily follows that $[e]_G$ is a normal subgroup of G. If ϕ is a path in G from x to y, and $a \in G$, ϕa (similarly, $a\phi$) will denote the path from xa to ya equal to the product of ϕ with the constant path at a.

A space is *locally connected* if it has a base of open connected sets. It is *connected im kleinen* at a point if the point has a neighborhood base of connected sets (not necessarily open). A space which is everywhere connected im kleinen is locally connected.

A function f is *almost open* if for each x in the domain and each open neighborhood U of x, $\overline{f(U)}$ is a neighborhood of $f(x)$.

3. Almost open

Lemma 1. *Let $C \subseteq D \subseteq G$. If there is a subset S' of G such that $T(S'C)$ is second $|S'|$-category in X, then $\overline{T(C)}$ has nonempty interior. If additionally $C^{-1}C \subseteq D$, then $\overline{T(D)}$ is a neighborhood of p.*

Proof. Since $T(S'C) = \bigcup_{s \in S'} T(sC)$, some $\overline{T(sC)}$ has nonempty interior. Since $(\overline{T(sC)})^\circ = s(\overline{T(C)})^\circ$, $\overline{T(C)}$ also has nonempty interior.

Note that $T(C)$ intersects $(\overline{T(C)})^\circ$. If y is a point in this intersection, then $y = cp$ for some $c \in C$. If $C^{-1}C \subseteq D$, then $p = c^{-1}y \in c^{-1}(\overline{T(C)})^\circ = (\overline{T(c^{-1}C)})^\circ \subseteq (\overline{T(C^{-1}C)})^\circ \subseteq (\overline{T(D)})^\circ$. □

Theorem 1. *T is almost open if and only if for each open neighborhood U of e there is a set S' in G such that $T(S'U)$ is second $|S'|$-category in X.*

Proof. If T is almost open, then for each open neighborhood U of e, $(\overline{T(U)})^\circ \neq \emptyset$, so we can take $S' = \{e\}$.

For the reverse implication, let aU be a basic neighborhood of $a \in G$, where U is an open neighborhood of the identity. Then $(\overline{T(aU)})^\circ = a(\overline{T(U)})^\circ$, so $\overline{T(aU)}$ will be a neighborhood of $T(a)$ if and only if $\overline{T(U)}$ is a neighborhood of p.

Let V be a symmetric open neighborhood of e such that $V^2 \subseteq U$. Then $VV^{-1} \subseteq U$. By hypothesis there is a subset S' of G such that $T(S'V)$ is second $|S'|$-category in X, so by Lemma 1, $\overline{T(U)}$ is a neighborhood of p. ☐

Theorem 2. *If G has subset S such that $T(S)$ is second $\mu(S)$-category in X, then T is almost open.*

Proof. Let U be an open neighborhood of e. There is a subset S' of S such that $|S'| \leq \mu(S)$ and $S \subseteq S'U$. Since $T(S)$ is second $\mu(S)$-category, it is second $|S'|$-category, as is the larger set $T(S'U)$. Thus T is almost open by Theorem 1. ☐

Lemma 2. *For each nonempty $S \subseteq G$, $\mu(S) \leq \min\{d(S), L(S)\}$.*

Proof. Let U be an open neighborhood of e.

Let D be a dense subset of S such that $|D| = d(S)$. Let $s \in S$. There is a net $\{d_\alpha\}$ in D converging to s. Thus $\{d_\alpha^{-1}s\}$ converges to e, and for some α, $d_\alpha^{-1}s \in U$. Therefore, $s \in d_\alpha U$. Hence, $S \subseteq DU$. Thus $\mu(S) \leq d(S)$.

