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SUMS OF NUMBERS WITH SMALL PARTIAL QUOTIENTS

S. ASTELS

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ABSTRACT. In a paper of James Hlavka it is stated that $F(3)+F(2)+F(2) \neq \mathbb{R}$. In this manuscript we show that this is false by establishing that $F(3) \pm F(2) \pm F(2) = \mathbb{R}$. We also describe the corresponding products and quotients.

1. Introduction

For any positive integer m let $F(m)$ be the set of numbers

$$F(m) = \{[t, a_1, a_2, \dots]\,;\, t \in \mathbb{Z},\, 1 \le a_i \le m \text{ for } i \ge 1\}$$

where by $[a_0, a_1, a_2, \dots]$ we denote the *continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

with *partial quotients* a_0, a_1, a_2 and so on. In 1947 Marshall Hall, Jr., [7] proved that

$$F(4) + F(4) = \mathbb{R}$$

where for two sets A and B of real numbers we denote by $A + B$ the set

$$A + B = \{a + b\,;\, a \in A \text{ and } b \in B\}.$$

James Hlavka improved this result in 1975 [8] establishing that in fact

$$F(4) + F(3) = \mathbb{R}.$$

Hlavka also proved several other similar results, including

$$F(7) + F(2) = \mathbb{R} \quad \text{and} \quad F(3) + F(3) + F(2) = \mathbb{R}.$$

However, there were several cases that Hlavka was unable to treat with his techniques. In [4] the author used a new approach to establish that $F(5) \pm F(2) = \mathbb{R}$, where for sets A and B we let $A - B$ denote the pointwise difference of the two sets. Hanno Schecker [9] and Gregory Freiman [6] independently showed that $F(3)+F(3)$ contains many large intervals (it is easy to show that $F(3)+F(3) \neq \mathbb{R}$), while in [1]

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the author used similar techniques to give a complete characterization of $F(3)+F(3)$ and prove that $F(3)-F(3)=\mathbb{R}$.

In this manuscript we deal with one of the few remaining cases. Hlavka claimed (without proof) that $F(3)+F(2)+F(2)\neq\mathbb{R}$, but in this paper we will show that this is not true. We shall establish the following result.

Theorem 1.1. *We have*

$$F(3)\pm F(2)\pm F(2)=\mathbb{R}.$$

Several cases still remain open. For example, at present we do not know whether or not either of the sets $F(3)+F(2)$ or $F(2)+F(2)+F(2)$ contain intervals. In fact, the best result concerning the Hausdorff dimensions of these sets (see [2]) is

$$\dim_H(F(3)+F(2))\geq 0.808 \quad \text{and} \quad \dim_H(F(2)+F(2)+F(2))\geq 0.886\ .$$

Using the ideas behind the proof of Theorem 1.1, in [5] the author proved the following theorem.

Theorem 1.2. *There exists a constant c such that*

$$F(3)F(2)F(2)\supseteq(-\infty,-c]\cup[c,\infty).$$

Further,

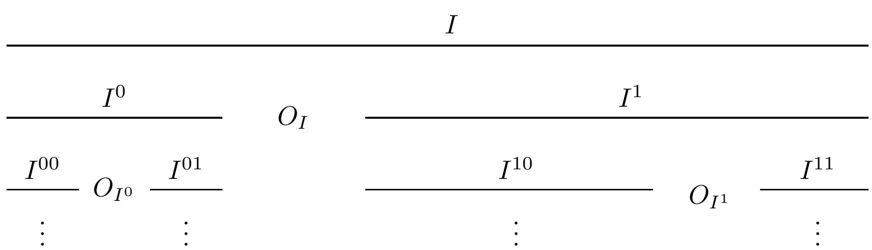
$$F(3)F(2)/F(2)=F(2)F(2)/F(3)=\mathbb{R}\backslash\{0\}.$$

2. Background

We define a *generalized Cantor set* (henceforth known as a *Cantor set*) to be any set C of real numbers of the form

$$C=I\setminus\bigcup_{i\geq 1}O_i$$

where I is a finite closed interval and $\{O_i\ ;\ i\geq 1\}$ is a countable collection of disjoint open intervals contained in I. Equivalently, we may describe a Cantor set C by construction from the interval I.



We have

$$C=I\backslash\bigcup_w O_{I^w}=\bigcap_{n\geq 0}\left(\bigcup_{|w|=n}I^w\right)$$

where the first union is over all binary words and the second is over all binary words of length n (for a more detailed explanation the reader is directed to [3]).