The open sets $sU, s \in S$, cover S. Thus, there is a subcover $\{s'U\}_{s' \in S'}$ with $|S'| \leq L(S)$. Since $S \subseteq S'U$, it follows that $\mu(S) \leq L(S)$. ☐

Corollary 1. *If G has a subspace S that is either separable or Lindelöf and such that $T(S)$ is second category in X, then T is almost open.*

4. Local connectedness

Lemma 3. *If X is regular, T is almost open, and G has a base each element of which has connected image under T, then X is connected im kleinen at each point of $T(G)$ and $T(G)$ is locally connected.*

Proof. Let $x \in U \cap T(G)$, where U is open in X. There is an open neighborhood V of x such that $\overline{V} \subseteq U$. By hypothesis, there is an open set C in $T^{-1}(V)$ such that $T(C)$ is connected and contains x. Since T is almost open, $\overline{T(C)}$ is a neighborhood of x. Since $\overline{T(C)} \subseteq \overline{V} \subseteq U$, it follows that X is connected im kleinen at x. Since $T(C)$ is connected and contained in $T(G)$, $\overline{T(C)} \cap T(G)$ is connected and is thus a connected neighborhood of x relative to $T(G)$. It follows that $T(G)$ is also connected im kleinen at x. Since $T(G) = Gp$, an orbit of G, it is homogeneous and thus locally connected. ☐

We now introduce a new topology τ_α on G in which sets of the form $a[e]_U$ form an open neighborhood base at $a \in G$, where U ranges over open neighborhoods of e. By [3, Theorem 4.5, p. 18], it is easily seen that these sets form a base for a topology under which G is still a topological group. Since τ_α is finer than the original topology, (G, τ_α) still acts continuously on X.

We will use μ_α, d_α, etc., when τ_α determines the invariant.

Lemma 4. *For each path component S of G, $\mu_\alpha(S) = \mu_\alpha([e]_G) \leq d_\alpha([e]_G) \leq \mathrm{w}([e]_G)$.*

Proof. Note that $[e]_G$ is a normal subgroup of G, and the various cosets of $[e]_G$ are precisely the path components of G. So each path component is a translation of $[e]_G$. Translations are still homeomorphisms with respect to τ_α, so the cardinal invariants of S and $[e]_G$ are equal. Thus by Lemma 2, all that need be shown is $d_\alpha([e]_G) \leq \mathrm{w}([e]_G)$.

By [1, Theorem 5, p. 484], $\mathrm{w}([e]_G^I) \leq \mathrm{w}([e]_G)$. Thus the subspace $B = \{\phi \in G^I \mid \phi(0) = e\}$ also has $\mathrm{w}(B) \leq \mathrm{w}([e]_G)$. Let $\{\phi_\gamma\}$ be a dense subset of B such that $|\{\phi_\gamma\}| \leq \mathrm{w}(B)$, and let $a_\gamma = \phi_\gamma(1)$. It suffices to show that $\{a_\gamma\}$ is dense in $([e]_G, \tau_\alpha)$.

Let $a[e]_U$ be a basic τ_α neighborhood of $a \in [e]_G$. There is a path $\phi \in B$ from e to a. Thus there is a net $\{\phi_{\gamma_\sigma}\}$ converging to ϕ. Hence $\{\phi^{-1}\phi_{\gamma_\sigma}\}$ converges to $\{e\}$, the constant path. Some $\phi^{-1}\phi_{\gamma_\sigma}$ is contained in U, so $a^{-1}a_{\gamma_\sigma} \in [e]_U$. Thus $a_{\gamma_\sigma} \in a[e]_U$. ☐

Theorem 3. *If X is regular and $T([e]_G)$ (equivalently, the image of any path component) is second $\mathrm{w}([e]_G)$-category in X, then X is connected im kleinen at each point of $T(G)$, and $T(G)$ is locally connected.*

Proof. Since $T([e]_G)$ is second $\mathrm{w}([e]_G)$-category, it is also second $\mu_\alpha([e]_G)$-category by Lemma 4. Thus by Theorem 2, T is almost open with respect to τ_α, and the result follows from Lemma 3. ☐

We mention that in Theorem 3 (and similarly for subsequent results), $T(G)$ can take the role of X, so if $T([e]_G)$ is second $\mathrm{w}([e]_G)$-category in $T(G)$, $T(G)$ will be locally connected.