This process is called a *derivation* of C from I. The intervals $I, I^0, \dots$ are called *bridges* of the derivation, while the open intervals $O_I, O_{I^0}, \dots$ are called *gaps* of C. We define the *thickness* of the derivation $\mathcal{D}$ to be

$$\tau(\mathcal{D}) = \inf_w \left(\frac{\min(|I^{w0}|, |I^{w1}|)}{|O_{I^w}|} \right).$$

We define the *thickness* of the Cantor set C to be

$$\tau(C) = \sup_{\mathcal{D}} \tau(\mathcal{D})$$

where the supremum is over all derivations $\mathcal{D}$ of C. It is not difficult to show that the supremum is attained if the sequence $\{|O_i|\}_i$ is non-decreasing (see Lemma 3.1 of [3]). We also define the *normalized thickness* of C, $\gamma(C)$, to be

$$\gamma(C) = \frac{\tau(C)}{\tau(C)+1}.$$

Let k be an integer which is at least 2, and assume that for $1 \le j \le k$, C_j is a Cantor set derived from I_j. We would like to determine when

$$C_1 + \dots + C_k = I_1 + \dots + I_k. \tag{1}$$

If $I_1, \dots, I_{k-1}$ are all much smaller than one of the gaps in C_k, then (1) cannot hold. Hence in our approach to finding sums of Cantor sets we will only consider sets that are approximately the same size, as follows. Let k be an integer which is at least 2, and assume that for $1 \le j \le k$, A_j is a bridge of the Cantor set C_j, with O_j a gap of C_j of maximal size contained in A_j. We say that the sequence of bridges $(A_1, \dots, A_k)$ is *compatible* if

$$|A_{r+1}| \ge |O_j| \quad \text{and} \quad |A_1| + \dots + |A_r| \ge |O_{r+1}|$$

for $r = 1, \dots, k-1$ and $j = 1, \dots, r$. Note that if $k = 2$, then this is equivalent to the condition

$$|A_1| \ge |O_2| \quad \text{and} \quad |A_2| \ge |O_1|.$$

In [3] the author derived a result concerning the sum of a finite number of Cantor sets.

Theorem 2.1. *Let k be a positive integer and for $j = 1, \dots, k$ let C_j be a Cantor set derived from I_j. Put $S_\gamma = \gamma(C_1) + \dots + \gamma(C_k)$, and assume that $(I_1, \dots, I_k)$ is compatible. If $S_\gamma \ge 1$, then*

$$C_1 + \dots + C_k = I_1 + \dots + I_k.$$

Otherwise

$$\gamma(C_1 + \dots + C_k) \ge S_\gamma \quad \textit{and} \quad \dim_H(C_1 + \dots + C_k) \ge \frac{\log 2}{\log(1 + 1/S_\gamma)}.$$

For any positive integer m we put

$$g(m) = \frac{-m + \sqrt{m^2 + 4m}}{2}, \qquad C(m) = [0,1] \cap F(m)$$

and let $I(m)$ be the closed interval

$$I(m) = \left[\ [0,\overline{m,1}], [0,\overline{1,m}]\ \right] = \left[\frac{g(m)}{m}, g(m)\right] = \left[\frac{-1+\sqrt{1+4/m}}{2}, \frac{-m+\sqrt{m^2+4m}}{2}\right].$$

We may characterize $C(m)$ as a Cantor set derived from $I(m)$ in the following manner. For any real a and b, we denote by $[[a,b]]$ and $((a,b))$ the intervals

$$[[a,b]] = [\min\{a,b\}, \max\{a,b\}]$$

and

$$((a,b)) = (\min\{a,b\}, \max\{a,b\}).$$

Assume that

(2) $$A = [[\ [0,a_1,\dots,a_r,b,\overline{m,1}], [0,a_1,\dots,a_r,m,\overline{1,m}]\ ]]$$

is a bridge of $C(m)$ of level n with $b < m$. We define A^0, A^1, and O_A by setting

$$A^0 = [[\ [0,a_1,\dots,a_r,b,\overline{m,1}], [0,a_1,\dots,a_r,b,\overline{1,m}]\ ]],$$
$$O_A = ((\ [0,a_1,\dots,a_r,b,\overline{1,m}], [0,a_1,\dots,a_r,b+1,\overline{m,1}]\ ))$$

and

$$A^1 = [[\ [0,a_1,\dots,a_r,b+1,\overline{m,1}], [0,a_1,\dots,a_r,m,\overline{1,m}]\ ]].$$

Note that A^0 is of the form (2) with $a_{r+1} = b$ and b replaced by 1. Similarly A^1 is also of the form (2). Since $I(m)$ is of the form (2) with $r = 0$ and $b = 1$, by induction we obtain a derivation of $C(m)$ from $I(m)$.

By calculation it can be shown (see Lemma 4.2 of [3]) that

$$\tau(C(m)) = \frac{g(m)(m-1)}{m-g(m)(m-1)} \cdot \frac{m+g(m)-1}{m+g(m)}.$$

Hence we may easily calculate $\tau(C(m))$ and $\gamma(C(m))$. For example,

(3) $$\gamma(C(2)) = 0.267\dots \quad \text{and} \quad \gamma(C(3)) = 0.451\dots .$$

3. Proof of the main result

Proof of Theorem 1.1. We first consider the set $C(3)+C(2)+C(2)$. Note that from (3) we have

$$\gamma(C(3)) + \gamma(C(2)) + \gamma(C(2)) = 0.98\dots < 1;$$

hence we cannot use Theorem 2.1 with the Cantor sets $C(3)$, $C(2)$, and $C(2)$ to find intervals in $C(3)+C(2)+C(2)$. However, since thickness (and therefore normalized thickness) is an infimum, we might hope to increase it by looking at Cantor sets properly contained in $C(3)$ or $C(2)$. For $m = 2$ or $m = 3$ let $w = a_1 \dots a_r$ be a word with digits a_i between 1 and m inclusive. We denote by $I(m;w)$ the bridge of $C(m)$

$$I(m;w) = [[\ [0,a_1,\dots,a_r,\overline{1,m}], [0,a_1,\dots,a_r,\overline{m,1}]\ ]]$$

and put $C(m;w) = C(m) \cap I(m;w)$. By calculation we find that if $w, v \neq \emptyset$, then

$$\gamma(C(2;v)) \geq 0.2800 \quad \text{and} \quad \gamma(C(3;w)) \geq 0.4565.$$

Note that this implies that

$$\gamma(C(3;w)) + \gamma(C(2)) + \gamma(C(2;v)) \geq 1.004 \tag{4}$$

and

$$\gamma(C(3)) + \gamma(C(2;v)) + \gamma(C(2;v)) \geq 1.011. \tag{5}$$

By calculation we have

$$\begin{aligned} |I(2)| &= 0.3660\ldots, & |O_{I(2)}| &= 0.1547\ldots, \\ |I(2;1)| &= 0.1547\ldots, & |O_{I(2;1)}| &= 0.0689\ldots, \\ |I(2;2)| &= 0.0566\ldots, & |O_{I(2;2)}| &= 0.0247\ldots, \\ |I(3)| &= 0.5275\ldots, & |O_{I(3)}| &= 0.1165\ldots, \\ |I(3;1)| &= 0.2330\ldots, & |O_{I(3;1)}| &= 0.0518\ldots, \\ |I(3;3)| &= 0.0426\ldots, & |O_{I(3;3)}| &= 0.0095\ldots . \end{aligned}$$

Therefore

$$(I(2;2), I(3;3), I(2;1)) \quad \text{and} \quad (I(2;1), I(2;1), I(3))$$

are compatible. By (4), (5), and Theorem 2.1 we have

$$C(2;2) + C(3;3) + C(2;1) = I(2;2) + I(3;3) + I(2;1) = [1.20\ldots, 1.46\ldots]$$

and

$$C(2;1) + C(2;1) + C(3) = I(2;1) + I(2;1) + I(3) \quad = [1.41\ldots, 2.25\ldots].$$

Thus

$$C(3) + C(2) + C(2) \supseteq [1.20\ldots, 2.25\ldots]$$

so

$$F(3) + F(2) + F(2) = \mathbb{Z} + C(3) + C(2) + C(2) = \mathbb{R}$$

as required.

The remaining results are proved in an analogous fashion. Note that if C is a Cantor set, then $-C$ is a Cantor set, and $\tau(-C) = \tau(C)$. Now,

$$(I(2;1), -I(3;1), I(2)) \quad \text{and} \quad (I(2;1), I(2;1), -I(3))$$

are compatible. Since

$$I(2;1) - I(3;1) + I(2) = [0.15\ldots, 0.90\ldots]$$

and

$$I(2;1) + I(2;1) - I(3) = [0.36\ldots, 1.20\ldots]$$

we have

$$F(2) + F(2) - F(3) = \mathbb{R}.$$

Similarly

$$(I(2;1), -I(2;1), I(3)) \quad \text{and} \quad (I(2;1), I(3;1), -I(2))$$

are compatible, with

$$I(2;1) - I(2;1) + I(3) = [0.10\ldots, 0.94\ldots]$$

and

$$I(2;1) + I(3;1) - I(2) = [0.40\dots, 1.15\dots].$$

Thus

$$F(2) - F(2) + F(3) = \mathbb{R}$$

and the theorem follows. □

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Department of Mathematics, University of Georgia, Athens, Georgia 30602
E-mail address: sastels@math.uga.edu