Theorem 3 can be nominally strengthened by looking at the paths in X that can be "raised" to G.

Lemma 5. *If $T(a) = x$, then $P_x = T(a[e]_G)$, the image of the path component of a in G.*

Proof. Let $y \in P_x$ and ϕ a path in G such that $T(\phi)$ contains x and $y = T(b)$. Then ϕ contains points ah, bh_1, where $h, h_1 \in H$. Let $\sigma = a^{-1}\phi h^{-1}$. Then σ is a path in $[e]_G$ and $x, y \in T(a\sigma)$. Thus $P_x \subseteq T(a[e]_G)$. The reverse inclusion holds since $a[e]_G$ is path connected. ☐

Theorem 3 and Lemma 5 immediately yield:

Corollary 2. *If X is regular, and some P_x is second $\mathrm{w}([e]_G)$-category in X, then X is connected im kleinen at each point of $T(G)$, and $T(G)$ is locally connected.*

The next two results, though weaker, give the cases which are probably of greatest interest.

Corollary 3. *If $[e]_G$ is second countable, X is regular, and some P_x is second category in X, then X is connected im kleinen at each point of $T(G)$, and $T(G)$ is locally connected.*

Corollary 4. *If $[e]_G$ is second countable, and X is homogeneous and regular with some second category path component $[x]_X$ such that for each path ϕ in $[x]_X$ there is a path ϕ' in G with $T(\phi') = \phi$, then X is locally connected.*

Proof. In this case, $[x]_X = P_x$. By Corollary 3, X is connected im kleinen at each point of $T(G)$. The result now follows by homogeneity. ☐

Since topological groups act on themselves and are always regular, we immediately have the following generalization of Ungar's result [4, Theorem 3.7, p. 396].

Corollary 5. *Every topological group with a second category, second countable path component is locally connected.*

We mention that G is also a topological group under a topology τ_c similar to τ_α except that connected components take the place of path components. Thus we can say, for instance, that if G has a subset S such that $T(S)$ is second $\mu_c(S)$-category in X, the conclusion of Theorem 3 holds. This is potentially a stronger result because $\mu_c(S) \leq \mu_\alpha(S)$. However, it is not clear what is gained, for there is no readily apparent choice for S that improves on letting $S = [e]_G$.

The main development of this paper could have been given as follows. If for each neighborhood U of e there is a set S' such that $T(S'[e]_U)$ is second $|S'|$-category, then each $T([e]_U)$ must be somewhere dense, and this in turn implies that $T(G)$ is locally connected. However, for each U, there is a set S' with $|S'| \leq \mathrm{w}([e]_G)$ such that $[e]_G \subseteq S'[e]_U$, and this leads to several results dependent on the "category" of $T([e]_G)$. In the next theorem, we give sets C_U that can be used in place of $[e]_U$ with similar result. Typically C_U is much larger than $[e]_U$, so we have a stronger result. However, we can as yet offer no better choice of set to take the role of $[e]_G$.

Let x be any fixed point of X. For each open neighborhood U of x, let $C_U = T^{-1}((x)_U)$.

Theorem 4. *Suppose X is completely metrizable and G acts transitively on X. Then X is locally connected if and only if for each open neighborhood U of x there is a subset S' of G such that $T(S'C_U)$ is second $|S'|$-category in X.*

Proof. If X is locally connected, then each $T(C_U)$ is open in X, so we can choose $S' = \{e\}$.

Conversely, assume that we can always find the sets S'. Suppose X is not locally connected. Then by [5, Lemma 2], x has a neighborhood U each component of which is nowhere dense. By Lemma 1, $\overline{T(C_U)}$ has nonempty interior. However, $\overline{T(C_U)} = \overline{(x)_U}$, contradicting that U has nowhere dense components. ☐

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Department of Mathematics, University of the Pacific, Stockton, California 95211
E-mail address: `kwhittin@uop.edu`